Game Theory and its Applications to Networks -

Part I: Strict Competition

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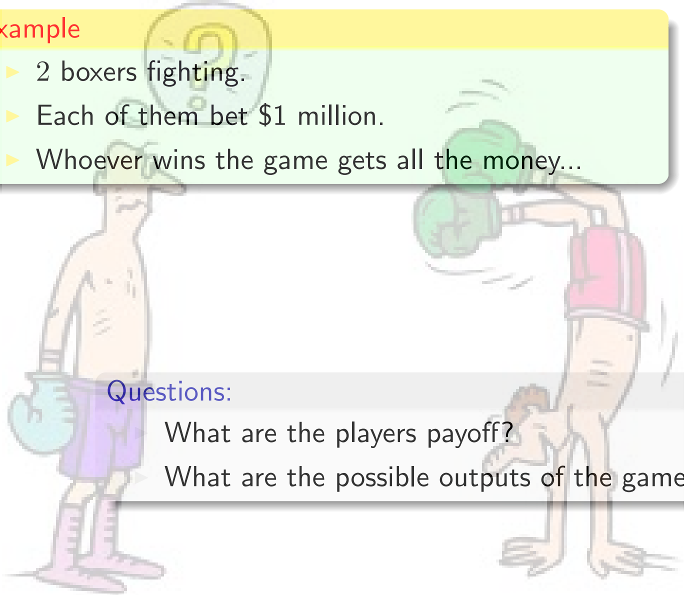
Master ENS Lyon, Fall 2010

What is Game Theory and what is it for?

Definition (Roger Myerson, *"Game Theory, Analysis of Conflicts"*)

"Game theory can be defined as the study of mathematical models of conflict and cooperation between intelligent rational decision-makers. Game theory provides general mathematical techniques for analyzing situations in which two or more individuals make decisions that will influence one another's welfare"

- Branch of optimization
- Multiple actors with different objectives
- Actors interact with each others



Example

- 2 boxers fighting.
- Each of them bet $1 million.
- Whoever wins the game gets all the money...

Questions:

- What are the players payoff?
- What are the possible outputs of the game?

Outline

Outline

Definition:
Two Players, Zero-Sum Game.

- 2 players, finite number of actions
- Payoffs of players are opposite

Modelization

- We call strategy a decision rule on the set of actions
- Payoffs can be represented by a matrix A where
$$\left.\begin{array}{l}\text{Player } 1 \text{ chooses } i, \\ \text{Player } 2 \text{ chooses } j\end{array}\right\} \Rightarrow \left\{\begin{array}{l}\text{player } 1 \text{ gets } a_{ij} \\ \text{player } 2 \text{ gets } -a_{ij}\end{array}\right.$$
- A solution point is such that no player has incentives to deviate

Solution of a Game

What is the solution of the game

Player 2, columns; Player 1, rows:

5	1	3
3	2	4
−3	0	1

?

What is the solution of the game

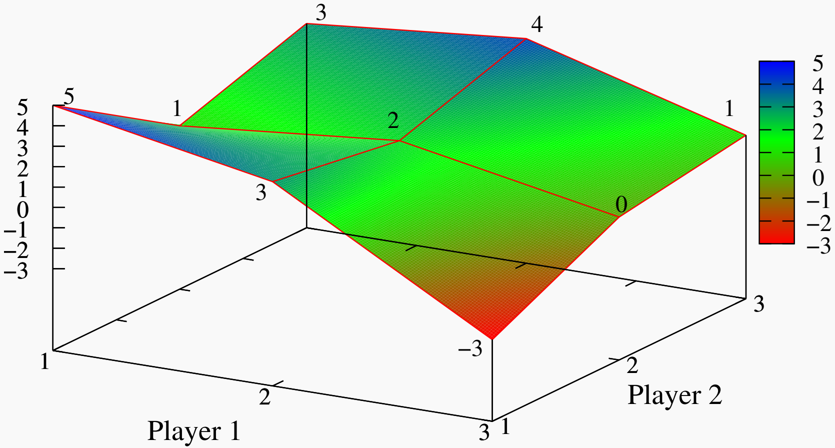
Player 2 (columns), Player 1 (rows)

5	1	3
3	2	4
-3	0	1

?

Solution of a Game

Spatial Representation



Solution of a Game

What is the solution of the game

Player 2

Player 1			
	5	1	3
	3	2	4
	−3	0	1

?

Interpretation:

- Solution point is a saddle point
- Value of a game: $V = \underbrace{\min_j \max_i a_{ij}}_{V_+} = \underbrace{\max_i \min_j a_{ij}}_{V_-}$

Games with no solution?

