Notes on General Frame Operators in Infinite Dimensions

1. a) A mapping $L : H \to G$ from a Hilbert space H into a Hilbert space G is called a **linear operator** if for all $\alpha, \beta \in \mathbb{C}$ and $f, g \in H$

$$L[\alpha f + \beta g] = \alpha L[f] + \beta L[g].$$

   b) The **operator norm** of L is defined as

$$\|L\| := \sup \{ \|L[f]\|_G : f \in H, \ \|f\|_H = 1 \},$$

   where $\|\cdot\|_H$ is the norm induced by the inner product $\langle \cdot, \cdot \rangle_H$ on H and similarly for $\|\cdot\|_G$.

   c) If $H = \mathbb{C}^N$ and $G = \mathbb{C}^M$, both equipped with the standard Euclidean inner product, then L can be represented by an $M \times N$ matrix and $\|L\|$ is the maximum singular value of L. (In a slight abuse of notation, in the finite case we use L to denote both the linear operator and the matrix which captures the action of this linear operator. This should not cause any confusion, though.)

2. a) The **adjoint** of a linear operator L is the unique linear operator $L^* : G \to H$ such that

$$\langle L[f], y \rangle_G = \langle f, L^*[y] \rangle_H$$

   for all $f \in H$ and $y \in G$. Proof that such a linear operator exists and is unique can be found in any text on functional analysis (see, for example Chapter 7 of N. Young's "Introduction to Hilbert Space").

b) If $H = \mathbb{C}^N$ and $G = \mathbb{C}^M$, both equipped with the standard Euclidean inner product, then L^* is represented by the conjugate transpose of L.

c) For general Hilbert spaces, it is always true that $(L^*)^* = L$ and that $\|L^*\| = \|L\|$. We call $L : H \to H$ **self-adjoint** or **Hermitian** if $L = L^*$ (for $H = \mathbb{C}^n$, this means the matrix is conjugate-symmetric).

3. a) We can define the range and the null space of a linear operator in much the same way as we do for matrices in finite dimensions:

$$\begin{aligned} \text{Null}(L) &= \{f \in H : \ L[f] = 0\} \\ \text{Range}(L) &= \{y \in G : \ y = L[f] \text{ for some } f \in H\}. \end{aligned}$$

b) As in finite dimensions:

$$\begin{aligned} &\text{if } y \in \text{Range}(L) \text{ and } v \in \text{Null}(L^*), \text{ then } \langle y, v\rangle_G = 0; \\ &\text{if } f \in \text{Range}(L^*) \text{ and } g \in \text{Null}(L), \text{ then } \langle f, g\rangle_H = 0. \end{aligned}$$

4. a) Given a sequence of signals $\{\psi_\gamma\}_{\gamma\in\Gamma}$ in a Hilbert space H, we can define the linear operator $\Psi : H \to \ell_2(\Gamma)$ by

$$\Psi[f] = \{\langle f, \psi_\gamma\rangle_H\}_{\gamma\in\Gamma}.$$

We call Ψ the **frame operator** associated with the sequence $\{\psi_\gamma\}_{\gamma\in\Gamma}$, even though $\{\psi_\gamma\}_{\gamma\in\Gamma}$ may or may not be a frame (see below).

b) As elsewhere in these notes, we are using Γ to represent an arbitrary countable index set. If the sequence $\{\psi_\gamma\}$ contains a finite number of elements n, then we might take $\Gamma = \{1, \ldots, n\}$ or $\Gamma = \{0, \ldots, n-1\}$ or maybe even some other set of n distinct integers depending on which indexing scheme is most natural. If $\{\psi_\gamma\}$ contains an infinite number of elements, then we might take $\Gamma = \mathbb{Z}$ or $\Gamma = \{0, 1, 2, \ldots\}$ or maybe some other countable set, depending on which indexing scheme is more natural. For these discrete spaces, we will simply write ℓ_2 in place of $\ell_2(\Gamma)$ or $\mathbb{C}^N$ or $\mathbb{R}^N$ when the meaning is clear.

c) When $H = \mathbb{C}^N$ and $|\Gamma| = M$ is finite, then Ψ corresponds to an $M \times N$ matrix whose rows are the ψ_k^*.

