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Mathematical Games

Games on line graphs and sand piles*

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Abstract

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The dynamics of several games on line graphs is studied. Relations between these games and a one-dimensional version of the sand pile model are established. We also study a generalization of the latter model, which we call the ice pile model. Specifically, we investigate the dynamical behavior of all these games and provide closed formulas for the transient time lengths they require to reach the steady state.

1. Introduction

1.1. Games on line graphs

Let $G=(V,E)$ be a graph where V is the set of vertices and $E\subseteq V\times V$ the set of edges. We associate to the graph G a nonnegative integer configuration set, $\mathbb{N}^V$ ($\mathbb{N}=\{0,1,\ldots\}$), and the local transition functions defined as follows:

$$\forall i\in V,\quad \theta_i:\mathbb{N}^V\to\mathbb{N}^V,\quad \theta_i(x)=y,$$

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where

$$y_i = \begin{cases} x_j - d_i \mathbb{1}(x_i - d_i) & \text{if } j = i \\ x_j + \mathbb{1}(x_i - d_i) & \text{if } j \in V_i \\ x_j & \text{otherwise,} \end{cases} \tag{1}$$

where $V_i = \{ j \in V\colon (i,j) \in E \}$ is the set of neighbors of the vertex i, $d_i = |V_i| < +\infty$ is the degree of vertex i, and $\mathbb{1}(u) = 1$ iff $u \geqslant 0$, 0 otherwise.

If the configuration of this model is $x \in \mathbb{N}^V$, then x_i represents the number of disks or chips stacked at vertex i. Whenever a vertex has as many chips as its degree we say that it is firing. The preceding model is called chip firing game (CFG).

The dynamics associated to the chip firing game can be sequential or parallel. The sequential one consists in updating the vertices, one by one either, in a random or a prescribed periodic order. The parallel dynamics, which is the most usual one in the context of cellular automata, consists in updating all the vertices synchronously. Formally, the parallel dynamics is specified by the following local rules:

$$x_i(t+1) = x_i(t) - d_i \mathbb{1}(x_i(t) - d_i) + \sum_{j \in V_i} \mathbb{1}(x_j(t) - d_j), \quad \forall i \in V, \tag{2}$$

i.e. if a vertex is firing, a chip is moved from it to each of its neighbors. In the parallel dynamics case, we define the global transition function Θ as follows:

$$\Theta : \mathbb{N}^V \to \mathbb{N}^V, \quad \Theta(x(t)) = x(t+1),$$

where $x(t+1)$ is defined in (2).

We shall always consider graphs $G = (V, E)$, where the vertex set V is a finite or infinite one-dimensional chain in $\mathbb{Z}$ (i.e. $V \subseteq \mathbb{Z}$, and if $i, j \in V$, $k \in \mathbb{Z}$ and $i < k < j$, then $k \in V$), and the edge set $E = \{(i,j)\colon |i-j| = 1\}$ (i.e. every vertex is connected to its nearest neighbors). We denominate this class of graphs as line graphs.

Examples of the sequential and parallel dynamics are given in Fig. 1.

$$\text{(a)}\; \begin{array}{llll} \underline{3} & & & \\ \underline{2} & 1 & & \\ \underline{1} & 2 & & \\ 0 & \underline{3} & & \\ \underline{1} & 1 & 1 & \\ 0 & \underline{2} & 1 & \\ \underline{1} & 0 & 2 & \\ 0 & 1 & \underline{2} & \\ 0 & \underline{2} & 0 & 1 \\ \underline{1} & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{array} \qquad \text{(b)}\; \begin{array}{llll} 3 & & & \\ 2 & 1 & & \\ 1 & 2 & & \\ 1 & 1 & 1 & \\ 0 & 2 & 1 & \\ 1 & 0 & 2 & \\ 0 & 2 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{array}$$

Fig. 1. Dynamics on the line graph $\mathbb{N}$: (a) sequential (updated vertices are underlined); (b) parallel.

The above-mentioned models have been studied by various authors. Spencer [10] introduced them to study perfect information games. Anderson et al. [1] analyzed a particular version of these models, in which the graph $G=(V,E)$ is such that $V=\mathbb{Z}$, and the initial configuration is $x_0(0)=N$, $x_j(0)=0$ if $j\neq 0$. In [1], the authors provide N-dependent bounds, for the transient time length in this particular case, although they do not study the more general initial configuration cases. Later on, Bitar and Goles [3] studied the steady-state behavior of such models in a general nonoriented, finite and connected graph, and proved, in the case of trees, that the parallel dynamics of such games converges to fixed points or cycles of period two. A typical dynamical behavior of this latter class of games is given in the example of Fig. 2.

1.2. The sand pile model

In this section we analyze a model proposed by Back et al. [2], called the sand pile model (SPM), which simulates several physical phenomena related to the *self-organized criticality* paradigm. The concept of *self-organized criticality* was introduced in [2], and has attracted some attention recently as it may explain various physical phenomena. The SPM simulates the avalanches produced in a one-dimensional profile of a sand pile. Several physical parameters, related to the dynamics of avalanches, have been studied in the literature [5, 8]. In this model a sand pile is represented by an ordered partition of N, i.e. an element of the set

$$S_N=\left\{\omega\in\mathbb{N}^{\mathbb{N}}:\ \omega_i\geqslant\omega_{i+1},\ \sum_{i\in\mathbb{N}}\omega_i=N\right\}.$$

Given a state of the model, $\omega\in S_N$, the number of sand grains stacked at site i are represented by ω_i. The dynamics is specified as follows: a grain of sand tumbles from site i to site $i+1$, iff the height difference, $\omega_i-\omega_{i+1}$, is at least z_c, where $z_c=2$. Clearly, z_c represents a critical slope of the sand pile. If the local slope of the sand pile at a specific site is at least z_c, then an avalanche will occur at that site. Formally, the SPM is defined by a positive integer N, the graph $G=(\mathbb{N},E)$, where $E=\{(i,j):\ |i-j|=1\}$, and the following local rules:

$$\forall i\in\mathbb{N},\quad f_i:S_N\to S_N,\quad f_i(\omega)=v,$$

t	1	2	3	4	5	6	7	8	9	10
0	0	1	2	1	2	2	2	0	2	0
1	0	2	0	3	1	2	1	2	0	1
2	1	0	2	1	3	0	3	0	2	0
3	0	2	0	3	1	2	1	2	0	1
4	1	0	2	1	3	0	3	0	2	0

Fig. 2. Two-periodic limit behavior of a game on the line graph $\{1,2,\dots,10\}$, with parallel dynamics.

where

$$v_j=\begin{cases}\omega_j-\mathbb{1}(\omega_i-\omega_{i+1}-2) & \text{if } j=i\\ \omega_j+\mathbb{1}(\omega_i-\omega_{i+1}-2) & \text{if } j=i+1\\ \omega_j & \text{otherwise.}\end{cases}$$

A graphical example of the preceding local rules is shown in Fig. 3.

Once more, the dynamics associated to this model can be sequential or parallel. Formally, the parallel dynamics is specified by the following local rules:

$$\omega_0(t+1)=\omega_0(t)-\mathbb{1}(\omega_0(t)-\omega_1(t)-2),$$

$$\omega_i(t+1)=\omega_i(t)+\mathbb{1}(\omega_{i-1}(t)-\omega_i(t)-2)-\mathbb{1}(\omega_i(t)-\omega_{i+1}(t)-2),\quad \forall i>0. \tag{3}$$

For this latter updating scheme, we define the global transition function F as

$$F:S_N\to S_N,\quad F(\omega(t))=\omega(t+1),$$

where $\omega(t+1)$ is defined in (3).

Examples of both types of dynamics are given in Fig. 4.

Clearly, for the sequential dynamics one may update a configuration in several manners. In Fig. 5, we give all possible trajectories of the SPM starting from the initial configuration $(N,0,\dots)$, for $N=10$.

On the other hand, the SPM can be coded in an alternative way by taking into account only the height differences between consecutive piles. That is to say, we associate with the sand pile configuration ω, the height difference configuration,

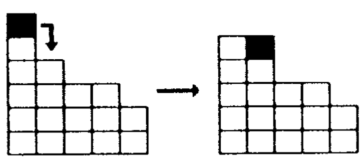


Fig. 3. Local update of a one-dimensional sand pile.

(a)

$\underline{7}$			
$\underline{6}$	1		
5	$\underline{2}$		
$\underline{5}$	1	1	
$\underline{4}$	2	1	
3	$\underline{3}$	1	
3	2	$\underline{2}$	
3	2	1	1

(b)

7			
6	1		
5	2		
4	2	1	
3	3	1	
3	2	2	
3	2	1	1

Fig. 4. Sand pile model dynamics: (a) sequential; (b) parallel.

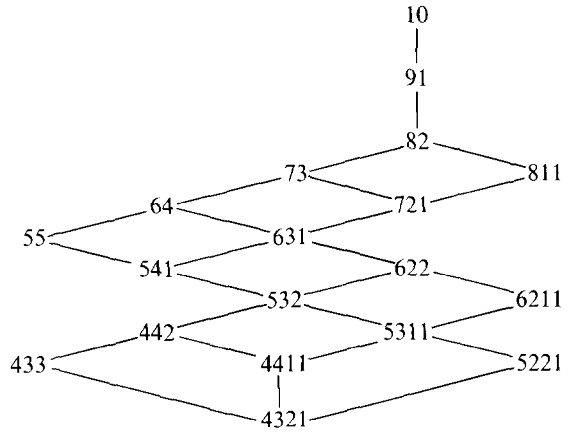


Fig. 5. Sequential update lattice for $N=10$.

defined as $\varphi(\omega)=(\omega_0-\omega_1,\ldots,\omega_i-\omega_{i+1},\ldots)\in\mathbb{N}^{\mathbb{N}}$. In this alternative coding of the model, the dynamics is given by the following local rules:

$$\forall i\in\mathbb{N},\quad \theta_i:\mathbb{N}^{\mathbb{N}}\to\mathbb{N}^{\mathbb{N}},\quad \theta_i(z)=y,$$

where

$$y_j=\begin{cases} z_j+\mathbb{I}(z_i-2) & \text{if } j=i-1,\, i>0\\ z_j-2\mathbb{I}(z_i-2) & \text{if } j=i\\ z_j+\mathbb{I}(z_i-2) & \text{if } j=i+1\\ z_j & \text{otherwise.}\end{cases}$$

Since $\varphi(f_i(\omega))=\theta_i(\varphi(\omega))$, it follows straightforwardly that both codings of the model are equivalent. Note that the alternative coding of the SPM is similar to the games defined in Section 1.1.