Proposition:

For any game, we can define:

$V_- = \max_i \min_j a_{ij}$ and $V_+ = \min_j \max_i a_{ij}$.

In general $V_- \leq V_+$

Proof.

$\forall i, \min_j \max_i a_{ij} \geq \min_j a_{ij}$ □

Example: $\begin{pmatrix} 4 & 2 \\ 1 & 3 \end{pmatrix}$ $2 \leftarrow V_-$, $3 \leftarrow V_+$

Interpretation of V_- and V_+

4	0	1
0	-1	2
-1	3	1

Interpretation 1: Security Strategy and Level

V_- is the utility that Player 1 can secure ("gain-floor").
V_+ is the "loss-ceiling" for Player 2.

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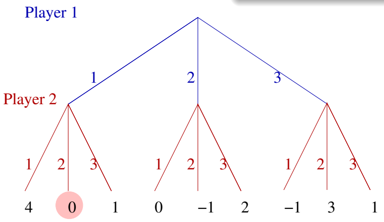

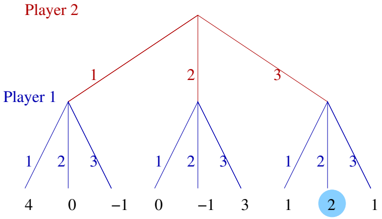

Interpretation 2: Ordered Decision Making

Suppose that there is a predefined order in which players take decisions. (Then, whoever plays second has an advantage.)
V_- is the solution value when Player 1 plays first.
V_+ is the solution value when Player 2 plays first.

Proposition: Uniqueness of Solution

A zero-sum game admits a unique V_- and V_+. If it exists V is unique.
A zero-sum game admits at most one (strict) saddle point

Proof.

Let (i,j) and (k,l) be two saddle points. $\begin{pmatrix} a_{ij} & \cdots & a_{il} \\ & \vdots & \\ a_{kj} & \cdots & a_{kl} \end{pmatrix}$

By definition of a_{ij} : $a_{ij} \leq a_{il}$ and $a_{ij} \geq a_{kj}$. Similarly, by definition of a_{kl} : $a_{kl} \leq a_{kj}$ and $a_{kl} \geq a_{il}$
Then, $a_{ij} \leq a_{il} \leq a_{kl} \leq a_{kj} \leq a_{ij}$ □

Extension to Mixed Strategies

Definition: Mixed Strategy.

A mixed strategy x is a probability distribution on the set of pure strategies: $\forall i, x_i \geq 0$, $\sum_i x_i = 1$

Optimal Strategies:

- Player 1 maximize its expected gain-floor with $x = \text{argmax} \min_y xAy^t$.
- Player 2 minimizes its expected loss-ceiling with $y = \text{argmin} \max_x xAy^t$.

Values of the game:

- $V_-^m = \max_x \min_y xAy^t = \max_x \min_j xA_{.j}$ and
- $V_+^m = \min_y \max_x xAy^t = \min_y \max_i A_{i.}y^t$.

The Minimax Theorem

Theorem 1: The Minimax Theorem.

In mixed strategies: $V_-^m = V_+^m \stackrel{\text{def}}{=} V^m$

Proof.

Lemma 1: Theorem of the Supporting Hyperplane.

Let B a closed and convex set of points in R^n and $x \notin B$ Then,
$\exists p_1, ...p_n, p_{n+1} : \sum_{i=1}^{n} x_i p_i = p_{n+1}$ and $\forall y \in B, p_{n+1} < \sum_{i=1}^{n} p_i y_i$.

Proof.

Consider z the point in B of minimum distance to x and consider
$\forall n, 1 \leq i \leq n, p_i = z_i - x_i$, $p_{n+1} = \sum_i z_i x_i - \sum_i x_i$ □

□

The Minimax Theorem

Theorem 1: The Minimax Theorem.

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Proof.

Lemma 1: Theorem of the Alternative for Matrices.

Let $A = (a_{ij})_{m \times n}$ Either (i) $(0, ..., 0)$ is contained in the convex hull of $A_{.1}, ..., A_{.n}, e_1, ...e_m$. Or (ii) There exists $x_1, ..., x_m$ s.t. $\forall i, x_i > 0, \sum_{i=1}^{m} x_i = 1, \forall j \in 1, ..., n, \sum_{i=1}^{m} a_{ij} x_i$.

Lemma 2.

Lemma 3: Let A be a game and $k \in R$. Let B the game such that $\forall i, j, b_{ij} = a_{ij} + k$. Then $V_-^m(A) = V_-^m(B) + k$ and $V_+^m(A) = V_+^m(B) + k$.

Theorem 1: The Minimax Theorem.

