Notes on General Frame Operators in Infinite Dimensions

1. a) A mapping $L : H \rightarrow G$ from a Hilbert space $H$ into a Hilbert space $G$ is called a **linear operator** if for all $\alpha, \beta \in \mathbb{C}$ and $f, g \in H$

$$L[\alpha f + \beta g] = \alpha L[f] + \beta L[g].$$

b) The **operator norm** of $L$ is defined as

$$\|L\| := \sup \{\|L[f]\|_G : f \in H, \|f\|_H = 1\},$$

where $\|\cdot\|_H$ is the norm induced by the inner product $\langle \cdot, \cdot \rangle_H$ on $H$ and similarly for $\|\cdot\|_G$.

c) If $H = \mathbb{C}^N$ and $G = \mathbb{C}^M$, both equipped with the standard Euclidean inner product, then $L$ can be represented by an $M \times N$ matrix and $\|L\|$ is the maximum singular value of $L$. (In a slight abuse of notation, in the finite case we use $L$ to denote both the linear operator and the matrix which captures the action of this linear operator. This should not cause any confusion, though.)

2. a) The **adjoint** of a linear operator $L$ is the unique linear operator $L^* : G \rightarrow H$ such that

$$\langle L[f], y \rangle_G = \langle f, L^*[y] \rangle_H$$

for all $f \in H$ and $y \in G$. Proof that such a linear operator exists and is unique can be found in any text on functional analysis (see, for example Chapter 7 of N. Young’s “Introduction to Hilbert Space”).
b) If $H = \mathbb{C}^N$ and $G = \mathbb{C}^M$, both equipped with the standard Euclidean inner product, then $L^*$ is represented by the conjugate transpose of $L$.

c) For general Hilbert spaces, it is always true that $(L^*)^* = L$ and that $\|L^*\| = \|L\|$. We call $L : H \to H$ self-adjoint or Hermitian if $L = L^*$ (for $H = \mathbb{C}^n$, this means the matrix is conjugate-symmetric).

3. a) We can define the range and the null space of a linear operator in much the same way as we do for matrices in finite dimensions:

$$\text{Null}(L) = \{ f \in H : L[f] = 0 \}$$
$$\text{Range}(L) = \{ y \in G : y = L[f] \text{ for some } f \in H \}.$$ 

b) As in finite dimensions:

if $y \in \text{Range}(L)$ and $v \in \text{Null}(L^*)$, then $\langle y, v \rangle_G = 0$;
if $f \in \text{Range}(L^*)$ and $g \in \text{Null}(L)$, then $\langle f, g \rangle_H = 0$.

4. a) Given a sequence of signals $\{\psi_{\gamma}\}_{\gamma \in \Gamma}$ in a Hilbert space $H$, we can define the linear operator $\Psi : H \to \ell_2(\Gamma)$ by

$$\Psi[f] = \{ \langle f, \psi_{\gamma} \rangle_H \}_{\gamma \in \Gamma}.$$ 

We call $\Psi$ the frame operator associated with the sequence $\{\psi_{\gamma}\}_{\gamma \in \Gamma}$, even though $\{\psi_{\gamma}\}_{\gamma \in \Gamma}$ may or may not be a frame (see below).

b) As elsewhere in these notes, we are using $\Gamma$ to represent an arbitrary countable index set. If the sequence $\{\psi_{\gamma}\}$ contains a finite number of elements $n$, then we might take $\Gamma = \{1, \ldots, n\}$ or $\Gamma = \{0, \ldots, n-1\}$ or maybe even some other set of $n$ distinct integers depending
on which indexing scheme is most natural. If \( \{ \psi_\gamma \} \)
contains an infinite number of elements, then we might take \( \Gamma = \mathbb{Z} \)
or \( \Gamma = \{ 0, 1, 2, \ldots \} \) or maybe some other countable set, depending on which indexing scheme is more natural. For these discrete spaces, we will simply write \( \ell_2 \) in place of \( \ell_2(\Gamma) \) or \( \mathbb{C}^N \) or \( \mathbb{R}^N \) when the meaning is clear.

c) When \( H = \mathbb{C}^N \) and \( |\Gamma| = M \) is finite, then \( \Psi \) corresponds to an \( M \times N \) matrix whose rows are the \( \psi_k^* \).

