A Survey of Compressive Sensing and Applications

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Signal processing trends

DSP: sample first, ask questions later

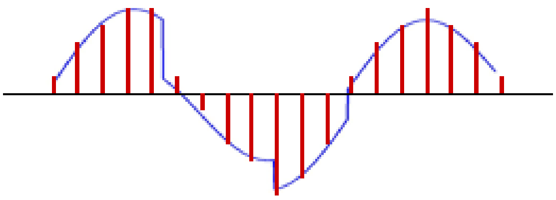
Explosion in sensor technology/ubiquity has caused two trends:

- Physical capabilities of hardware are being stressed, increasing speed/resolution becoming *expensive*
  - gigahertz+ analog-to-digital conversion
  - accelerated MRI
  - industrial imaging
- Deluge of data
  - camera arrays and networks, multi-view target databases, streaming video...

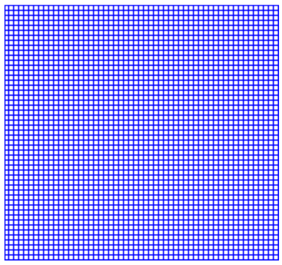
Compressive Sensing: sample smarter, not faster

Classical data acquisition

- *Shannon-Nyquist sampling theorem* (Fundamental Theorem of DSP): "if you sample at twice the bandwidth, you can perfectly reconstruct the data"



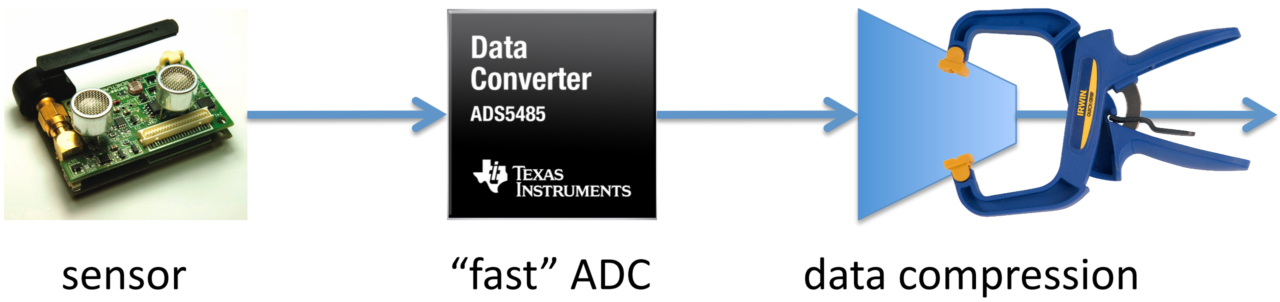
time



space

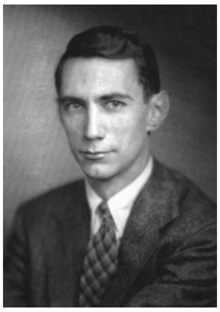
- Counterpart for "indirect imaging" (MRI, radar): *Resolution is determined by bandwidth*

Sense, sample, process...



Compressive sensing (CS)

- Shannon/Nyquist theorem is *pessimistic*
  - 2×bandwidth is the worst-case sampling rate — holds uniformly for *any* bandlimited data
  - sparsity/compressibility is irrelevant
  - Shannon sampling based on a linear model, compression based on a nonlinear model

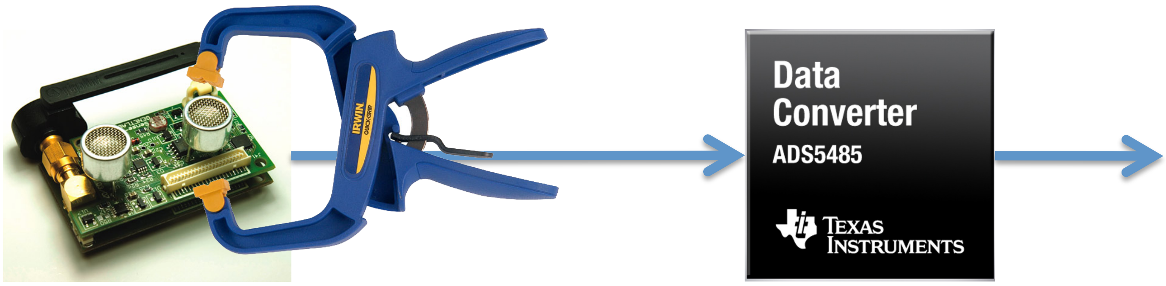


Shannon

- Compressive sensing
  - new sampling theory that *leverages compressibility*
  - key roles played by new *uncertainty principles* and *randomness*