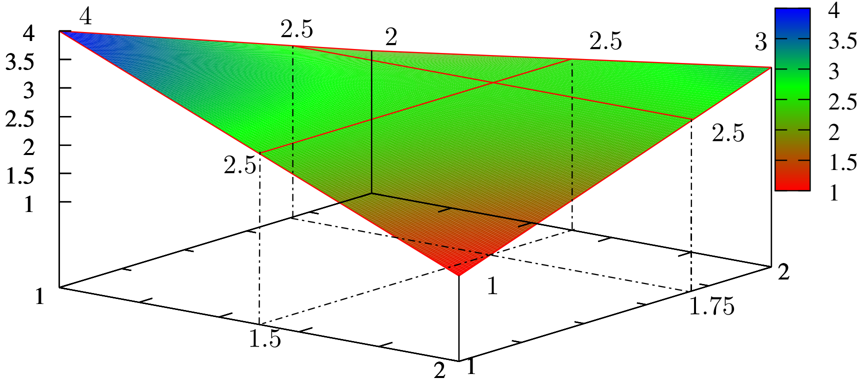
In mixed strategies: $V_-^m = V_+^m \stackrel{\text{def}}{=} V^m$

Proof.

From Lemma 2, we get that for any game, either (i) from lemma 2 and $V_+^m \leq 0$ or (ii) and $V_-^m > 0$. Hence, we cannot have $V_-^m \leq 0 < V_+^m$. With Lemma 3 this implies that $V_-^m = V_+^m$. □

The Minimax Theorem - Illustration

Example: $\begin{pmatrix} 4 & 2 \\ 1 & 3 \end{pmatrix}$



A Note on Symmetric Games

Definition: Symmetric Game.

A game is symmetric if its matrix is skew-symmetric

Proposition:

The value of a symmetric game is 0 and any strategy optimal for player 1 is also optimal for player 2.

Proof.

Note that $xAx^t = -xA^tx^t = -(xAx^t)^t = -xAx^t = 0$. Hence $\forall x, \min_y xAy^t \leq 0$ and $\max_y yAx^t \geq 0$ so $V = 0$.

If x is an optimal strategy for 1 then $0 \leq xA = x(-A^t) = -xA^t$ and $Ax^t \leq 0$. □

Outline

Definition: Dominating (pure) Strategies.

We say that pure strategy i dominates strategy j for player 1 if

$$\forall k,\ a_{ik} \geq a_{jk}.$$

A similar definition holds for dominating strategies for player 2.

Example:

$$\begin{matrix} 4 & 7 & 5 & 5 \\ 6 & 6 & 7 & 6 \\ 5 & 4 & 8 & 10 \\ 6 & 1 & 2 & 9 \end{matrix}$$

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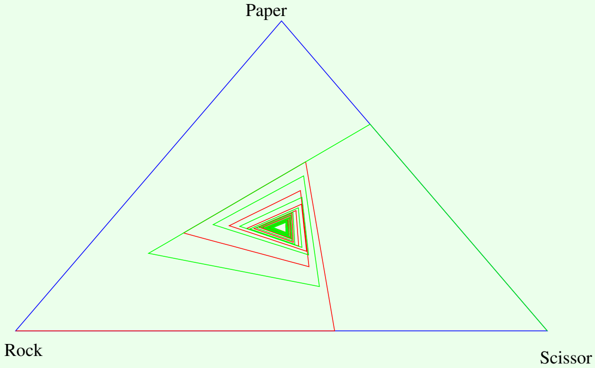
Fictitious Play

Learning each other's behavior

- The players play a number of times
- Each time, each player plays so as to maximize its expected return against its opponent's observed empirical probability distribution
- The empirical distribution converge to optimal strategies.

Example (Rock-Paper-Scissor)



Problem for player 1: Maximize its "gain-floor", i.e. $\max_x \min_p \sum_i a_{pi} x_i$ We re-write this as a linear program:

$$\max_{x,\underline{v}} \underline{v} \text{ s.t. } \begin{cases} x_i \geq 0, \; \forall i \\ \sum_{i=1}^{n} x_i = 1 & \leftarrow \overline{v} \\ \underline{v} \leq \sum_{i=1}^{n} a_{ij} x_i, \; \forall j & \leftarrow y \end{cases}$$

The dual problem is:

$$\min_{y,\overline{v}} \overline{v} \text{ s.t. } \begin{cases} y_j \geq 0, \forall j \\ \sum_{j=1}^{m} y_j = 1 & \leftarrow \underline{v} \\ \overline{v} \geq \sum_{j=1}^{m} a_{ij} y_j, \; \forall i & \leftarrow x \end{cases}$$

which is the optimization problem of the second player!