5. a) The adjoint \( \Psi^* \) of the frame operator \( \Psi \) maps sequences of numbers to signals in \( H \) by taking the corresponding superposition of the \( \psi_\gamma; \Psi^*: \ell_2(\Gamma) \to H \) by

\[
\Psi^*[\alpha] = \sum_{\gamma \in \Gamma} \alpha_\gamma \psi_\gamma.
\]

b) We will sometimes refer to \( \Psi \) as the analysis operator and \( \Psi^* \) as the synthesis operator corresponding to \( \{ \psi_\gamma \}_{\gamma \in \Gamma} \).

6. a) If there exist real numbers \( 0 < A \leq B < \infty \) such that

\[
A \| f \|_H^2 \leq \| \Psi[f] \|_{\ell_2}^2 = \sum_{\gamma \in \Gamma} |\langle f, \psi_\gamma \rangle_H|^2 \leq B \| f \|_H^2,
\]

then the \( \{ \psi_\gamma \}_{\gamma \in \Gamma} \) form a frame for \( H \) and \( A \) and \( B \) are called the frame bounds.

b) If the \( \psi_\gamma \) are linearly independent, then \( \{ \psi_\gamma \}_{\gamma \in \Gamma} \) is a Riesz basis for \( H \).

c) If \( \| \psi_\gamma \|_H = 1 \) for all \( \gamma \), then \( A \leq 1 \leq B \).

d) If \( \| \psi_\gamma \|_H = 1 \) for all \( \gamma \) and \( A = B = 1 \), then \( \{ \psi_\gamma \}_{\gamma \in \Gamma} \) is an orthobasis for \( H \).
e) If $A = B$, then $\{\psi_\gamma\}_{\gamma \in \Gamma}$ is a **tight frame** for $H$.

7. a) At the very least, for $\{\psi_\gamma\}_{\gamma \in \Gamma}$ to qualify as a basis or a frame, it must be **complete**. That is, it must be true that

$$\text{if } \langle f, \psi_\gamma \rangle_H = 0 \text{ for all } \gamma \in \Gamma, \text{ then } f = 0.$$ 

In other words, there is no $f \in H$ that is orthogonal to all of the $\psi_\gamma$. An equivalent statement is that span($\{\psi_\gamma\}_{\gamma \in \Gamma}$) is **dense** in $H$; that is, for every $f \in H$ and every $\epsilon > 0$ there exists a $f' \in \text{span}(\{\psi_\gamma\}_{\gamma \in \Gamma})$ such that $\|f - f'\|_H \leq \epsilon$.

b) If $H$ has finite dimension $n$, then any sequence $\{\psi_\gamma\}_{\gamma \in \Gamma}$ is complete if at least $n$ of the $\psi_\gamma$ are linearly independent (so clearly we need $|\Gamma| \geq n$). We also do not have this technicality of “dense” in finite dimensions: it will simply be true that $H = \text{span}(\{\psi_\gamma\}_{\gamma \in \Gamma})$.

c) If $\{\psi_\gamma\}_{\gamma \in \Gamma}$ is not complete, then the lower frame bound $A = 0$. In finite dimensions, the converse is true: if $A = 0$, then $\{\psi_\gamma\}_{\gamma \in \Gamma}$ is not complete. In infinite dimensions, we can have complete $\{\psi_\gamma\}_{\gamma \in \Gamma}$ for which $A = 0$; see the examples below in the notes.

d) The sequence $\{\psi_\gamma\}_{\gamma \in \Gamma}$ is complete if and only if Null($\Psi$) = {0}. This is equivalent to the closure of Range($\Psi^*$) being the entirety of $H$ (i.e. Range($\Psi^*$) is dense in $H$).