In this paper we study the dynamics of the models mentioned so far. This analysis is carried out in the case of arbitrary initial configurations of finite support. We provide bounds for the transient time length of these models, and characterize the fixed points to which they converge. We also introduce a generalizaton of the SPM, we call the ice pile model, and give closed formulas for the minimum and maximum transient time length of its dynamics. In addition, we provide strategies to achieve such extreme transient times. Furthermore, a physical interpretation of this model is provided. To accomplish the above-mentioned tasks, we establish several relations between games on graphs, the SPM, the ice pile model, and the lattice structure of S_N studied by Brilawsky [4].

2. General results about sand piles

To study the sand pile dynamics we associate with each sand pile configuration, a quantity we call the "energy" E defined as follows:

$$E: S_N \to \mathbb{N}, \qquad E(\omega) = \sum_{i \in \mathbb{N}} i\omega_i.$$

Since $\sum_{i\in\mathbb{N}} \omega_i = N$, $E(\omega) < \infty$.

It is important to note that E is monotone as a function of time, since $E(f_i(\omega)) - E(\omega) = 1$ if $\omega_i - \omega_{i+1} \geqslant 2$; hence, the dynamics of the SPM is driven by this operator.

On the other hand, let us define over S_N the relation $\leqslant$ as follows:

$$\omega \leqslant v \Leftrightarrow \sum_{j \geqslant i} \omega_j \geqslant \sum_{j \geqslant i} v_j, \quad \forall i \in \mathbb{N}.$$

It is easy to see that $\leqslant$ is a partial-order relation on S_N. In the following lemma we establish a link between the operator E and the relation $\leqslant$.

Lemma 2.1. *Let ω, $v \in S_N$, then*

$$E(v) \geqslant E(\omega) \text{ and } \omega \leqslant v \Rightarrow \omega = v.$$

Proof. The proof is direct by recalling that $E(v) = \sum_{i\in\mathbb{N}} \sum_{j > i} v_j$. □

Clearly, ω is a fixed point iff $\omega = f_i(\omega)$, $\forall i \in \mathbb{N}$, or equivalently, $\omega_i - \omega_{i+1} \leqslant 1$, $\forall i \in \mathbb{N}$. Given ω, we define the set

$$S(\omega) = \{v \in S_N \colon v \leqslant \omega \text{ and } v \text{ is a fixed point}\}.$$

Since $\bar{1} = (1, \ldots, 1, 0, \ldots) \in S(\omega)$, it follows that $S(\omega) \neq \Phi$. We now give an example of the set defined above

$$S(3321) = \{32211, 22221, 222111, 321111, 2211111, 21111111, 111111111\}.$$

Lemma 2.2. *If $\omega \in S_N$ is such that $\omega_i - \omega_{i+1} \geqslant 2$, then*

(i) $f_i(\omega) \leqslant \omega$,

(ii) $v \in S(\omega) \Rightarrow v \leqslant f_i(\omega)$.

Proof. (i) The proof is straightforward.

(ii) Note that $\sum_{j \geqslant l} (f_i(\omega))_j = \sum_{j \geqslant l} \omega_j \leqslant \sum_{j \geqslant l} v_j$, $\forall l \neq i+1$; hence, we only have to prove that $\sum_{j \geqslant i+1} (f_i(\omega))_j \leqslant \sum_{j \geqslant i+1} v_j$. Suppose $\sum_{j \geqslant i+1} \omega_j \geqslant \sum_{j \geqslant i+1} v_j$. Since $v \leqslant \omega$, then

$$\sum_{j \geqslant i+1} \omega_j = \sum_{j \geqslant i+1} v_j. \tag{4}$$

Moreover,

$$\sum_{j\geqslant i}\omega_j\leqslant\sum_{j\geqslant i}v_j \quad\text{and}\quad -\sum_{j\geqslant i+2}\omega_j\geqslant-\sum_{j\geqslant i+2}v_j. \tag{5}$$

From (4) and (5) we conclude that $\omega_i\leqslant v_i$ and $\omega_{i+1}\geqslant v_{i+1}$. It follows that

$$v_i-v_{i+1}\geqslant\omega_i-\omega_{i+1}\geqslant 2,$$

which is a contradiction, since v is a fixed point. We then have that $\sum_{j\geqslant i+1}\omega_j<\sum_{j\geqslant i+1}v_j$, hence,

$$\sum_{j\geqslant i+1}(f_i(\omega))_j=\sum_{j\geqslant i+1}\omega_j+1<\sum_{j\geqslant i+1}v_j+1\Rightarrow\sum_{j\geqslant i+1}(f_i(\omega))_j\leqslant\sum_{j\geqslant i+1}v_j,$$

which completes the proof of the lemma. □

From the time monotonousness of E, and the finite cardinal of S_N, it follows that starting from any initial distribution of grains, the SPM converges towards a fixed point.

Proposition 2.3. *Let $\omega(0)\in S_N$ and $\underline{\omega}\in S(\omega(0))$ be such that $E(\underline{\omega})=\min_{v\in S(\omega(0))}E(v)$, then, the SPM sequential dynamics starting from $\omega(0)$ converges towards the fixed point $\underline{\omega}$, in $T_{\mathrm{sec}}(\omega(0))=E(\underline{\omega})-E(\omega(0))$ time steps, independently of the order in which the sites are updated.*

Proof. Suppose the sites are updated sequentially in an arbitrary order, and the fixed point v is finally reached. Lemma 2.2(i), insures that $v\in S(\omega(0))$; hence, from the definition of $\underline{\omega}$, we conclude that $E(v)\geqslant E(\underline{\omega})$. On the other hand, since $\underline{\omega}\in S(\omega(0))$, Lemma 2.2(ii), insures that $\underline{\omega}\leqslant v$. Lemma 2.1 implies then that $v=\underline{\omega}$, i.e. the sequential iteration converges to $\underline{\omega}$.

Finally, since E grows by exactly one unit each time step, the number of steps required to reach $\underline{\omega}$ is $E(\underline{\omega})-E(\omega(0))$, independently of the order in which the sites are updated. □

Corollary 2.4. *Given any initial configuration, both, the SPM sequential and parallel dynamic converge towards the same fixed point.*

Proof. From Proposition 2.3, we have that the sequential dynamics converges towards the same fixed point independently of the order in which sites are updated. To prove that the sequential and parallel dynamic converge towards the same fixed point, it suffices to show that any parallel update can be simulated by a sequence of sequential updates. In fact, if $\omega(t)$ is the sand pile configuration at the tth time step, and $i_1,\dots,i_m$ are such that $\omega_{i_j}(t)-\omega_{i_j+1}(t)\geqslant 2$, then

$$\omega(t+1)=(f_{i_1}\circ f_{i_2}\circ\cdots\circ f_{i_m})(\omega(t)).\qquad\square$$

Let us define $T_{\mathrm{par}}(\omega(0))$ as the number of time steps required to reach the fixed point starting from $\omega(0)$, using the parallel updating scheme. We also define the length of ω as $|\omega| = \min_{i \in \mathbb{N}} \{i\colon \omega_i \neq 0\}$.

We now give a lower bound for $T_{\mathrm{par}}(\omega(0))$.

Corollary 2.5. *If starting from $\omega(0)$ the SPM converges towards $\underline{\omega}$, then*

$$T_{\mathrm{par}}(\omega(0)) = 0 \quad \text{if } |\underline{\omega}| = 1 \text{ or } \omega_0(0) = 1,$$

$$T_{\mathrm{par}}(\omega(0)) \geqslant \max\left\{ \left\lceil \frac{T_{\mathrm{sec}}(\omega(0))}{|\underline{\omega}|-1} \right\rceil, \frac{T_{\mathrm{sec}}(\omega(0))}{\lfloor \omega_0(0)/2 \rfloor} \right\},$$

where $\lfloor \cdot \rfloor$ and $\lceil \cdot \rceil$ are the floor and ceiling functions, respectively.

Proof. If $|\underline{\omega}| = 1$ or $\omega_0(0) = 1$, then $\omega(0) = \underline{\omega}$, i.e. $T_{\mathrm{par}}(\omega(0)) = 0$. Suppose now these cases do not hold. We shall show that at each time step, no more than $\lfloor \omega_0(0)/2 \rfloor$ $(|\underline{\omega}|-1)$ grains can be moved. It follows that the growth rate of E is at the most $\lfloor \omega_0(0)/2 \rfloor$ $(|\underline{\omega}|-1)$ units per time step, hence,

$$T_{\mathrm{par}}(\omega(0)) \geqslant \frac{E(\underline{\omega}) - E(\omega(0))}{\lfloor \omega_0(0)/2 \rfloor} \quad \left(\left\lceil \frac{E(\underline{\omega}) - E(\omega(0))}{|\underline{\omega}|-1} \right\rceil \right).$$

Suppose

$$s = E(\omega(t+1)) - E(\omega(t)) \geqslant \left\lfloor \frac{\omega_0(0)}{2} \right\rfloor + 1,$$

hence,

$$\exists i_1 < i_2 < \cdots < i_s \text{ such that } \omega_{i_j}(t) - \omega_{i_j+1}(t) \geqslant 2, \quad j = 1, \ldots, s$$

$$\Rightarrow \omega_0(t) \geqslant \omega_{i_1}(t) \geqslant \omega_{i_1+1}(t) + 2 \geqslant \omega_{i_2}(t) + 2 \geqslant \cdots$$

$$\geqslant \omega_{i_s}(t) + 2(s-1) \geqslant \omega_{i_s+1}(t) + 2s \geqslant 2\left\lfloor \frac{\omega_0(0)}{2} \right\rfloor + 2 \geqslant \omega_0(0) + 1,$$

which contradicts the fact that $(\omega_0(t))_{t \geqslant 0}$ is a decreasing sequence.