Example and Geometrical Interpretation:

$$\begin{pmatrix} 4 & 2 \\ 1 & 3 \end{pmatrix}$$

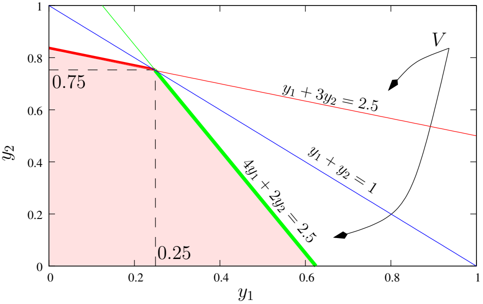
In pure strategies:

V_- (2)

V_+ (3)

In mixed strategies:

$$\begin{cases} V_- = V_+ = 2.5, \\ x_{\text{opt}} = (0.5, 0.5), \\ y_{\text{opt}} = (0.25, 0.75). \end{cases}$$



Outline

If the set of actions is infinite, the value V_m may not exists, and even if it does, there may not be a solution strategy.

Example

2	0
$1/2$	$1/2$
$1/3$	$2/3$
$1/4$	$3/4$
$1/5$	$4/5$
$\cdots$	$\cdots$

For the truncated game, the solution is: $(x_1 = \frac{k-2}{3k-2}, x_k = \frac{2k}{3k-2}), (y_1 = \frac{k-1}{3k-2}, y_2 = \frac{2k-1}{3k-2})$ with the game value $V_m = 2\frac{k-1}{3k-2}$.
The value of the infinite game is $V_\infty = 2/3$ which cannot be attained. Yet, the players can secure a value arbitrarily close to.

Definition: ε-saddle point.

For a given $\varepsilon \geq 0$, the pair $(x_\varepsilon, y_\varepsilon) \in U^1 \times U^2$ is called an ε-saddle point if $J(x, y_\varepsilon) - \varepsilon \leq J(x_\varepsilon, y_\varepsilon) \leq J(x_\varepsilon, y) + \varepsilon$ for all $(x, y) \in U^1 \times U^2$

If the set of actions is infinite, the value V_m may not exists, and even if it does, there may not be a solution strategy.

Example

2	[illegible]
$1/2$	[illegible]
$1/3$	[illegible]
$1/4$	[illegible]
$1/5$	$4/5$
$\cdots$	$\cdots$

For the truncated game, the solution is: $(x_1 =$ [illegible] $)$ with the [illegible] $= 2/3$ which cannot be attained. Yet, the players can secure a value arbitrarily close to.

Theorem 2: Finite Value of Infinite Game.

An infinite game has a finite value if and only if, $\forall \varepsilon > 0$, an ε-saddle point exists.

Definition: ε-saddle point.

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Definition: Continuous Game.

The strategy set is $[0,1] \times [0,1]$. The payoff function is $A : [0,1] \times [0,1] \to \mathbb{R}$.
⚠ A is sometimes called the kernel.

Definition: Mixed Strategy.

A probability distribution over the set of pure strategies. Can be represented by the cumulative distribution function F, continuous, non-decreasing, with $F\colon [0,1] \to [0,1], F(0) = 0, F(1) = 1$.

- Expected payoff for pure strategy x for player 1:
$$E(x,G) = \int_0^1 A(x,y)dG(y)$$
- Expected payoff for pure strategy y for player 2:
$$E(F,y) = \int_0^1 A(x,y)dF(x)$$
- Value of the game:
$$V_-^m = \sup_F \inf_y E(F,y) \text{ and } V_+^m = \inf_G \sup_x E(x,G)$$
- Expected reward
$$E(F,G) = \int_0^1 \int_0^1 A(x,y)dF(x)dG(y)$$

Theorem 3.

If A is continuous, then the forms $\sup\inf$ and $\inf\sup$ may be replaced by $\max\min$ and $\min\max$.

Proof.

$y \mapsto E(F, y)$ is continuous over a compact (the interval $[0, 1]$). By definition of V_-, there exists F_n s.t. $\min_y E(F_n, y) > V_- - 1/n$. As the set of functions from $[0 : 1]$ to itself is compact, there exists a convergent subsequence of F. The limit F_0 can be extended to a continuous function attaining maximum V_-. □

Theorem 4.

If A is continuous, then $V_- = V_+$.

Proof.

Consider the sequence of matrices A_n, with $\forall i, j$, $a^n_{ij} = A(i/n, j/n)$. It has a value and optimal strategies. From the uniform continuity of function A over $[0, 1] \times [0, 1]$, the value of the continuous game is the limit of the value of the sequence of finite games. □

Definition: Concave-Convex Games.