5. a) The adjoint Ψ^* of the frame operator Ψ maps sequences of numbers to signals in H by taking the corresponding superposition of the ψ_γ; $\Psi^* : \ell_2(\Gamma) \to H$ by

$$\Psi^*[\alpha] = \sum_{\gamma \in \Gamma} \alpha_\gamma \psi_\gamma.$$

b) We will sometimes refer to Ψ as the **analysis operator** and Ψ^* as the **synthesis operator** corresponding to $\{\psi_\gamma\}_{\gamma \in \Gamma}$.

6. a) If there exist real numbers $0 < A \leq B < \infty$ such that

$$A\|f\|_H^2 \;\leq\; \|\Psi[f]\|_{\ell_2}^2 = \sum_{\gamma \in \Gamma} |\langle f, \psi_\gamma \rangle_H|^2 \;\leq\; B\|f\|_H^2,$$

then the $\{\psi_\gamma\}_{\gamma \in \Gamma}$ form a **frame** for H and A and B are called the **frame bounds**.

b) If the ψ_γ are linearly independent, then $\{\psi_\gamma\}_{\gamma \in \Gamma}$ is a **Riesz basis** for H.

c) If $\|\psi_\gamma\|_H = 1$ for all γ, then $A \leq 1 \leq B$.

d) If $\|\psi_\gamma\|_H = 1$ for all γ and $A = B = 1$, then $\{\psi_\gamma\}_{\gamma \in \Gamma}$ is an **orthobasis** for H.

e) If $A = B$, then $\{\psi_\gamma\}_{\gamma\in\Gamma}$ is a **tight frame** for H.

7. a) At the very least, for $\{\psi_\gamma\}_{\gamma\in\Gamma}$ to qualify as a basis or a frame, it must be **complete**. That is, it must be true that

$$\text{if } \langle f, \psi_\gamma\rangle_H = 0 \text{ for all } \gamma \in \Gamma, \text{ then } f = 0.$$

In other words, there is no $f \in H$ that is orthogonal to all of the ψ_γ. An equivalent statement is that $\text{span}(\{\psi_\gamma\}_{\gamma\in\Gamma})$ is **dense** in H; that is, for every $f \in H$ and every $\epsilon > 0$ there exists a $f' \in \text{span}(\{\psi_\gamma\}_{\gamma\in\Gamma})$ such that $\|f - f'\|_H \le \epsilon$.

b) If H has finite dimension n, then any sequence $\{\psi_\gamma\}_{\gamma\in\Gamma}$ is complete if at least n of the ψ_γ are linearly independent (so clearly we need $|\Gamma| \ge n$). We also do not have this technicality of "dense" in finite dimensions: it will simply be true that $H = \text{span}(\{\psi_\gamma\}_{\gamma\in\Gamma})$.

c) If $\{\psi_\gamma\}_{\gamma\in\Gamma}$ is not complete, then the lower frame bound $A = 0$. In finite dimensions, the converse is true: if $A = 0$, then $\{\psi_\gamma\}_{\gamma\in\Gamma}$ is not complete. In infinite dimensions, we can have complete $\{\psi_\gamma\}_{\gamma\in\Gamma}$ for which $A = 0$; see the examples below in the notes.

d) The sequence $\{\psi_\gamma\}_{\gamma\in\Gamma}$ is complete if and only if $\text{Null}(\Psi) = \{0\}$. This is equivalent to the closure of $\text{Range}(\Psi^*)$ being the entirety of H (i.e. $\text{Range}(\Psi^*)$ is dense in H).