8. Notice that

$$\|\Psi[f]\|_{\ell_2}^2 = \langle \Psi[f], \Psi[f] \rangle_{\ell_2} = \langle f, \Psi^*\Psi[f] \rangle_H,$$

where $\Psi^*\Psi : H \rightarrow H$ is a self-adjoint linear operator. We can use this fact to re-write the frame bounds as solutions
to the following optimization programs:

\[
A = \inf \{ \langle f, \Psi \Psi^* f \rangle_H : \| f \|_H = 1 \}
\]

\[
B = \sup \{ \langle f, \Psi \Psi^* f \rangle_H : \| f \|_H = 1 \}.
\]

9. a) Given the sequence \( \{ \psi_\gamma \}_{\gamma \in \Gamma} \) we can form the **Gram matrix** \( \Psi \Psi^* \) with

\[
(\Psi \Psi^*)_{j,k} = \langle \psi_j, \psi_k \rangle_H, \quad j, k \in \Gamma.
\]

b) If \( \{ \psi_\gamma \}_{\gamma \in \Gamma} \) is a finite sequence of length \( N \), then \( \Psi \Psi^* \) is an \( N \times N \) matrix which is Hermitian and positive-semidefinite (i.e. has real eigenvalues that are non-negative).

c) If \( \{ \psi_\gamma \}_{\gamma \in \Gamma} \) is an infinite sequence, then \( \Psi \Psi^* \) is a “matrix” with an infinite number of rows and columns. More precisely, it is a linear operator from \( \ell_2(\Gamma) \) into \( \ell_2(\Gamma) \) whose action on \( \alpha \in \ell_2(\Gamma) \) is given by

\[
(\Psi \Psi^*[\alpha])_j = \sum_{k \in \Gamma} \langle \psi_j, \psi_k \rangle_H \alpha_k.
\]

d) We can relate the Gram matrix to the frame bounds \( A, B \) in the following way. If the sequence \( \{ \psi_k \}_{k \in \Gamma} \) is complete, then

\[
A\| \alpha \|_{\ell_2}^2 \leq \| \Psi^* \alpha \|_H^2 \leq B\| \alpha \|_{\ell_2}^2
\]

for all \( \alpha \in \text{Range}(\Psi) \).

Notice the condition \( \alpha \in \text{Range}(\Psi) \); it cannot be ignored. There are plenty of perfectly good frames both in finite and infinite dimensions for which \( \Psi^* \) has a null space; restricting the above to \( \text{Range}(\Psi) \) only allows us to consider vectors orthogonal to this null space.
e) Other ways to write $\|\Psi^*\alpha\|_H^2$ include
\[
\|\Psi^*\alpha\|_H^2 = \left\| \sum_{\gamma \in \Gamma} \alpha_\gamma \psi_\gamma \right\|_H^2 = \langle \Psi^*\alpha, \Psi^*\alpha \rangle_H = \langle \alpha, \Psi\Psi^*\alpha \rangle_{\ell_2},
\]
and so if $\{\psi_\gamma\}_{\gamma \in \Gamma}$ is complete,
\[
A = \inf \{ \langle \alpha, \Psi\Psi^*\alpha \rangle_{\ell_2} : \alpha \in \text{Range}(\Psi), \|\alpha\|_{\ell_2} = 1 \},
B = \sup \{ \langle \alpha, \Psi\Psi^*\alpha \rangle_{\ell_2} : \alpha \in \text{Range}(\Psi), \|\alpha\|_{\ell_2} = 1 \}.
\]
Actually, we do not need the restriction $\alpha \in \text{Range}(\Psi)$ for $B$; we can take the supremum over all $\alpha \in \ell_2(\Gamma)$ that have unit norm (why?).