A game is said concave-convex it $\forall y, x \mapsto A(x,y)$ is concave and $\forall x, y \mapsto A(x,y)$ is convex.

Proposition:

A continuous concave-convex game always have pure strategy solutions.

Proof.

... □

Outline

Outline

Multiple Round Games

- Simplest case of dynamic games
- Given number of rounds
- Ex: TicTacTo

Resolution techniques:

- Backward Induction (exact solution)
- Behavioral strategy: collection of probability distributions for each possible information set (approximate solution)

Multiple Round Games

Definition: Strategies.

In a game in extensive form, a strategy for a player is a sequence of actions.
⚠Actions are different from strategies.

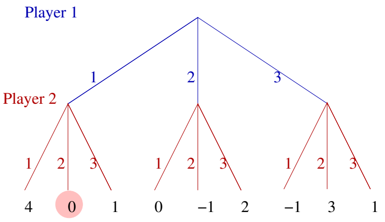
Definition: Behavorial Strategies.

N_i is the set of decision nodes for player i.
A behavioral strategy for player i is a mapping from each node in N_i to the set of (probability distributions on the) possible actions.

Proposition: Existence of Solution

Any zero-sum game in extensive form (finite, with full information, and without chance move) admits a saddle point in behavioral strategies. It is also a saddle point in mixed strategies.

☺ The behavioral strategy saddle point can be found recursively.



- Strategies for player 1 are $\{\{1\},\{2\},\{3\}\}$
- Strategies for player 2: $\{1,1,1\},\{1,1,2\},\{1,1,3\},\{2,1,1\}\ldots$ (overall: 27 pure strategies).

Corresponding normal form game (partial -without action 3 for player 1-, for display reasons ☺):

4	4	4	0	0	0	1	1	1
0	−1	2	0	−1	2	0	−1	2

Behavioral Strategies: For player one: a choice of p_1, p_2, p_3 (with $p_1 + p_2 + p_3 = 1$)

For player two: a choice of $q_1^1, q_2^1, q_3^1, q_1^2, q_2^2, q_3^2, q_1^3, q_2^3, q_3^3$, with $q_1^1 + q_2^1 + q_3^1 = 1$, $q_1^2 + q_2^2 + q_3^2 = 1$, $q_1^3 + q_2^3 + q_3^3 = 1$

Games with Repeated Decisions

There exists two (distinct) states of the game with identical (chosen) action.

Extensive Forms with Cycles

There exist some cycles in the state graph.

Games with Partial Information

The players do not have perfect information about each other's actions.

Information Set depending on the Actions

The knowledge of the system for a player depends on its actions.

- There exists p states plus a state 0 representing the end of the game.
- In each state k, a game is played, characterized by matrix A^k in R^{m_k,n_k} and a matrix of probability vectors $(q^k)_{1\leq i\leq m_k,1\leq j\leq n_k}$ over the set of states.
- The matrix game is (by a great abuse of notations)

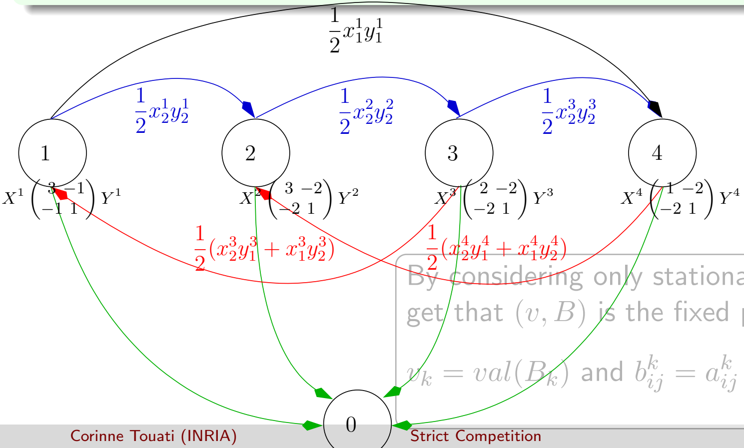
$$\alpha^k_{ij} = a^k_{ij} + \sum_0^p q^{kl}_{ij} S_l \text{ with } \sum_0^p q^{kl}_{ij} = 1,\ q^l_{ij} \geq 0,\ q^{k0}_{ij} > 0$$