8. Notice that

$$\|\Psi[f]\|_{\ell_2}^2 = \langle \Psi[f], \Psi[f]\rangle_{\ell_2} = \langle f, \Psi^*\Psi[f]\rangle_H,$$

where $\Psi^*\Psi : H \to H$ is a self-adjoint linear operator. We can use this fact to re-write the frame bounds as solutions to the following optimization programs:

$$A = \inf\{\ \langle f, \Psi^*\Psi[f]\rangle_H :\ \|f\|_H = 1\}$$
$$B = \sup\{\ \langle f, \Psi^*\Psi[f]\rangle_H :\ \|f\|_H = 1\}.$$

9. a) Given the sequence $\{\psi_\gamma\}_{\gamma\in\Gamma}$ we can form the **Gram matrix** $\Psi\Psi^*$ with

$$(\Psi\Psi^*)_{j,k} = \langle \psi_j, \psi_k\rangle_H, \quad j,k \in \Gamma.$$

b) If $\{\psi_\gamma\}_{\gamma\in\Gamma}$ is a finite sequence of length N, then $\Psi\Psi^*$ is an $N \times N$ matrix which is Hermitian and positive-semidefinite (i.e. has real eigenvalues that are non-negative).

c) If $\{\psi_\gamma\}_{\gamma\in\Gamma}$ is an infinite sequence, then $\Psi\Psi^*$ is a "matrix" with an infinite number of rows and columns. More precisely, it is a linear operator from $\ell_2(\Gamma)$ into $\ell_2(\Gamma)$ whose action on $\alpha \in \ell_2(\Gamma)$ is given by

$$(\Psi\Psi^*[\alpha])_j = \sum_{k\in\Gamma} \langle \psi_j, \psi_k\rangle_H \alpha_k.$$

d) We can relate the Gram matrix to the frame bounds A, B in the following way. If the sequence $\{\psi_k\}_{k\in\Gamma}$ is complete, then

$$A\|\alpha\|_{\ell_2}^2 \ \le\ \|\Psi^*\alpha\|_H^2 \ \le\ B\|\alpha\|_{\ell_2}^2 \quad \text{for all} \quad \alpha \in \text{Range}(\Psi).$$

Notice the condition $\alpha \in \text{Range}(\Psi)$; it cannot be ignored. There are plenty of perfectly good frames both in finite and infinite dimensions for which Ψ^* has a null space; restricting the above to Range(Ψ) only allows us to consider vectors orthogonal to this null space.

e) Other ways to write $\|\Psi^*\alpha\|_H^2$ include

$$\|\Psi^*\alpha\|_H^2 \;=\; \left\|\sum_{\gamma\in\Gamma}\alpha_\gamma\psi_\gamma\right\|_H^2 \;=\; \langle\Psi^*\alpha,\Psi^*\alpha\rangle_H \;=\; \langle\alpha,\Psi\Psi^*\alpha\rangle_{\ell_2},$$

and so if $\{\psi_\gamma\}_{\gamma\in\Gamma}$ is complete,

$$\begin{aligned} A &= \inf\{\ \langle\alpha,\Psi\Psi^*\alpha\rangle_{\ell_2}\ : \alpha\in\text{Range}(\Psi),\ \|\alpha\|_{\ell_2}=1\}, \\ B &= \sup\{\ \langle\alpha,\Psi\Psi^*\alpha\rangle_{\ell_2}\ : \alpha\in\text{Range}(\Psi),\ \|\alpha\|_{\ell_2}=1\}. \end{aligned}$$

Actually, we do not need the restriction $\alpha\in\text{Range}(\Psi)$ for B; we can take the supremum over all $\alpha\in\ell_2(\Gamma)$ that have unit norm (why?).