Suppose now that $s = E(\omega(t+1)) - E(\omega(t)) \geqslant |\underline{\omega}|$, hence, at instant t there are at least $|\underline{\omega}|$ grains that can tumble from one site to another, so $|\omega(t)| \geqslant |\underline{\omega}|$. On the other hand, Lemma 2.2(ii) insures that $\underline{\omega} \leqslant \omega(t)$; hence, $|\omega(t)| \leqslant |\underline{\omega}|$. It follows then that $\omega_{i+1}(t) - \omega_i(t) \geqslant 2$, $\forall i \in \{1, \ldots, |\omega(t)|\}$, hence,

$$|\omega(t+1)| = |\omega(t)| + 1 = |\underline{\omega}| + 1,$$

which is a contradiction, since $\underline{\omega} \leqslant \omega(t+1)$ implies that $|\omega(t+1)| \leqslant |\underline{\omega}|$. $\square$

3. Maximum transient time in sand piles

Using the results of the preceding sections, we now provide a closed formula for $T_{\text{sec}}(\bar{N})$, where $\bar{N}=(N,0,\ldots)$, i.e. the number of time steps required by the SPM to reach a fixed point starting from $\bar{N}$, when the iteration scheme is sequential.

We first note that any positive integer N can be written in the following manner:

$$N=\tfrac{1}{2}k(k+1)+k', \quad 0\leqslant k'\leqslant k.$$

Here we shall assume that N, k and k' are such that the above equality holds. Let us also denote

$$\pi=(k,k-1,\ldots,k'+1,k',k',k'-1,\ldots,2,1,0,\ldots).$$

Proposition 3.1. *If $\omega\in S_N$ is a fixed point, then*

$$E(\pi)\leqslant E(\omega).$$

Proof. Suppose ω is a fixed point, and define $I(\omega)=\{i\colon \omega_i=\omega_{i+1}>0\}$.

If $|I(\omega)|\geqslant 2$, let $m=\min_{i\in I} i$ and $M=\max_{i\in I} i+1$. Let us now construct ω' from ω, by moving a grain from site M to site m, i.e. we define

$$\omega'=(\omega_0,\ldots,\omega_m+1,\ldots,\omega_M-1,\omega_{M+1},\ldots).$$

Clearly, $\omega'\in S_N$, and since ω is a fixed point such that $|I(\omega)|\geqslant 2$ we have that ω' is also a fixed point. Furthermore,

$$E(\omega')=E(\omega)-M+m<E(\omega).$$

Repeating this procedure we finally reach a fixed point $\tilde{\omega}\in S_N$, such that $|I(\tilde{\omega})|<2$ and $E(\tilde{\omega})<E(\omega)$. The proposition follows from the fact that the only fixed point $\omega\in S_N$ such that $|I(\omega)|<2$ is π. $\square$

We can now characterize the fixed point reached by the SPM starting from $\bar{N}$, as well as the transient time length of the sequential dynamics.

Lemma 3.2. *If the SPM starts from $\bar{N}\in S_N$, it converges towards π and*

$$T_{\text{sec}}(\bar{N})=\binom{k+1}{3}+kk'-\binom{k'}{2}.$$

Proof. Since $\pi\leqslant\bar{N}$, and π is a fixed point, then $\pi\in S(\bar{N})$. This fact, together with Proposition 3.1, insures that

$$E(\pi)\leqslant \min_{\omega\in S(\bar{N})} E(\omega)\leqslant E(\pi).$$

Then, it follows from Proposition 2.3, that if the SPM starts from $\bar{N}$ it converges toward π, and

$$T_{\text{sec}}(\bar{N}) = E(\pi) - E(\bar{N}) = \binom{k+1}{3} + kk' - \binom{k'}{2}. \qquad \square$$

Recalling Corollary 2.5 and the obvious fact that $T_{\text{sec}}(\bar{N}) \geqslant T_{\text{par}}(\bar{N})$, we deduce from the preceding lemma that

$$\mathrm{O}(N^{3/2}) \geqslant T_{\text{par}}(\bar{N}) \geqslant \Omega(N).$$

4. Chip firing games and sand piles

In this section we shall study the chip firing game on the line graph $\mathbb{K}$, which we denote CFG($\mathbb{K}$), for the cases in which $\mathbb{K} = \mathbb{Z}$ or $\mathbb{N}$. In both cases the update rule is the following:

$$x_i \leftarrow x_i - d_i \mathbb{1}(x_i - d_i),$$

$$x_j \leftarrow x_j + \mathbb{1}(x_i - d_i) \quad \forall j \in V_i,$$

where $d_i = |V_i|$ and $V_i = \{i-1, i+1\} \cap \mathbb{K}$.

We shall always consider the case of an initial configuration of chips distributed on a finite number of vertices of $\mathbb{K}$, i.e. when $x(0) \in \mathbb{N}^{\mathbb{K}}$ is such that $0 < \sum_{i \in \mathbb{K}} x_i(0) < \infty$.

We are interested in the study of the dynamical behavior of the above-mentioned games. To carry out this study we shall establish a morphism between each one of this games and the SPM. The existence of such morphisms implies that the CFG($\mathbb{Z}$) and the CFG($\mathbb{N}$) can be interpreted as particular cases of the SPM.

4.1. The CFG($\mathbb{Z}$) game

As already mentioned, Anderson et al. [1] studied the CFG($\mathbb{Z}$) in the particular case in which the initial configuration $x(0) \in \mathbb{N}^{\mathbb{Z}}$ is such that $x_0(0) = N$ and $x_i(0) = 0$, $\forall i \neq 0$ (see Fig. 6).

For the above-mentioned particular initial configuration, Anderson et al. showed that the dynamics, either sequential or parallel, converges towards a unique fixed point. They also provide N-dependent bounds for the transient time length of the game. We take here a different approach than the one followed in [7]. In fact, we shall establish a morphism between the CFG($\mathbb{Z}$) and the SPM. The existence of this morphism implies that the CFG($\mathbb{Z}$) can be interpreted as a particular case of the SPM. Then, the results of Section 2 allow us to characterize the dynamical behavior of the CFG($\mathbb{Z}$) for general initial configurations.

...	−3	−2	−1	0	1	2	3	...
				7				
			1	5	1			
			2	3	2			
		1	1	3	1	1		
		1	2	1	2	1		
		2	0	3	0	2		
	1	0	2	1	2	0	1	
	1	1	0	3	0	1	1	
	1	1	1	1	1	1	1	

Fig. 6. CFG($\mathbb{Z}$) using the initial configuration of [1].

As in Section 1.1, let θ_i, $\forall i \in \mathbb{Z}$, denote the local transition functions, and Θ the global update rule of the CFG($\mathbb{Z}$). We then define

$$T_{\text{seq}}(x) = \min\{t \geqslant 0\colon\ \theta_{i_0} \circ \theta_{i_1} \circ \cdots \circ \theta_{i_t}(x) \text{ is a fixed point}\},$$

$$T_{\text{par}}(x) = \min\{t \geqslant 0\colon\ \Theta^t(x) \text{ is a fixed point}\}.$$

Observe that there is a sequence of sequential updates that has the same effect as a single parallel update. In fact, if $x \in \mathbb{N}^{\mathbb{Z}}$ is such that the firing vertices are $i_1, i_2, \ldots, i_s$, it follows that

$$\Theta(x) = \theta_{i_1} \circ \theta_{i_2} \circ \cdots \circ \theta_{i_s}(x). \tag{6}$$

For $x \in \mathbb{N}^{\mathbb{Z}}$, let us define $l(x) = \min\{i \in \mathbb{Z}\colon\ x(i) \neq 0\}$, i.e. $l(x)$ is the leftmost vertex at which chips are piled. Furthermore, let $\tilde{x} = (\ldots, \tilde{x}_{-1}, \tilde{x}_0, \tilde{x}_1, \ldots)$, where $\tilde{x}_i = \sum_{j \geqslant i} x_j$.

It is important to point out that starting from the initial configuration $y_0(0) = N$, $y_i(0) = 0$, $\forall i \neq 0$, the sequential dynamics converges towards the fixed point $\underline{y}$ [1], where

$$\begin{array}{l} \underline{y} = (\ldots, 0, \underset{\substack{\uparrow \\ -\lfloor \frac{N}{2} \rfloor}}{1}, \ldots, 1, \underset{\substack{\uparrow \\ 0}}{N \bmod 2}, 1, \ldots, \underset{\substack{\uparrow \\ \lfloor \frac{N}{2} \rfloor}}{1}, 0, \ldots). \end{array}$$

In this context we have the following result.

Proposition 4.1. *Let $x(0) \in \mathbb{N}^{\mathbb{Z}}$ be any initial configuration such that $N = \sum_{i \in \mathbb{Z}} x_i(0)$, and let $y(0) \in \mathbb{N}^{\mathbb{Z}}$ be such that*

$$y_{l(x(0))}(0) = N, \qquad y_i(0) = 0 \quad \forall i \neq l(x(0)).$$

Then

(i) $\tilde{y}(t) \leqslant \tilde{x}(t)$, $0 \leqslant t \leqslant T_{\text{par}}(x(0))$.

(ii) $\tilde{y}(t) \leqslant \tilde{x}(t) \Rightarrow l(x(t)) \geqslant l(y(t))$.