- A strategy vector is $\sum_{i=1}^{m_k} x^{kl}_i = 1, x^{kl}_i \geq 0$

Example

$$A_1 = \begin{pmatrix} 3 + S_4/2 & -1 \\ -1 & 1 + \frac{1}{2}S_2 \end{pmatrix} \quad A_2 = \begin{pmatrix} 3 & -2 \\ -2 & 1 + \frac{1}{2}S_3 \end{pmatrix}$$

$$A_3 = \begin{pmatrix} 2 & -2 + \frac{1}{2}S_1 \\ -2 + \frac{1}{2}S_1 & 1 + \frac{1}{2}S_4 \end{pmatrix} \quad A_4 = \begin{pmatrix} 1 & -2 + \frac{1}{2}S_2 \\ -2 + \frac{1}{2}S_2 & 1 \end{pmatrix}$$



By considering only stationary strategies, we get that (v, B) is the fixed point solution of:

$$v_k = val(B_k) \text{ and } b_{ij}^k = a_{ij}^k + \sum_0^p q_{ij}^{kl} v_l.$$

Example

$$A_1 = \begin{pmatrix} 3 + S_4/2 & -1 \\ -1 & 1 + \frac{1}{2}S_2 \end{pmatrix} \quad A_2 = \begin{pmatrix} 3 & -2 \\ -2 & 1 + \frac{1}{2}S_3 \end{pmatrix}$$

$$A_3 = \begin{pmatrix} 2 & -2 + \frac{1}{2}S_1 \\ -2 + \frac{1}{2}S_1 & 1 + \frac{1}{2}S_4 \end{pmatrix} \quad A_4 = \begin{pmatrix} 1 & -2 + \frac{1}{2}S_2 \\ -2 + \frac{1}{2}S_2 & 1 \end{pmatrix}$$

Solving:

$v_0 = (0, 0, 0, 0)$

$$B_0 = \left(\begin{pmatrix} 3 & -1 \\ -1 & 1 \end{pmatrix}, \begin{pmatrix} 3 & -2 \\ -2 & 1 \end{pmatrix}, \begin{pmatrix} 2 & -2 \\ -2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & -2 \\ -2 & 1 \end{pmatrix} \right)$$

$v_1 = (\frac{1}{3}, -\frac{1}{8}, -\frac{2}{7}, -\frac{1}{2})$ $B_1 =$

$$\left(\begin{pmatrix} \frac{11}{4} & -1 \\ -1 & \frac{15}{16} \end{pmatrix}, \begin{pmatrix} 3 & -2 \\ -2 & \frac{6}{7} \end{pmatrix}, \begin{pmatrix} 2 & \frac{-11}{6} \\ \frac{-11}{6} & \frac{3}{4} \end{pmatrix}, \begin{pmatrix} 1 & \frac{-33}{16} \\ -\frac{33}{16} & 1 \end{pmatrix} \right) \ldots$$

- Extension of stochastic games where the probability of infinite play is positive.
- The payoff is obtained only when the game terminates. There exists also a payoff for infinite game.
- The matrix game is

$$\alpha_{ij}^{k} = q_{ij}^{k0} a_{ij}^{k} + \sum_{1}^{p} q_{ij}^{kl} S_l \text{ with } \sum_{0}^{p} q_{ij}^{kl} = 1,\ q_{ij}^{kl} \geq 0$$

- A strategy vector is $\sum_{i=1}^{m_k} x_i^{kt} = 1, x_i^{kt} \geq 0$

⚠The value iteration method can be used: the error does not vanish to 0.

⚠There may not be an optimal strategy, but only ε-optimal strategy.

Example: $\begin{pmatrix} A & A & A \\ A & A & 1 \\ A & 1 & -1 \\ 1 & -1 & -1 \end{pmatrix}$ $v = 1$, ε-optimal strategy for player 1: $(0, 1-\delta-\delta^2, \delta, \delta^2)$

Differential Games

- Limit case of a stochastic or recursive game where time interval between stages vanishes.
- State space x continuous in time (of dimension n)
- Player 1 chooses ϕ, Player 2 chooses ψ
- The system evolves according to $x = f(x, \phi, \psi)$ (kinematic equations)
- The game stops either when x attains a given closed subset of R^n or at given time epoch T.
- The payoff is either a function of the terminal state $x(T)$ or an integral: $\int_0^T G(x)dt$.

This kind of problems have been studied widely in the domain of optimal control theory.

- State of the system (x, y): x mobiles that have a file (y mobiles do not)
- the source is in contact with mobiles without a file at rate η
- mobiles join the system at a rate λ
- mobiles with the file die at rate νx
- State evolves according to: $\begin{cases} x_t = \eta_t y_t - \nu_t x_t \\ y_t = -\eta_t y_t + \lambda_t \end{cases}$
- Player 1 (source) chooses η, player 2 (nature) chooses ν.