Heisenberg

Compressive sensing



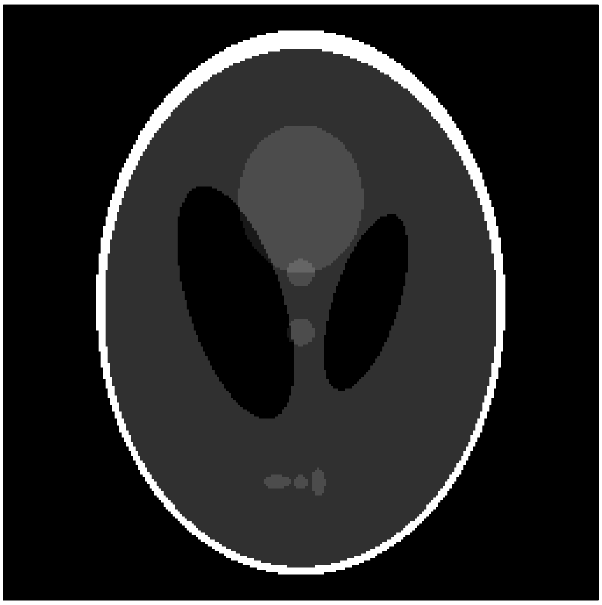
“compressive” sensor

“slow” ADC

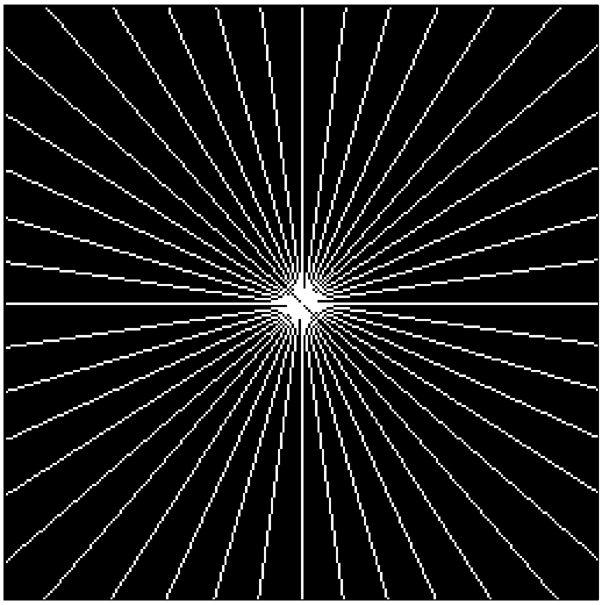
- Essential idea:
  “pre-coding” the signal in analog makes it “easier” to acquire
- Reduce power consumption, hardware complexity, acquisition time

A simple underdetermined inverse problem

Observe a subset Ω of the 2D discrete Fourier plane



phantom (hidden)