(iii) $l(x(t)) \geqslant l(x(0)) - \lfloor \frac{N}{2} \rfloor$, $0 \leqslant t \leqslant T_{\text{par}}(x(0))$. *Furthermore, if N is fixed, there exists an initial configuration of N chips such that the preceding bound is reached for some t.*

Proof. (i) For $t=0$ the inequality obviously holds. Suppose it is true for t. To prove that it holds for $t+1$, it suffices to verify [9], since (6) holds, that

$$\tilde{y}(t) \leqslant \tilde{x}(t) \;\Rightarrow\; \tilde{\theta}_i(y(t)) \leqslant \tilde{\theta}_i(x(t)).$$

(ii) In fact,

$$N = \tilde{y}_{l(y(t))}(t) \leqslant \tilde{x}_{l(y(t))}(t) \leqslant N \;\Rightarrow\; x_i(t) = 0 \quad \forall i < l(y(t))$$
$$\Rightarrow l(x(t)) \geqslant l(y(t)).$$

(iii) Since $(l(y(t)))_{t \geqslant 0}$ is a decreasing sequence that converges to $l(\underline{y})$, we have from (i) and (ii) that

$$l(x(t)) \geqslant l(y(t)) \geqslant l(\underline{y}) = l(x(0)) - \left\lfloor \frac{N}{2} \right\rfloor, \quad 0 \leqslant t \leqslant T_{\text{par}}(x(0)).$$

Finally, if N is fixed, the preceding bound is reached for the initial configuration

$$y_0(0) = N, \qquad y_i(0) = 0 \quad \forall i \neq 0.$$

in $T_{\text{par}}(y(0))$ steps. □

Thus, if the CFG($\mathbb{Z}$) starts from $x(0)$, we can see that vertex $l(x(0)) - \lfloor N/2 \rfloor$ never fires. In fact, if it fires at a given moment, then at the next time step, the vertex immediately to the left of it will be occupied by a chip, which contradicts Proposition 4.1(iii). So, we may change the local update rule of vertex $l(x(0)) - \lfloor N/2 \rfloor$, without changing the global dynamics of the game, associating to this vertex the following local rule:

$$\theta' : \mathbb{N}^{\mathbb{Z}} \to \mathbb{N}^{\mathbb{Z}}, \quad \theta'(x) = y,$$

where

$$y_j = \begin{cases} x_j - 2\mathbb{1}(x_j - 2) & \text{if } j = l(x(0)) - \left\lfloor \dfrac{N}{2} \right\rfloor \\ x_j + \mathbb{1}(x_{j-1} - 2) & \text{if } j = l(x(0)) - \left\lfloor \dfrac{N}{2} \right\rfloor + 1 \\ x_j & \text{otherwise.} \end{cases}$$

In fact, we may suppose that the dynamics takes place on the line graph $\mathbb{Z} \setminus \{i:\ i < l(x(0)) - \lfloor N/2 \rfloor\}$. Furthermore, $\forall j \in \mathbb{N}$, let us rename the vertex $l(x(0)) - \lfloor N/2 \rfloor + j$ as the vertex j. Thus, we obtain a new game which corresponds to the height difference coding of the SPM, and is equivalent to the CFG($\mathbb{Z}$). In Fig. 7, we illustrate the procedure followed in the construction of this new game.

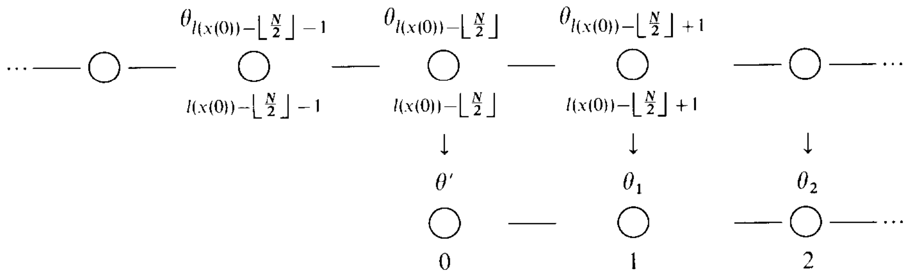


Fig. 7.

Without loss of generality, we may suppose that $l(x(0))=0$; we define the operator

$$\mathcal{M}:\mathbb{N}^{\mathbb{Z}}\to S_N,\quad \mathcal{M}(x)=\omega,$$

where

$$\omega_i=\sum_{j\geq i-\lfloor N/2\rfloor} x_j\quad \forall i\in\mathbb{N},$$

$$N=\sum_{i\in\mathbb{N}}\omega_i.$$

In the following lemma we prove that $\mathcal{M}$ is a morphism.

Lemma 4.2. *The operator $\mathcal{M}$ is a morphism between the* CFG($\mathbb{Z}$) *and the SPM, i.e. $\mathcal{M}$ is one to one, and $\mathcal{M}(\Theta(x))=F(\mathcal{M}(x))$, where, Θ and F are the global transition functions of the* CFG($\mathbb{Z}$) *and the SPM, respectively.*

Proof. If $\mathcal{M}(x)=\mathcal{M}(y)$, then

$$x_i=[\mathcal{M}(x)]_{i+\lfloor N/2\rfloor}-[\mathcal{M}(x)]_{i+\lfloor N/2\rfloor+1}=y_i\quad \forall i\in\mathbb{N}.$$

Thus, $\mathcal{M}$ is one to one. Furthermore, since

$$\mathcal{M}(\theta_{i-\lfloor N/2\rfloor}(x))=f_i(\mathcal{M}(x))\quad \forall i\in\mathbb{N}$$

it follows that $\mathcal{M}(\Theta(x))=F(\mathcal{M}(x))$. □

An example of the morphism established above is shown in Fig. 8.

Thus, we conclude that the CFG($\mathbb{Z}$) can be interpreted as a particular case of the SPM. Since we have already characterized the dynamical behavior of this latter model, we can use the known results, related to the SPM, in the study of the CFG($\mathbb{Z}$) game.

For instance, let us consider the CFG($\mathbb{Z}$) in the case of an initial configuration $x(0)$ like the one studied by Anderson et al. In fact, if $x_0(0)=2n+1$ (or $2n$), and $x_i(0)=0$, $\forall i\neq 0$, then, the sequential dynamics of the SPM starting, respectively, from

$$\omega(0)=\mathcal{M}(x(0))=(2n+1,2n+1,\ldots,2n+1,0,\ldots)\in S_{(2n+1)(n+1)},\quad |\omega(0)|=n,$$

...	-3	-2	-1	0	1	2	3...		0	1	2	3	4	5	6	...
				7					7	7	7	7				
			1	5	1				7	7	7	6	1			
			2	3	2				7	7	7	5	2			
		1	1	3	1	1			7	7	6	5	2	1		
		1	2	1	2	1		$\xrightarrow{\mathscr{M}}$	7	7	6	4	3	1		
		2	0	3	0	2			7	7	5	5	2	2		
	1	0	2	1	2	0	1		7	6	6	4	3	1	1	
	1	1	0	3	0	1	1		7	6	5	5	2	2	1	
	1	1	1	1	1	1	1		7	6	5	4	3	2	1	

Fig. 8. CFG($\mathbb{Z}$) parallel evolution starting from $(\ldots,0,7,0,\ldots)$ and SPM-equivalent dynamics.

or

$$\omega(0)=\mathscr{M}(x(0))=(2n,2n,\ldots,2n,0,\ldots)\in S_{(2n)(n+1)},\quad |\omega(0)|=n,$$

converges towards

$$\pi=(2n+1,2n,\ldots,2,1,0,\ldots)$$

or

$$\pi=(2n,2n-1,\ldots,n+1,n,n,n-1,\ldots,2,1,0,\ldots),$$

respectively, in $T_{\mathrm{sec}}(\omega(0))=E(\pi)-E(\omega(0))=\frac{1}{6}n(n+1)(2n+1)$ steps, as implied by Propositions 2.3 and 3.1, and the fact that $\pi\in S(\omega(0))$. From the above facts, we conclude that the CFG($\mathbb{Z}$) sequential and parallel dynamics, starting from $x(0)$, converges towards

$$(\ldots,0,1,\ldots,1,1,1,\ldots,1,0\ldots)$$

or

$$(\ldots,0,1,\ldots,1,0,1,\ldots,1,0,\ldots),$$

respectively, in exactly $\frac{1}{6}n(n+1)(2n+1)$ time steps, for the sequential dynamics case, and, that the parallel dynamics, converges towards a fixed point in no more than $O(n^3)$ steps, and not less than $\Omega(n^2)$ time steps.

The preceding analysis, can be carried out in the case of an arbitrary initial configuration of finite support. Thus, we have complemented the results obtained in [1].

4.2. *The CFG($\mathbb{N}$) game*

To interpret this game as a particular case of the SPM, we first establish a morphism $\mathscr{M}'$, between the CFG($\mathbb{N}$) and the CFG($\mathbb{Z}$). Then, it follows that $\mathscr{M}\circ\mathscr{M}'$ is a morphism between the CFG($\mathbb{N}$) and the SPM, through which the desired interpretation can be obtained.

The idea underlying the definition of $\mathcal{M}'$, is that any configuration of $\mathbb{N}^{\mathbb{N}}$ can be seen as a coding of a configuration of $\mathbb{N}^{\mathbb{Z}}$, symmetric with respect to the vertex $0\in\mathbb{Z}$. More precisely, we define

$$\mathcal{M}':\mathbb{N}^{\mathbb{N}}\to\mathbb{N}^{\mathbb{Z}},\quad \mathcal{M}'(x')=x,$$

where $x=(\ldots,x_2',x_1',2x_0',x_1',x_2',\ldots)$.

Lemma 4.3. *The operator $\mathcal{M}'$ is a morphism between the* CFG($\mathbb{N}$) *and the* CFG($\mathbb{Z}$), *i.e. $\mathcal{M}'$ is one to one, and*

$$\mathcal{M}'(\Theta(x'))=\Theta(\mathcal{M}'(x')),$$

where, the function Θ that appears at the left (right) of the equality is defined over $\mathbb{N}^{\mathbb{N}}$ ($\mathbb{N}^{\mathbb{Z}}$).

Proof. It is obvious that $\mathcal{M}'$ is one to one. Hence, the lemma follows directly from the following fact:

$$\mathcal{M}'(\theta_i(x'))=\begin{cases}\theta_i(x) & \text{if } i=0\\ \theta_{-i}\circ\theta_i(x) & \text{if } i>0,\end{cases}$$

where the θ_i functions that appear at the left (right) of the equality are defined over $\mathbb{N}^{\mathbb{N}}$ ($\mathbb{N}^{\mathbb{Z}}$). $\square$

An example of the morphism established above is shown in Fig. 9.