10. a) If the $\{\psi_\gamma\}_{\gamma \in \Gamma}$ form a frame for $H$, we can recover any $f \in H$ from its expansion coefficients $\Psi[f] = \{ \langle f, \psi_\gamma \rangle_H \}_{\gamma \in \Gamma}$. It is a fact that if the lower frame bound $A > 0$, then the self-adjoint linear operator $\Psi^*\Psi$ is invertible. We define the dual frame $\{\tilde{\psi}_\gamma\}_{\gamma \in \Gamma}$ as
\[
\tilde{\psi}_\gamma = (\Psi^*\Psi)^{-1}\psi_\gamma.
\]
b) The sequence $\{\tilde{\psi}_\gamma\}_{\gamma \in \Gamma}$ is also a frame for $H$. The associated frame operator is denoted $\tilde{\Psi}$. For all $f \in H$,
\[
\frac{1}{B} \|f\|_H^2 \leq \|\tilde{\Psi}[f]\|_{\ell_2}^2 = \sum_{\gamma \in \Gamma} |\langle f, \tilde{\psi}_\gamma \rangle_H|^2 \leq \frac{1}{A} \|f\|_H^2.
\]
c) We can re-write the expression $f = (\Psi^*\Psi)^{-1}\Psi^*\Psi[f]$ as the following reproducing formula:
\[
f = \sum_{\gamma \in \Gamma} \langle f, \psi_\gamma \rangle_H \tilde{\psi}_\gamma.
\]
We can also switch the roles of the primal and dual frames:

\[ f = \sum_{\gamma \in \Gamma} \langle f, \tilde{\psi}_\gamma \rangle_H \psi_\gamma. \]

In the first reproducing formula above, \( \{\psi_\gamma\}_{\gamma \in \Gamma} \) is the **analysis frame** and \( \{\tilde{\psi}_\gamma\}_{\gamma \in \Gamma} \) is the **synthesis frame**.

In the second reproducing formula, \( \{\tilde{\psi}_\gamma\}_{\gamma \in \Gamma} \) is the analysis frame while \( \{\psi_\gamma\}_{\gamma \in \Gamma} \) is the synthesis frame.

d) If the \( \{\psi_\gamma\}_{\gamma \in \Gamma} \) are a Riesz basis, then \( \Psi \) itself is invertible, and \( \tilde{\psi}_\gamma = \Psi^{-1} \delta_\gamma \), where \( \delta_\gamma \) is a sequence of numbers in \( \ell_2(\Gamma) \) with \( \delta_\gamma[n] = 1 \) if \( n = \gamma \) and is zero elsewhere. In this case, it is easy to check that

\[ \langle \psi_k, \tilde{\psi}_j \rangle_H = \begin{cases} 1 & k = j \\ 0 & \text{otherwise} \end{cases}. \]

Together, the sequences \( \{\psi_\gamma\}_{\gamma \in \Gamma} \) and \( \{\tilde{\psi}_\gamma\}_{\gamma \in \Gamma} \) are called **biorthogonal bases**.

e) If the \( \{\psi_\gamma\}_{\gamma \in \Gamma} \) are a tight frame \((A = B)\), then \( \Psi^*\Psi = AI \) where \( I \) is the identity operator on \( H \). Then \( \tilde{\psi}_\gamma = \frac{1}{A} \psi_\gamma \), and the reproducing formula becomes

\[ f = \frac{1}{A} \sum_{\gamma \in \Gamma} \langle f, \phi_\gamma \rangle_H \phi_\gamma. \]

f) Notice that a tight frame analysis operator preserves the inner product (to within a scaling) as well as the norm:

\[ \langle \Psi[f], \Psi[g]\rangle_{\ell_2} = \langle f, \Psi^*\Psi[g]\rangle_H = A \langle f, g\rangle_H \quad \text{for all } f, g \in H. \]
11. A word of caution: while it is true for orthonormal bases that $\Psi\Psi^* = A I$, this is not true for tight frames in general. For tight frames that are not orthonormal bases, the synthesis operator $\frac{1}{A}\Psi^*$ has a non-trivial null space, and so norms and inner products in coefficient space are not necessarily preserved after synthesis. That is

$$\|\alpha\|_{\ell_2}^2 \neq A \|\Psi^*\alpha\|_H^2 \quad \text{for} \quad \alpha \notin \text{Range}(\Psi)$$
$$\langle \alpha, \beta \rangle_{\ell_2} \neq A \langle \Psi^*\alpha, \Psi^*\beta \rangle_H \quad \text{for general} \quad \alpha, \beta \notin \text{Range}(\Psi).$$

Notes by J. Romberg – January 8, 2012 – 16:46