white star = sample locations

$N := 512^2 = 262,144$ pixel image

observations on 22 radial lines, $10,486$ samples, $\approx 4\%$ coverage

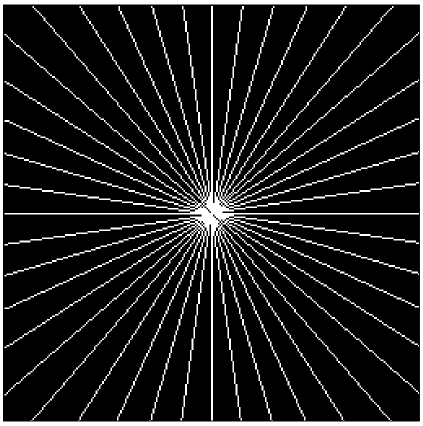
Minimum energy reconstruction

Reconstruct g^* with

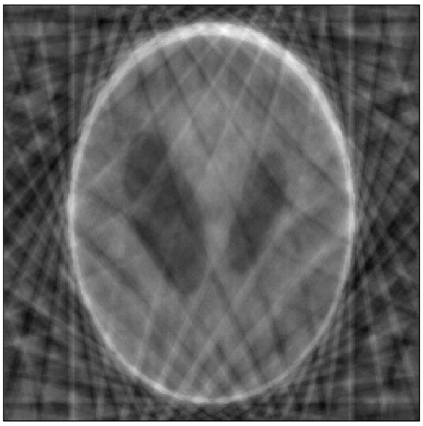
$$\hat{g}^*(\omega_1, \omega_2) = \begin{cases} \hat{f}(\omega_1, \omega_2) & (\omega_1, \omega_2) \in \Omega \\ 0 & (\omega_1, \omega_2) \notin \Omega \end{cases}$$

Set unknown Fourier coeffs to zero, and inverse transform

original



Fourier samples



g^*

Total-variation reconstruction

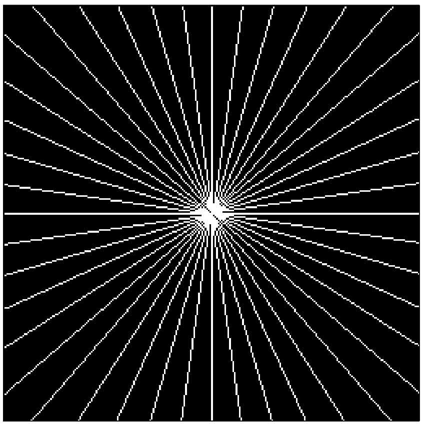
Find an image that

- Fourier domain: *matches observations*
- Spatial domain: has a *minimal amount of oscillation*

Reconstruct g^* by solving:

$$\min_g \sum_{i,j} |(\nabla g)_{i,j}| \quad \text{s.t.} \quad \hat{g}(\omega_1, \omega_2) = \hat{f}(\omega_1, \omega_2), \quad (\omega_1, \omega_2) \in \Omega$$

original



Fourier samples

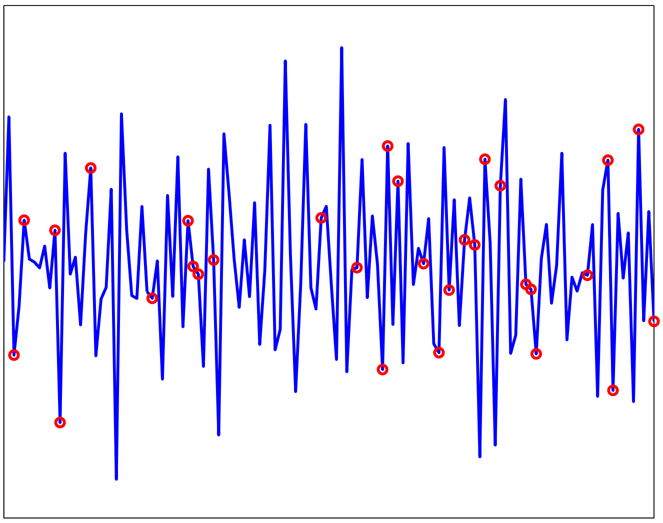
g^* = original

perfect reconstruction

Sampling a superposition of sinusoids

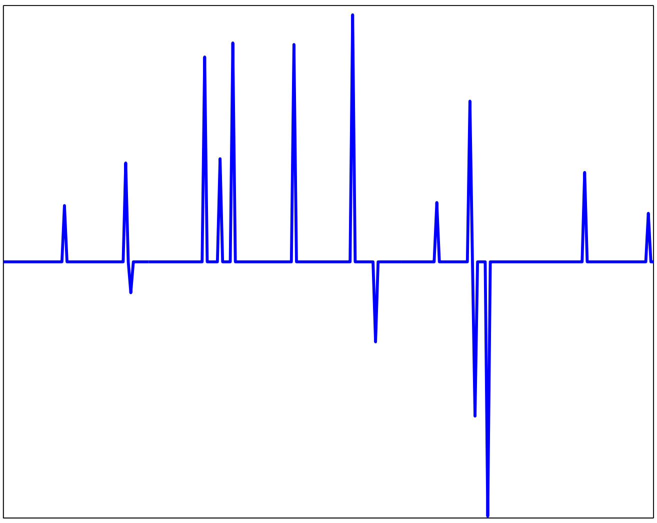
We take M samples of a superposition of S sinusoids:

Time domain $x_0(t)$



Measure M samples
(red circles = samples)

Frequency domain $\hat{x}_0(\omega)$

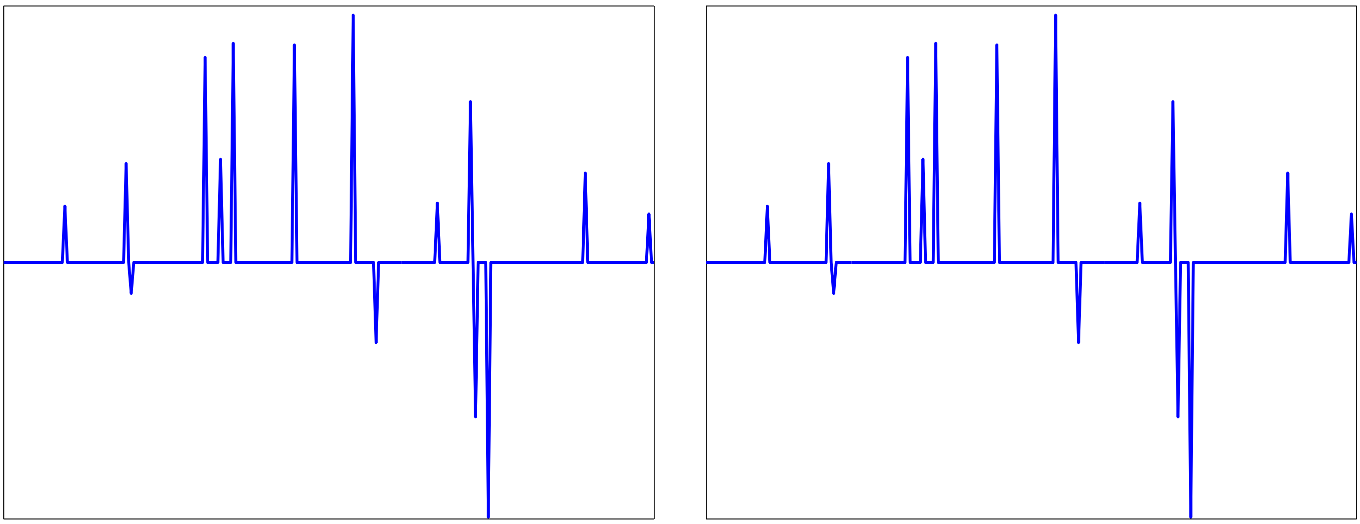


S nonzero components

Sampling a superposition of sinusoids

Reconstruct by solving

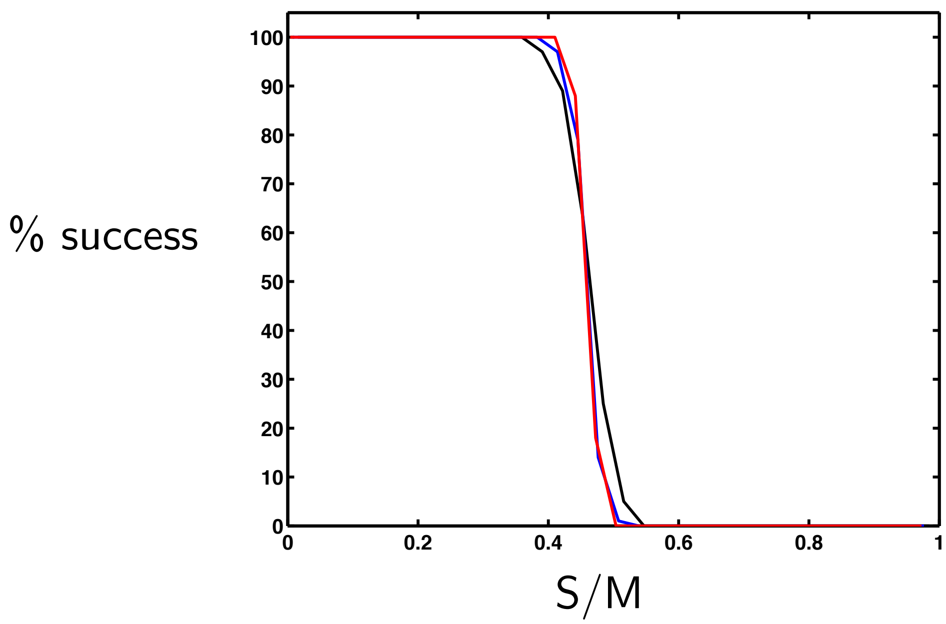
$$\min_{x} \ \|\hat{x}\|_{\ell_1} \quad \text{subject to} \quad x(t_m) = x_0(t_m), \quad m = 1, \ldots, M$$