Again, we conclude that the CFG($\mathbb{N}$) can be interpreted as a particular case of the SPM. Since we have characterized the dynamical behavior of this latter model, we can use the results related to the SPM, in the analysis of the CFG($\mathbb{N}$).

For instance, following the same scheme of analysis as the one carried out at the end of Section 3 for the CFG($\mathbb{Z}$), we conclude that the CFG($\mathbb{N}$) sequential dynamics,

0	1	2	3	...		...	−3	−2	−1	0	1	2	3	...
3										6				
2	1								1	4	1			
1	2								2	2	2			
1	1	1						1	1	2	1	1		
0	2	1			$\xrightarrow{\mathcal{M}'}$			1	2	0	2	1		
1	0	2						2	0	2	0	2		
0	2	0	1				1	0	2	0	2	0	1	
1	0	1	1				1	1	0	2	0	1	1	
0	1	1	1				1	1	1	0	1	1	1	

Fig. 9. CFG($\mathbb{N}$) parallel evolution starting from (3, 0, ...) and CFG($\mathbb{Z}$)-equivalent dynamics.

starting from $x'(0)$, where $x'_0(0)=n$ and $x'_i(0)=0$, $\forall i>0$, converges towards

$$x'=(0,1,\ldots,1,0,\ldots), \quad \text{where } x'_n=1,\ x'_{n+1}=0$$

in $\frac{1}{6}n(n+1)(n+2)$ steps. Furthermore, we deduce that the parallel dynamics starting from $x'(0)$ also converges towards x', in no more than $O(n^3)$ steps and not less than $\Omega(n^2)$ time steps.

5. The ice pile model

In this section we study the dynamical behavior of the ice pile model (IPM), which is a generalization of the SPM. This former model was introduced in [6], and is of interest both in physics and mathematics.

We define now the IPM. Let n be a positive integer and

$$L_n=\left\{\omega\in\mathbb{N}^n\colon\ \omega_{i+1}\geqslant\omega_i,\ \sum_{i=1}^{n}\omega_i=n\right\}$$

be the states set of the model (i.e. L_n is the set of ordered partitions of n), and consider the dynamics induced by the following rules (see Fig. 10):

Sand pile rule: It consists in applying operator T_i to ω, if $\omega_i-\omega_{i+1}\geqslant 2$, where

$$T_i(\omega)=(\ldots,\omega_{i-1},\omega_i-1,\omega_{i+1}+1,\ldots).$$

Staircase rule: It consists in applying operator $T_{i,k}$, $k>i+1$, to ω, if $\omega_i-1=\omega_{i+1}=\cdots=\omega_{k-1}=\omega_k+1$, where

$$T_{i,k}(\omega)=(\ldots,\omega_{i-1},\omega_i-1,\omega_{i+1},\ldots,\omega_{k-1},\omega_k+1,\omega_{k+1},\ldots).$$

Clearly, the operators $\{T_i\}$ of the IPM are equivalent to the local operators $\{f_i\}$ of the SPM. The operators $\{T_{i,k}\}$ allow an updating scheme not feasible in the SPM. Note also, that since the staircase update rule allows arbitrary far sites to interact, the IPM is not a cellular automaton.

We shall always associate to the IPM, a sequential dynamic (see Fig. 11). Clearly, the unique fixed point of this model is $(1,\ldots,1)\in L_n$.

The IPM can be physically interpreted as piles of ice-cubes which interact from left to right. Whenever two consecutive piles have a height difference of at least 2, a cube

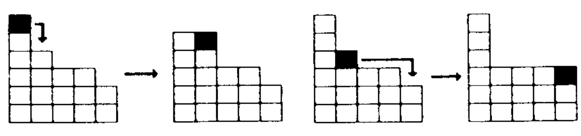


Fig. 10. Update rules of the IPM: (i) sand pile rule; (ii) staircase rule.

t	1	2	3	4	5	6	7	8
0	$\underline{8}$							
1	$\underline{7}$	1						
2	$\underline{6}$	2						
3	$\underline{5}$	3						
4	4	$\underline{4}$						
5	4	$\underline{3}$	1					
6	4	2	$\underline{2}$					
7	4	$\underline{2}$	1	1				
8	$\underline{4}$	1	1	1	1			
9	3	$\underline{2}$	1	1	1			
10	$\underline{3}$	1	1	1	1	1		
11	2	$\underline{2}$	1	1	1	1		
12	$\underline{2}$	1	1	1	1	1	1	
13	1	1	1	1	1	1	1	1

Fig. 11. IPM evolution in L_8.

tumbles from the left to the right pile. Otherwise, when piles i, k have a height difference equal to 2, a cube slides without friction from pile i to k.

Now, as in Section 2, we define a partial-order relation over the set L_n as follows:

$$\omega \leqslant v \Leftrightarrow \sum_{j \geqslant i} \omega_j \geqslant \sum_{j \geqslant i} v_j, \quad 1 \leqslant i \leqslant n.$$

Brylawski [4] showed that $(L_n, \leqslant)$ is a lattice with maximum $\bar{n} = (n, 0, \ldots)$ and minimum $\bar{1} = (1, \ldots, 1)$, and proved that ω covers v iff

$$v = (\omega_1, \ldots, \omega_{i-1}, \omega_i - 1, \ldots, \omega_k + 1, \omega_{k+1}, \ldots, \omega_n), \quad \text{where}$$

$$k = i + 1 \quad (\text{transition } T_i),$$

or

$$k > i + 1 \quad \text{and} \quad \omega_i - 1 = \omega_{i+1} = \cdots = \omega_{k-1} = \omega_k + 1 \quad (\text{transition } T_{i,k}).$$

We exhibit below examples of both types of transitions

$$6\,3\,2\,1 \to 5\,4\,2\,1 \quad (\text{transition } T_1),$$

$$6\,3\,2\,1 \to 6\,2\,2\,2 \quad (\text{transition } T_{2,4}).$$

It should be clear from the above facts that there is a strong relation between the lattice structure of $(L_n, \leqslant)$ and the IPM.

In the lattice analogy of the IPM we shall refer to the trajectories as chains. Formally, $\mathscr{C} = (\omega_i)_{i=0}^{q-1} \subset L_n$ is a chain of $(L_n, \leqslant)$, between μ and v, of length $l(\mathscr{C}) = q$, if

$$\mu = \omega_0 \geqslant \cdots \geqslant \omega_i \geqslant \omega_{i+1} \geqslant \cdots \geqslant \omega_{q-1} = v$$

and

$$\omega_{i-1} \text{ covers } \omega_i, \quad \forall i \in \{1, \ldots, q-1\}.$$

The problem of determining the minimum length of a chain between $\bar{n}$ and $\bar{1}$ was solved by Brylawski. Such minimum length is [4]

$$2n-3 \quad \forall n>2. \tag{7}$$

Below, we show a procedure used to obtain minimal chains (i.e. chains of minimum length between $\bar{n}$ and $\bar{1}$),

$$\bar{n} \to n-1,1 \to n-2,2 \to n-2,1,1 \to n-3,2,1 \to n-3,1,1,1 \to \cdots$$
$$\cdots \to 3,1,1,\ldots,1 \to 2,2,1,\ldots,1 \to 2,\ldots,1 \to \bar{1}.$$

We shall prove here that the maximum length of a chain between $\bar{n}$ and $\bar{1}$ is

$$2\binom{k+1}{3}+kk'+1, \quad \text{where } n=\frac{1}{2}k(k+1)+k', \ 0 \leqslant k' \leqslant k. \tag{8}$$

We also provide a family of maximal chains (i.e. chains of maximum length between $\bar{n}$ and $\bar{1}$). The maximal chain characterization problem was proposed to one of us (E.G.) by M.P. Schützenberger.

Clearly, since any chain can be interpreted as a trajectory of the IPM, we can deduce from (9) and (10) closed formulas for the extreme lengths of the IPM transients starting from the initial configuration $\bar{n}$.

5.1. *Lattices and sublattices*

Let us define the maximum (minimum) of $\nu, \mu \in L_n$ as follows:

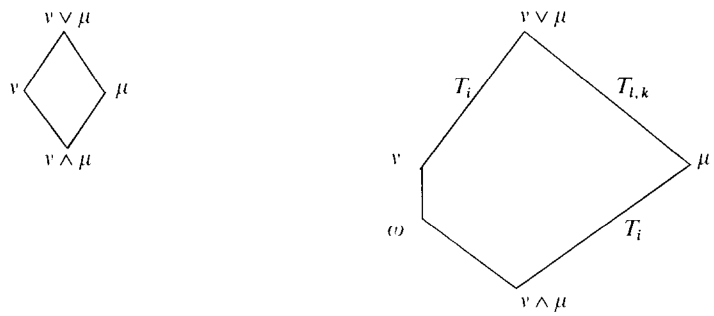
$$\nu \vee \mu = \min\{\omega \in L_n\colon \mu \leqslant \omega, \ \nu \leqslant \omega\},$$
$$(\nu \wedge \mu = \max\{\omega \in L_n\colon \omega \leqslant \mu, \ \omega \leqslant \nu\}).$$

The lattice structure of $(L_n, \leqslant)$ insures the existence of these partitions. Furthermore, it was proved in [4] that if $\nu \neq \mu$ then the set $\{\nu, \mu, \nu \vee \mu, \nu \wedge \mu\}$ belongs to one of the sublattices of Fig. 12a and b.

It is important to point out that the pentagon structure of Fig. 12a appears from the IPM point of view, iff the height of a specific column of the ice pile can be changed using either the sand pile rule, or the staircase rule.