Outline

Games with Chance Move

4	1	1	3
3	0	2	5

$i = 2$ chance moves

1. No user knows the output of the chance move. Then $V = 2.5$
2. Both users know the output of the chance move: $V = 1.5$
3. Only first player knows, and he plays first

4	1
3	0

1	3
2	5

$i = 2$ chance moves

1. No user knows the output of the chance move. Then $V = 2.5$
2. Both users know the output of the chance move: $V = 1.5$
3. Only first player knows, and he plays first

Equivalent game: (of size $n^i \times n^m$)

	$A\|A, A\|A$	$A\|A, B\|A$	$B\|A, A\|A$	$B\|A, B\|A$
$A\|1, A\|2$	2.5	2.5	2	2
$A\|1, B\|2$	3	4.5	1.5	3
$B\|1, A\|2$	2	0.5	3	1.5
$B\|1, B\|2$	2.5	2.5	2.5	2.5

Then, $V = 2.5$. Optimal strategy: $x = (0, 0, 0, 1)$, (i.e. player 1 does not reveal any info) and $y = (\hat{y}, 0, 1 - \hat{y}, 0)$ with $\hat{y} \in \left\{\frac{1}{2}, \frac{2}{3}\right\}$.

$$\begin{array}{|cc|}\hline 4 & 1 \\ 3 & 0 \\ \hline\end{array}\quad\begin{array}{|cc|}\hline 1 & 3 \\ 2 & 5 \\ \hline\end{array}$$

$i = 2$ chance moves

1. No user knows the output of the chance move. Then $V = 2.5$
2. Both users know the output of the chance move: $V = 1.5$
3. Only first player knows, and he plays first

In behavioral strategies:

$$\text{Player 1 chooses}\begin{cases} p_1 = Prob(\text{Take action A}|\text{Chance move is 1}), \\ p_2 = Prob(\text{Take action A}|\text{Chance move is 2}). \end{cases}$$

$$\text{Player 2 chooses}\begin{cases} q_1 = Prob(\text{Take action A}|\text{Player 1 plays A}), \\ q_2 = Prob(\text{Take action A}|\text{Player 1 plays B}). \end{cases}$$

Player 1:

$$\max_{\underline{v_1}+\underline{v_2}} \text{ s.t. } \begin{array}{l} \underline{v_1} \leq 2p_1 + 0.5p_2 \\ \underline{v_1} \leq 0.5p_1 + 1.5p_2 \\ \underline{v_2} \leq 1.5(1-p_1) + 1(1-p_2) \\ \underline{v_2} \leq 2.5(1-p_2) \\ 0 \leq p_1, p_2 \leq 1 \end{array}$$

Player 2:

$$\min_{\overline{v_1}+\overline{v_2}} \text{ s.t. } \begin{array}{l} \overline{v_1} \leq 2q_1 + 0.5(1-q_1) \\ \overline{v_1} \leq 1.5q_2 \\ \overline{v_2} \leq 0.5q_1 + 1.5(1-q_1) \\ \overline{v_2} \leq 1q_2 + 2.5(1-q_2) \\ 0 \leq q_1, q_2 \leq 1 \end{array}$$

Games with Chance Move

4	1		1	3
3	0		2	5

$i = 2$ chance moves

1. No user knows the output of the chance move. Then $V = 2.5$
2. Both users know the output of the chance move: $V = 1.5$
3. Only first player knows, and he plays first

In behavioral strategies:

Player 1 chooses $\begin{cases} p_1 = Prob(\text{Take action A}|\text{Chance move is 1}), \\ p_2 = Prob(\text{Take action A}|\text{Chance move is 2}). \end{cases}$

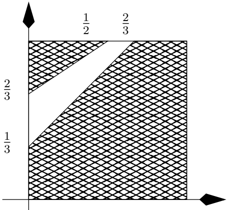
Player 2 chooses $\begin{cases} q_1 = Prob(\text{Take action A}|\text{Player 1 plays A}), \\ q_2 = Prob(\text{Take action A}|\text{Player 1 plays B}). \end{cases}$

Solution:

$V = 2.5$

$p_1 = p_2 = 0$

$q_2 - q_1 \geq 1/3,\ 1.5q_2 - q_1 \leq 1$

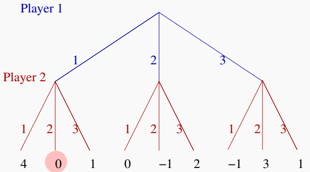


Definition: Extensive Form Game.

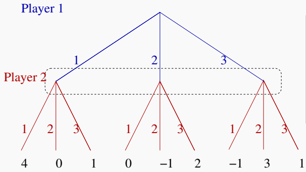
An extensive form game is a finite tree structure with:

- A vertex indicating the starting point of the game,
- A pay-off function assigning a real number to each terminal vertex of the tree,
- A partition of the nodes of the tree into two player sets (with N^i the set of player i),
- A subpartition of each player set N^i into information sets η_j^i such that all nodes of a information set has the same number of children and that no node follows another node of the same information set.