10. a) If the $\{\psi_\gamma\}_{\gamma\in\Gamma}$ form a frame for H, we can recover any $f\in H$ from its expansion coefficients $\Psi[f] = \{\langle f,\psi_\gamma\rangle_H\}_{\gamma\in\Gamma}$. It is a fact that if the lower frame bound $A>0$, then the self-adjoint linear operator $\Psi^*\Psi$ is invertible. We define the **dual frame** $\{\tilde{\psi}_\gamma\}_{\gamma\in\Gamma}$ as

$$\tilde{\psi}_\gamma = (\Psi^*\Psi)^{-1}\psi_\gamma.$$

b) The sequence $\{\tilde{\psi}_\gamma\}_{\gamma\in\Gamma}$ is also a frame for H. The associated frame operator is denoted $\tilde{\Psi}$. For all $f\in H$,

$$\frac{1}{B}\|f\|_H^2 \;\le\; \|\tilde{\Psi}[f]\|_{\ell_2}^2 = \sum_{\gamma\in\Gamma}|\langle f,\tilde{\psi}_\gamma\rangle_H|^2 \;\le\; \frac{1}{A}\|f\|_H^2.$$

c) We can re-write the expression $f=(\Psi^*\Psi)^{-1}\Psi^*\Psi[f]$ as the following **reproducing formula**:

$$f = \sum_{\gamma\in\Gamma}\langle f,\psi_\gamma\rangle_H\,\tilde{\psi}_\gamma.$$

We can also switch the roles of the primal and dual frames:

$$f = \sum_{\gamma\in\Gamma} \langle f, \tilde{\psi}_\gamma \rangle_H \, \psi_\gamma.$$

In the first reproducing formula above, $\{\psi_\gamma\}_{\gamma\in\Gamma}$ is the **analysis frame** and $\{\tilde{\psi}_\gamma\}_{\gamma\in\Gamma}$ is the **synthesis frame**. In the second reproducing formula, $\{\tilde{\psi}_\gamma\}_{\gamma\in\Gamma}$ is the analysis frame while $\{\psi_\gamma\}_{\gamma\in\Gamma}$ is the synthesis frame.

d) If the $\{\psi_\gamma\}_{\gamma\in\Gamma}$ are a Riesz basis, then Ψ itself is invertible, and $\tilde{\psi}_\gamma = \Psi^{-1}\delta^\gamma$, where δ^γ is a sequence of numbers in $\ell_2(\Gamma)$ with $\delta^\gamma[n] = 1$ if $n = \gamma$ and is zero elsewhere. In this case, it is easy to check that

$$\langle \psi_k, \tilde{\psi}_j \rangle_H = \begin{cases} 1 & k = j \\ 0 & \text{otherwise} \end{cases}.$$

Together, the sequences $\{\psi_\gamma\}_{\gamma\in\Gamma}$ and $\{\tilde{\psi}_\gamma\}_{\gamma\in\Gamma}$ are called **biorthogonal bases**.

e) If the $\{\psi_\gamma\}_{\gamma\in\Gamma}$ are a tight frame ($A = B$), then $\Psi^*\Psi = AI$ where I is the identity operator on H. Then $\tilde{\psi}_\gamma = \frac{1}{A}\psi_\gamma$, and the reproducing formula becomes

$$f = \frac{1}{A}\sum_{\gamma\in\Gamma} \langle f, \phi_\gamma \rangle_H \, \phi_\gamma.$$

f) Notice that a tight frame analysis operator preserves the inner product (to within a scaling) as well as the norm:

$$\langle \Psi[f], \Psi[g] \rangle_{\ell_2} = \langle f, \Psi^*\Psi[g] \rangle_H = A\langle f, g \rangle_H \quad \text{for all} \quad f, g \in H.$$

11. A word of caution: while it is true for orthobases that $\Psi\Psi^* = AI$, this is **not true for tight frames in general**. For tight frames that are not orthobases, the synthesis operator $\frac{1}{A}\Psi^*$ has a non-trivial null space, and so norms and inner products in coefficient space are not necessarily preserved after synthesis. That is

$$
\begin{aligned}
\|\alpha\|_{\ell_2}^2 &\neq A\|\Psi^*\alpha\|_H^2 \quad \text{for} \;\; \alpha \notin \text{Range}(\Psi) \\
\langle \alpha, \beta \rangle_{\ell_2} &\neq A\langle \Psi^*\alpha, \Psi^*\beta \rangle_H \quad \text{for general} \;\; \alpha, \beta \notin \text{Range}(\Psi).
\end{aligned}
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