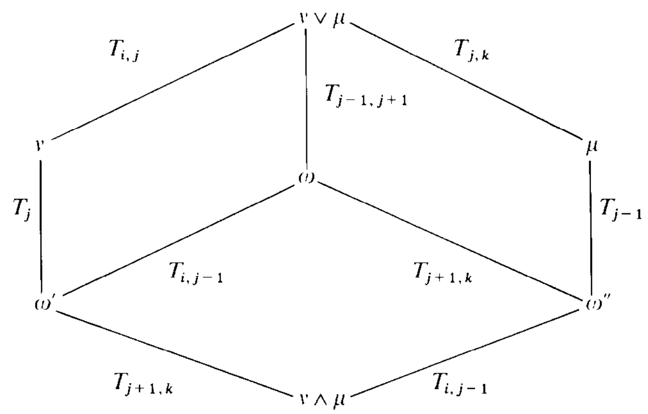
5.2. *Maximal chains*

Let $\mathscr{C} = (\nu_i)_{i=0}^{q-1} \subset L_n$ be a chain. We shall say that ν_{i+1} is obtained from ν_i by a premodular transition denoted by $(\nu_i \to_{\mathrm{pm}} \nu_{i+1})$, if $\nexists\ \omega, \omega' \in L_n$, $\omega \neq \nu_{i+1}$, ν_i cover of ω (i.e. $\nu_i = \nu_{i+1} \vee \omega$), such that $\nu_i, \nu_{i+1}, \omega, \omega'$ and $\omega \wedge \nu_{i+1}$ belong to the sublattice shown in Fig. 13, i.e. ν_i and ν_{i+1} cannot belong to the shortest branch of the pentagon sublattice.



$k > l+1$
$k = i \vee l = i+1$
$\omega \in L_n$

Fig. 12a. Sublattice structures that contain $\nu, \mu, \nu \vee \mu$ and $\nu \wedge \mu$, where $\nu \neq \mu$.



$j > i+1$
$k > j+1$
$\omega, \omega', \omega'' \in L_n$

Fig. 12b. Sublattice structure that contains $\nu, \mu, \nu \vee \mu$ and $\nu \wedge \mu$, where $\nu \neq \mu$.

Analogously, we shall say that ν_{i+1} is obtained from ν_i by a modular transition (denoted by $\nu_i \rightarrow_{\mathrm{m}} \nu_{i+1}$), if $\nu_i \rightarrow_{\mathrm{pm}} \nu_{i+1}$, and $\nexists \omega \in L_n$, $\omega \neq \nu_{i+1}$, ν_i cover of ω (i.e. $\nu_i = \nu_{i+1} \vee \omega$), such that ν_i, ν_{i+1}, ω and $\nu_{i+1} \wedge \omega$ belong to the sublattice shown in Fig. 14, i.e. a modular transition corresponds to a transition based on the sand pile rule or to a transition based on the staircase rule, in the case that the associated state of the IPM cannot be updated using the sand pile rule.

Finally, we say that a chain is modular (premodular), if all of its partitions are obtained by modular (premodular) transitions. The evolution of the IPM shown in Fig. 15 is associated to a modular chain of $(L_8, \leqslant)$.

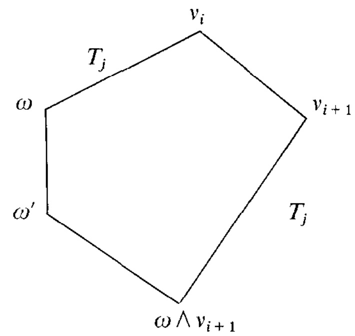
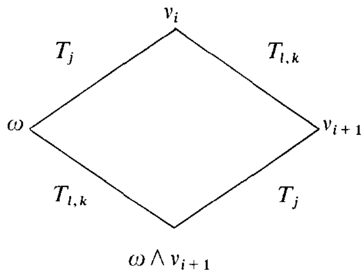


Fig. 13.



$l > j+1 \vee j > k$

$k > l+1$

Fig. 14.

We shall now prove that there exists a maximal modular chain between $\bar{n}$ and $\bar{1}$. The importance of this result follows from the fact that determining the length of a modular chain is not difficult.

First, we need to introduce several definitions. Let C^n be the set of chains in L_n between $\bar{n}$ and $\bar{1}$, and consider the following operator:

$$\mathcal{N}: C^n \to \mathscr{P}(\mathbb{N}),$$

$$\mathscr{C} = (\omega_i)_{i=0}^{l(\mathscr{C})-1} \to \mathcal{N}(\mathscr{C}) = \{i\colon \omega_i \not\to_m \omega_{i+1}\}.$$

Clearly, $\mathscr{C} \in C^n$ is a modular chain iff $\mathcal{N}(\mathscr{C}) = \Phi$.

Furthermore, given two chains $\mathscr{C}_1 = (v_i)_{i=0}^{q-1}$ and $\mathscr{C}_2 = (v_i)_{i=q}^{q+q'-1}$, where v_{q-1} covers v_q, we denote by $\mathscr{C}_1 \cup \mathscr{C}_2$ the chain $(v_i)_{i=0}^{q+q'-1}$.

Theorem 5.1. *There exists a maximal modular chain $\tilde{\mathscr{C}} \in C^n$.*

Proof. It suffices to prove that if $\mathscr{C} \in C^n$ is a nonmodular maximal chain, then there exists a modular chain $\tilde{\mathscr{C}} \in C^n$, such that $l(\mathscr{C}) \leqslant l(\tilde{\mathscr{C}})$.

t	1	2	3	4	5	6	7	8
0	$\underline{8}$							
1	$\underline{7}$	1						
2	$\underline{6}$	2						
3	5	$\underline{3}$						
4	$\underline{5}$	2	1					
5	4	$\underline{3}$	1					
6	$\underline{4}$	2	2					
7	3	3	$\underline{2}$					
8	3	$\underline{3}$	1	1				
9	3	2	$\underline{2}$	1				
10	3	$\underline{2}$	1	1	1			
11	$\underline{3}$	1	1	1	1	1		
12	2	$\underline{2}$	1	1	1	1		
13	$\underline{2}$	1	1	1	1	1	1	
14	1	1	1	1	1	1	1	1

Fig. 15. Evolution of the IPM associated to a modular chain.

Let $\mathscr{C} \in C^n$ be a nonmodular maximal chain, and $i_0 = \inf_{i \in \mathscr{N}(\mathscr{C})} i < +\infty$ (we assume that $\inf_{i \in \Phi} i = +\infty$). We shall prove that

$$\exists \mathscr{C}' \in C^n \text{ such that } l(\mathscr{C}) \leqslant l(\mathscr{C}') \quad \text{and} \quad i_0' = \inf_{i \in \mathscr{N}(\mathscr{C}')} i > i_0. \tag{9}$$

If $\mathscr{N}(\mathscr{C}') = \Phi$ then $\mathscr{C}'$ satisfies the theorem. On the other hand, if $\mathscr{N}(\mathscr{C}') \neq \Phi$, then considering in (9) the chain $\mathscr{C}'$ instead of $\mathscr{C}$, it follows that

$$\exists \mathscr{C}'' \in C^n \text{ such that } l(\mathscr{C}) \leqslant l(\mathscr{C}') \leqslant l(\mathscr{C}'') \quad \text{and} \quad i_0'' = \inf_{i \in \mathscr{N}(\mathscr{C}'')} i > i_0' > i_0.$$

Since L_n is of finite cardinal, we obtain, in a finite number of steps, a chain $\tilde{\mathscr{C}}$ satisfying the theorem. Thus, to complete the proof of the theorem we only have to show that (9) is verified.

We specify now an algorithm which yields a chain $\mathscr{C}'$ satisfying (9).

If $\mathscr{C} = (v_i)_{i=0}^{l(\mathscr{C})-1}$ and $i_0 = \inf_{i \in \mathscr{N}(\mathscr{C})} i$, then one of the following cases holds:

(i) If $v_{i_0} \not\to_{\mathrm{pm}} v_{i_0+1}$, note that $\exists \omega_{i_0+1}, \omega_{i_0}' \in L_n$, $\omega_{i_0+1} \neq v_{i_0+1}$, and $\exists i$ such that the situation shown in Fig. 16 is verified.

So, let $L = i_0 + 1, \omega_{i_0+2} = \omega_{i_0+1} \wedge v_{i_0+1}$, and consider the chain between v_0 and ω_{i_0+2}, defined as follows: $\mathscr{C}_L' = (v_i)_{i=0}^{L-1} \cup (\omega_{i_0+1}, \omega_{i_0}', \omega_{i_0+2})$.

(ii) If $v_{i_0} \not\to_{\mathrm{m}} v_{i_0+1}$ and $v_{i_0} \to_{\mathrm{pm}} v_{i_0+1}$, note that $\exists \omega_{i_0+1} \in L_n$, $\omega_{i_0+1} \neq v_{i_0+1}$, and $\exists i$ such that the situation shown in Fig. 17 is verified.

So, let $L = i_0 + 1$, $\omega_{i_0+2} = \omega_{i_0+1} \wedge v_{i_0+1}$, and consider the chain between v_0 and ω_{i_0+2}, defined as follows: $\mathscr{C}_L' = (v_i)_{i=0}^{L-1} \cup (\omega_{i_0+1}, \omega_{i_0+2})$.

We give now a procedure that completes the construction of the chain $\mathscr{C}'$ that satisfies (9).

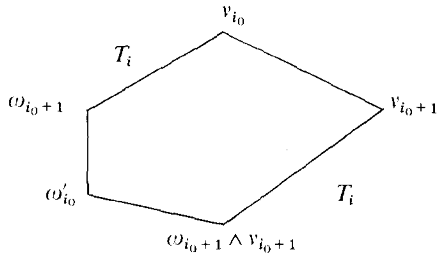


Fig. 16.

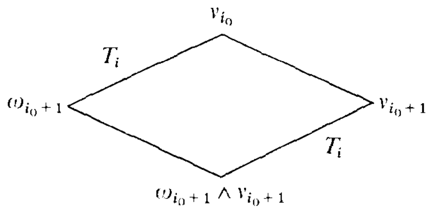


Fig. 17.

(I) So far, we have constructed a chain $\mathscr{C}'_L = (\mu_i^L)_{i=0}^{l(\mathscr{C}'_L)-1}$, and the situation shown in Fig. 18 is verified.