Full Information

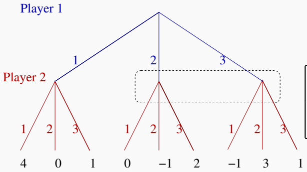


No Information



4	0	1
0	−1	2
−1	3	1

Partial Information



0	−1	2
−1	3	1

Definition: **Behavioral Strategy**.

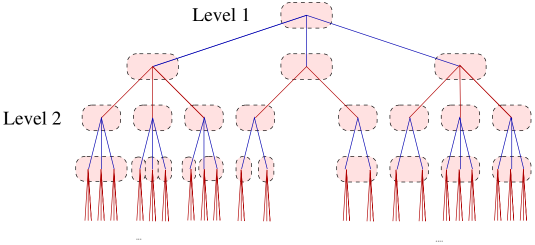
A strategy for player i is a mapping that assigns an action (resp. a distribution probability over the actions) to each information set.

Definition: Feedback Games.

A game in extensive form is a feedback game is (i) each player has perfect information of the current level of play (ii) each player knows the state of the game at every level of play.

Proposition: Solution of Feedback Games

Every finite feedback game admits a saddle point in behavioral strategies.

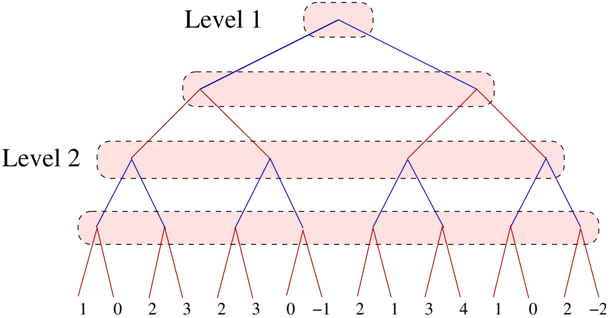


☺ The behavioral solution strategy can be obtained using simple recursive procedures (by solving a number of normal form games).

Proposition:

Any two-person zero-sum finite game in extensive form admits a saddle-point in mixed strategies. (But not necessarily in behavioral strategies.)

Example (open-loop game):



The solution is $\gamma_1 = \begin{cases} LL \text{ with proba } 3/5 \\ RR \text{ with proba } 2/5 \end{cases}$ $\gamma_2 = \begin{cases} LL \text{ with proba } 4/5 \\ RR \text{ with proba } 1/5 \end{cases}$.

Proposition:

In games where each player recall all their past actions but are ignorant of the actions of their opponent admits a solution in behavioral strategies.

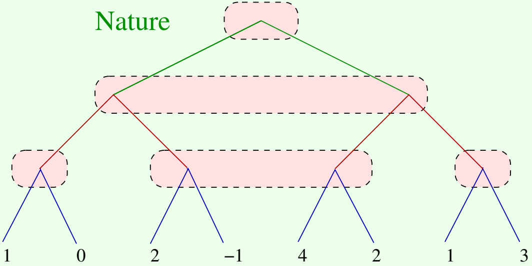
Proposition:

Every finite game admits a saddle point in randomized strategies. (A randomized strategy is a probability distributions over the (possibly mixed strategy) behavioral strategies.)

Definition: Games with chance move & partial information.

Can be seen as a 3 player game with the extra player ("nature") having a fixed mixed strategy.

Example



Such games admit a mixed strategy equilibrium. There is no systematic way to solve them.

Outline

Conclusion

Zero-Sum Games

- Two-player Zero-sum games are games of pure competition
- A solution point (saddle point) of the game is such that no player has incentive to deviates from. The value of the game is the corresponding payoff for player 1.

Basic Results

In games with perfect information and no chance moves:

- Finite zero-sum games always admit a value and solution point(s) in mixed-strategies.
- Infinite zero-sum games with continuous payoff have a value. If the payoff function is further concave-convex, then it also have a solution point in pure strategies.
- The players interests can be seen as two optimization problems that are dual from each other.

Conclusion

Zero-Sum Games

- Two-player Zero-sum games are games of pure competition
- A solution point (saddle point) of the game is such that no player has incentive to deviates from. The value of the game is the corresponding payoff for player 1.

Extensions

Extensions of zero-sum games include

- Multistage Games: the game is repeated over time
- Games with Chance Move: chance is modeled as an extra player with known fixed strategy
- Incomplete Information Games: where players have partial information about the system actual state or each other's actions.