Hence,

(i) If $\mu^L_{l(\mathscr{C}'_L)-1} = \nu_{L+1}$, then the chain that satisfies (9) is $\mathscr{C}' = \mathscr{C}'_L \cup (\nu_i)_{i=L+2}^{l(\mathscr{C})-1}$.

(ii) If $\mu^L_{l(\mathscr{C}'_L)-1} \neq \nu_{L+1}$, then only one of the following cases holds:

(a) The relations shown in Fig. 19 are verified.

In this case, we define the chain $\mathscr{C}'_{L+1} = \mathscr{C}'_L \cup (\mu^L_{l(\mathscr{C}'_L)-1} \wedge \nu_{L+1})$, and let $L \leftarrow L+1$. Then we return to (I).

(b) The relations shown in Fig. 20 are verified.

In this case, we define the chain $\mathscr{C}'_{L+1} = \mathscr{C}'_L \cup (\omega'_L, \mu^L_{l(\mathscr{C}'_L)-1} \wedge \nu_{L+1})$, and let $L \leftarrow L+1$. Then we return to (I).

The preceding procedure stops after a finite number of steps. In fact, if $L = l(\mathscr{C}) - 2$, then $\mu^L_{l(\mathscr{C}'_L)-1} = \nu_{L+1}$, since $\nu_{L+1} = \bar{1} \leqslant \omega$, $\forall \omega \in L_n$. $\square$

Observe that from the demonstration of theorem 5.1, we obtain the following corollary.

Corollary 5.2. *A pre-modular chain of C^n cannot be a maximal chain.*

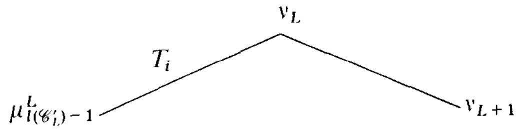


Fig. 18.

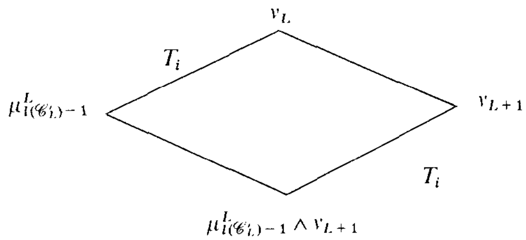
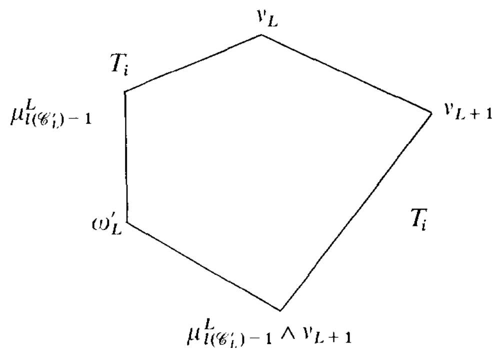


Fig. 19.



$\omega'_L \in L_n$

Fig. 20.

5.3. *Length of maximal chains*

In this section we shall determine the length of a maximal modular chain of C^n. To accomplish this task we need to define the dual or conjugate of $\omega \in L_n$, denoted ω^*, as follows:

$$\omega_i^* = |\{ j\colon \omega_j \geqslant i\}|,$$

i.e. the dual of ω is obtained reflecting the diagram of the ω ice pile on the 45° axis.

Clearly, $\omega \in L_n$ implies that $\omega^* \in L_n$. If $\omega^* = \omega$, then ω is called self-dual.

Brylawski [4] proved that given $v, \omega \in L_n$, then v covers ω, if and only if, ω^* covers v^*. Hence, given a chain $\mathscr{C}=(v_i)_{i=0}^{\ell(\mathscr{C})-1}$, we can define its dual chain $\mathscr{C}^*$ as $\mathscr{C}^*=(v^*_{l(\mathscr{C})-1}, \ldots, v^*_1, v^*_0)$.

Considering again the energy, functional $E(\omega)=\sum_{i=1}^n i\omega_i$, we have

$$\text{if } v=T_{i,k}(\omega), \quad k>i+1 \;\Rightarrow\; E(v)=E(\omega)+(k-i),$$

$$\text{if } v=T_i(\omega) \;\Rightarrow\; E(v)=E(\omega)+1.$$

An immediate consequence of this, is that $v\in L_n$ can be reached from $\bar{n}$ in at most $E(v)-E(\bar{n})$ transitions.

As in Section 3, let us decompose, for use in the remainder of this section, n in the following manner:

$$n=\tfrac{1}{2}k(k+1)+k', \quad 0\leqslant k'\leqslant k.$$

We are now ready to determine the length of the maximal chains of C^n. First, we shall exhibit a chain, $\mathscr{C}=(v_i)_{i=0}^{l(\mathscr{C})-1}\in C^n$, where

$$v_m=(k, k-1, \ldots, k'+1, k', k', k'-1, \ldots, 2, 1, 0, \ldots, 0), \quad m=\binom{k+1}{3}-\binom{k'}{2}+kk', \tag{10}$$

$$v_s=v_m^*=(k+1, k, \ldots, k-k'+3, k-k'+2, k-k', k-k'-1, \ldots, 2, 1, 0, \ldots, 0),$$
$$s=\binom{k+1}{3}+\binom{k'}{2}.$$

The following four-step procedure yields such a chain:

(i) We first provide a modular chain between $\bar{n}$ and

$$\alpha=(k+k', k-1, k-2, \ldots, 2, 1, 0, \ldots, 0), \quad \alpha_k=1.$$

In fact, we exhibit an evolution of the IPM starting from $\bar{n}$ that reaches α using only updates based on the sand pile rule. More precisely, consider the SPM evolution obtained by moving $k-1$ ice cubes from pile 1 to pile 2, then $k-2$ ice cubes from pile 1 to pile 3, and so on, until an ice cube is moved from pile 1 to pile k. The following illustration shows the chain associated to such evolution:

$$\begin{aligned}
\bar{n}&=(\tfrac{1}{2}k(k+1)+k', 0, \ldots)\geqslant(\tfrac{1}{2}k(k+1)+k'-1, 1, 0, \ldots)\\
&\geqslant(\tfrac{1}{2}k(k+1)+k'-2, 2, 0, \ldots)\geqslant\cdots\\
&\geqslant(\tfrac{1}{2}k(k+1)+k'-(k-1), k-1, 0, \ldots)\geqslant(\tfrac{1}{2}k(k+1)+k'-(k-1)-1, k, 0, \ldots)\\
&\geqslant(\tfrac{1}{2}k(k+1)+k'-(k-1)-1, k, 1, 0, \ldots)\geqslant\cdots\\
&\geqslant(\tfrac{1}{2}k(k+1)+k'-(k-1)-(k-2), k-1, k-2, 0, \ldots)\geqslant\cdots\\
&\geqslant(k+k'+s+(s-1)+\cdots+2+1, k-1, k-2, \ldots, s+1, 0, \ldots)\geqslant\cdots\\
&\geqslant(k+k', k-1, k-2, \ldots, 2, 1, 0, \ldots)=\alpha.
\end{aligned}$$

Clearly, all the transitions of the preceding chain are based on the sand pile rule.

(ii) We provide now a modular chain between α and

$$\beta=(k+1,k,\dots,k-k'+3,k-k'+2,k-k',k-k'-1,\dots,2,1,0,\dots,0),\quad \beta_k=1.$$

We accomplish this by exhibiting an evolution of the IPM that starts from α and reaches β using only updates based on the sand pile rule. Particularly, we consider in this case the SPM evolution obtained by moving 1 ice cube from pile 1 to pile k', then 1 ice cube from pile 1 to pile $k'-1$, and so on, until an ice cube is moved from pile 1 to pile 2. The illustration below shows the chain associated to such evolution:

$$\begin{aligned}
\alpha&=(k+k',k-1,k-2,\dots,2,1,0,\dots)\geqslant(k+k'-1,k,k-2,\dots,2,1,0,\dots)\\
&\geqslant(k+k'-1,k-1,k-1,\dots,2,1,0,\dots)\geqslant\cdots\\
&\geqslant(k+k'-1,k-1,\dots,k-k'+2,k-k'+2,k-k',k-k'-1,\dots,2,1,0,\dots)\\
&\geqslant(k+k'-2,k,\dots,k-k'+2,k-k'+2,k-k',k-k'-1,\dots,2,1,0,\dots)\geqslant\cdots\\
&\geqslant(k+k'-2,k-1,\dots,k-k'+3,k-k'+3,k-k'+2,k-k',\\
&\qquad k-k'-1,\dots,2,1,0,\dots)\geqslant\cdots\\
&\geqslant(k+k'-s,k-1,\dots,k-k'+s+2,k-k'+s+1,k-k'+s+1,\dots,\\
&\qquad k-k'+2,k-k',k-k'-1,\dots,2,1,0,\dots)\geqslant\cdots\\
&\geqslant(k+1,k,\dots,k-k'+3,k-k'+2,k-k',k-k'-1,\dots,2,1,0,\dots)=\beta.
\end{aligned}$$

Clearly, all the transitions of the preceding chain are based on the sand pile rule.

(iii) We now construct a modular chain between β and

$$\pi=(k,k-1,\dots,k'+1,k',k',k'-1,\dots,2,1,0,\dots,0).$$

Again, we proceed as in the preceding steps. In this case, we consider the SPM evolution obtained by moving 1 ice cube from pile k' to pile $k+1$, then another ice cube from pile $k'-1$ to pile k, and so on, until an ice cube is moved from pile 1 to pile $k-k'+2$. The following illustration shows the chain associated to such evolution:

$$\begin{aligned}
\beta&=(k+1,k,\dots,k-k'+3,k-k'+2,k-k',k-k'-1,\dots,1,0,\dots)\\
&\geqslant(k+1,k,\dots,k-k'+3,k-k'+1,k-k'+1,k-k'-1,\dots,1,0,\dots)\geqslant\cdots\\
&\geqslant(k+1,k,\dots,k-k'+3,k-k'+1,k-k',k-k'-1,\dots,2,1,1,0,\dots)\\
&\geqslant(k+1,k,\dots,k-k'+2,k-k'+2,k-k',k-k'-1,\dots,2,1,1,0,\dots)\geqslant\cdots\\
&\geqslant(k+1,k,\dots,k-k'+2,k-k'+1,k-k',k-k'-1,\dots,2,2,1,0,\dots)\geqslant\cdots\\
&\geqslant(k+1,k,\dots,k-k'+s+2,k-k'+s,k-k'+s-1,\dots,s+1,s,\\
&\qquad s,s-1,\dots,2,1,0,\dots)\geqslant\cdots\\
&\geqslant(k,k-1,\dots,k'+1,k',k',k'-1,\dots,2,1,0,\dots)=\pi.
\end{aligned}$$

Clearly, all the transitions of the preceding chain are based on the sand pile rule.

(iv) Finally, let $\tilde{\mathscr{C}}$ be the chain between $\bar{n}$ and β, associated to the evolution obtained following the same updating scheme as in steps (i) and (ii). Then $\tilde{\mathscr{C}}^*$ is a chain between $\beta^* = \pi$ and $\bar{n}^* = \bar{1}$.

Thus, we conclude that there exists a chain $\mathscr{C} = (v_i)_{i=0}^{l(\mathscr{C})-1} \in C^n$, such that $v_s = \beta = \pi^*$, $v_m = \pi$ for some s and m, $s < m$, and such that v_m is reached using only transitions based on the sand pile updating rule. Hence,

$$\begin{aligned} s &= E(v_s) - E(\bar{n}) = E(\pi^*) - E(\bar{n}) = \binom{k+1}{3} + \binom{k'}{2}, \\ m &= E(v_m) - E(\bar{n}) = E(\pi) - E(\bar{n}) = \binom{k+1}{3} - \binom{k'}{2} + kk'. \end{aligned} \tag{11}$$

Observe also, that we have shown that the chain $\mathscr{C}$ can be constructed in a way such that $l(\mathscr{C}) = m + s + 1$.

We now prove that the $m+1$ partition of any modular chain of C^n is π.

Lemma 5.3. *If $\mathscr{C} = (v_i)_{i=0}^{l(\mathscr{C})-1} \in C^n$ is a modular chain, then $v_m = \pi$, where*

$$m = \binom{k+1}{3} - \binom{k'}{2} + kk'.$$

Proof. Suppose $\mathscr{C} = (\omega_i)_{i=0}^{l(\mathscr{C})-1}$ is a modular chain. Let p be the first integer such that ω_{p+1} is obtained from ω_p through a transition based on the staircase updating rule. Thus, the chain $(\omega_i)_{i=0}^{p}$ can be interpreted as a SPM sequential evolution between the starting configuration $\bar{n}$ and the SPM fixed point v_p. From Lemma 3.2, we conclude immediately that $v_p = \pi$, and

$$p = m = \binom{k+1}{3} - \binom{k'}{2} + kk'. \qquad \square$$

We can now prove the main result of this section.

Theorem 5.4. *If $\mathscr{C} \in C^n$ is a maximal chain, then*

$$l(\mathscr{C}) = 2\binom{k+1}{3} + kk' + 1.$$

Proof. Note that the procedure specified prior to Lemma 5.3 yields a chain of length $m+s+1$, where m and s are as in (11). Hence, since $\mathscr{C}$ is maximal; it follows that

$$l(\mathscr{C}) \geqslant m + s + 1 = 2\binom{k+1}{3} + kk' + 1.$$

On the other hand, suppose $\mathscr{C}=(\omega_i)_{i=0}^{l(\mathscr{C})-1}$ is modular, to demonstrate that $l(\mathscr{C})\leqslant m+s+1$, it suffices to prove that $\mathscr{C}'=(\omega_i)_{i=m}^{l(\mathscr{C})-1}$ is such that $l(\mathscr{C}')\leqslant s+1$. In fact, from Lemma 5.3, $\omega_m=\pi$; hence, $\mathscr{C}'^*$ is a chain between $\omega^*_{l(\mathscr{C})-1}=\bar{1}^*=\bar{n}$ and $\omega^*_m=\pi^*$. However, since π^* can be obtained from $\bar{n}$ in at most $E(\pi^*)-E(\bar{n})=s$ transitions, we conclude that $l(\mathscr{C}')=l(\mathscr{C}'^*)\leqslant s+1$. □

A straightforward consequence of the preceding theorem is that the maximum transient length of the IPM, with sequential dynamics starting from $\bar{n}$, is

$$2\binom{k+1}{3}+kk'\sim\Theta(n\sqrt{n}).$$

Finally, we provide a family of maximal chains between $\bar{n}$ and $\bar{1}$. Recall first that Corollary 5.2 insures that nonpremodular chains are not maximal. On the other hand, one could be tempted to think that premodular chains are maximal. In Fig. 21 we provide a sublattice of $(L_{11},\leqslant)$ that clearly shows that premodularity does not insure the maximality of chains.

In fact, we have

$$5\,3\,2\,1\longrightarrow_{\mathrm{pm}}5\,3\,1\,1\,1\longrightarrow_{\mathrm{m}}5\,2\,2\,1\,1\longrightarrow_{\mathrm{m}}4\,3\,2\,1\,1,$$

$$5\,3\,2\,1\longrightarrow_{\mathrm{m}}4\,4\,2\,1\longrightarrow_{\mathrm{m}}4\,3\,3\,1\longrightarrow_{\mathrm{m}}4\,3\,2\,2\longrightarrow_{\mathrm{m}}4\,3\,2\,1\,1.$$

Thus, the introduction of a stronger property than premodularity, that is modularity, is warranted. We shall now prove that modularity insures the maximality of chains.

Theorem 5.5. *Any modular chain of C^n is a maximal chain.*

Proof. Let m and s be as in (11), and let $\mathscr{C}=(\omega_i)_{i=0}^{l(\mathscr{C})-1}\in C^n$ be a modular chain. Lemma 5.3, implies that $\omega_m=\pi$.

To prove the theorem, it suffices to show that $\mathscr{C}'=(\omega_i)_{i=m}^{l(\mathscr{C})-1}$ is such that $l(\mathscr{C}')=s+1$, since from this it follows that

$$l(\mathscr{C})=l((\omega_i)_{i=0}^{m-1})+l(\mathscr{C}')=m+(s+1)=2\binom{k+1}{3}+kk'+1,$$

i.e. $\mathscr{C}$ is a maximal chain of C^n.

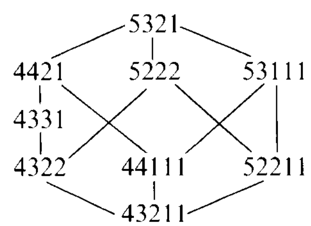


Fig. 21.

Note that $\mathscr{C}'$ and $(\mathscr{C}')^*$ are chains of equal length; hence, we shall show that $l((\mathscr{C}')^*)=s+1$. In fact, since $\mathscr{C}'$ is a modular chain and $\omega_m=\pi$, we have that if $i\in\{m+1,\ldots,l(\mathscr{C})-1\}$, then $\omega_{i-1}(j)=\omega_{i-1}(k)+2$. Hence, ω_{i-1}^* can be obtained from ω_i^* through a transition of type T_l, thus

$$E(\omega_{i-1}^*)-E(\omega_i^*)=1, \quad \forall i\in\{m+1,\ldots,l(\mathscr{C})-1\}.$$

On the other hand, $\omega_m^*=\pi^*$ and $\omega_{l(\mathscr{C})-1}^*=\bar{1}^*=\bar{n}$, hence,

$$l(\mathscr{C}')=l((\mathscr{C}')^*)=E(\omega_m^*)-E(\omega_{l(\mathscr{C})-1}^*)+1=E(\pi^*)-E(\bar{n})+1=s+1. \qquad \square$$

The preceding theorem implies that at each step, a local rule exists which allows to choose a maximal chain of the integer partition lattice. This property does not hold for arbitrary graphs. In this latter case, the search of an extremal chain between two arbitrary configurations is a global property.

We also deduce from the preceding theorem that an optimum strategy to obtain a maximum transient length of the IPM, starting from $\bar{n}$, consists in updating the model using the sand pile rule. Eventually, the IPM will reach a stable state for the SPM updating scheme. In the latter case, the staircase rule can be used to perturb this stable state. Returning back to a sand pile updating scheme, and so on and so forth. This strategy finally leads to the stable state $\bar{1}$ in a maximum number of time steps. Hence, an optimum strategy to obtain a maximum transient length of the IPM consists in updating the model in a way such that at each time step the increase of the energy functional E is the least possible.

6. Conclusions

We have introduced the IPM, a natural generalization of the SPM first proposed by P. Bak. We have also shown that the analysis of the IPM transient behavior can be used to characterize the dynamics of the one-dimensional SPM. Two cases of CFGs were proved to be instances of the SPM. The latter result allows to generalize the bounds on the length of the transient evolution of the CFG($\mathbb{Z}$) obtained in [1] without imposing conditions on the initial configuration of the game. In the case of the IPM, strategies that provide maximum and minimum lengths of the transient evolution were given. Furthermore, these strategies determine the update rule to be performed at a particular time step, based solely on the state that the model is in at that moment. From our analysis of the IPM we have deduced an exact formula for the height of the integer partition lattice; it has been recently called to our attention that this formula had been also obtained in [7].

Finally, it will be interesting to study the dynamic behavior of natural generalizations of sand piles to arbitrary graphs; i.e. whenever a vertex has as many chips as its degree it gives one chip to each neighbor. For the parallel update on trees the parallel dynamics converges to fixed points or cycles of period two [3]. For other graphs the periodicity in steady state remains an open problem.

Acknowledgment

The authors would like to thank Prof. M.P. Schützenberger (Santiago, 1981), for introducing one of us (E.G.) to the work of Brylawski as well as to the maximal chain characterization problem.

Note added in proof

A recent article by Björner, Lovász and Shor addresses the study of the generalization of the sand pile model mentioned in Section 6. Particularly interesting are the new techniques used to bound the sequential running time of this latter generalization. Interesting open questions still remain regarding the parallel evolution of this model.

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