original $\hat{x}_0$, $S = 15$ — *perfect* recovery from 30 samples

Numerical recovery curves

- Resolutions $N = 256, 512, 1024$ (black, blue, red)
- Signal composed of S randomly selected sinusoids
- Sample at M randomly selected locations



- In practice, perfect recovery occurs when $M \approx 2S$ for $N \approx 1000$

A nonlinear sampling theorem

Exact Recovery Theorem (Candès, R, Tao, 2004):

- Unknown $\hat{x}_0$ is supported on set of size S
- Select M sample locations $\{t_m\}$ "at random" with

$$M \;\geq\; \text{Const} \cdot S \log N$$

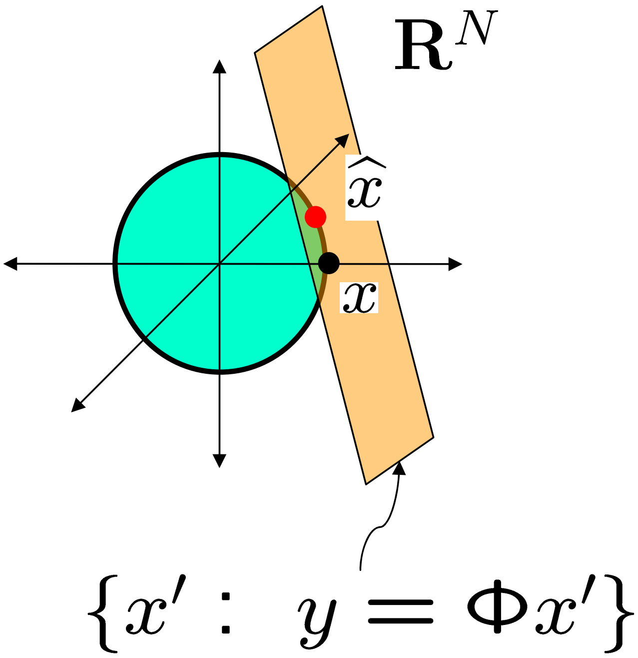
- Take time-domain samples (measurements) $y_m = x_0(t_m)$
- Solve

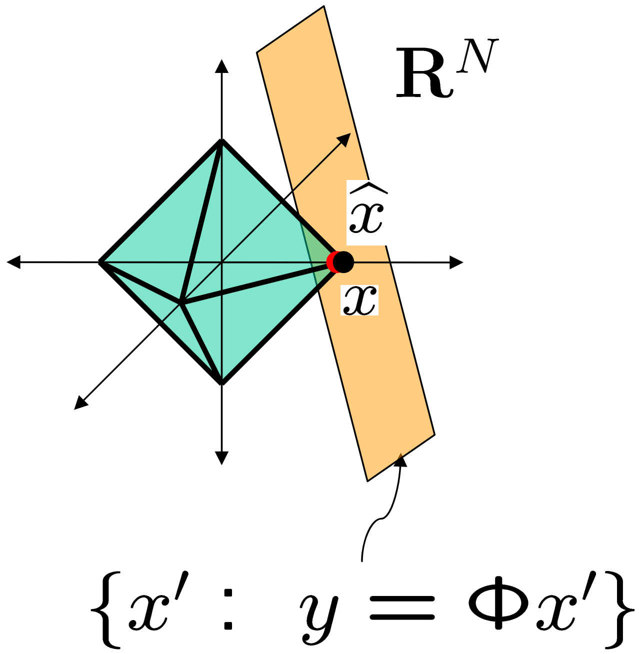
$$\min_x \|\hat{x}\|_{\ell_1} \quad \text{subject to} \quad x(t_m) = y_m, \quad m = 1, \ldots, M$$

- Solution is *exactly* f with extremely high probability
- In total-variation/phantom example, S=number of jumps

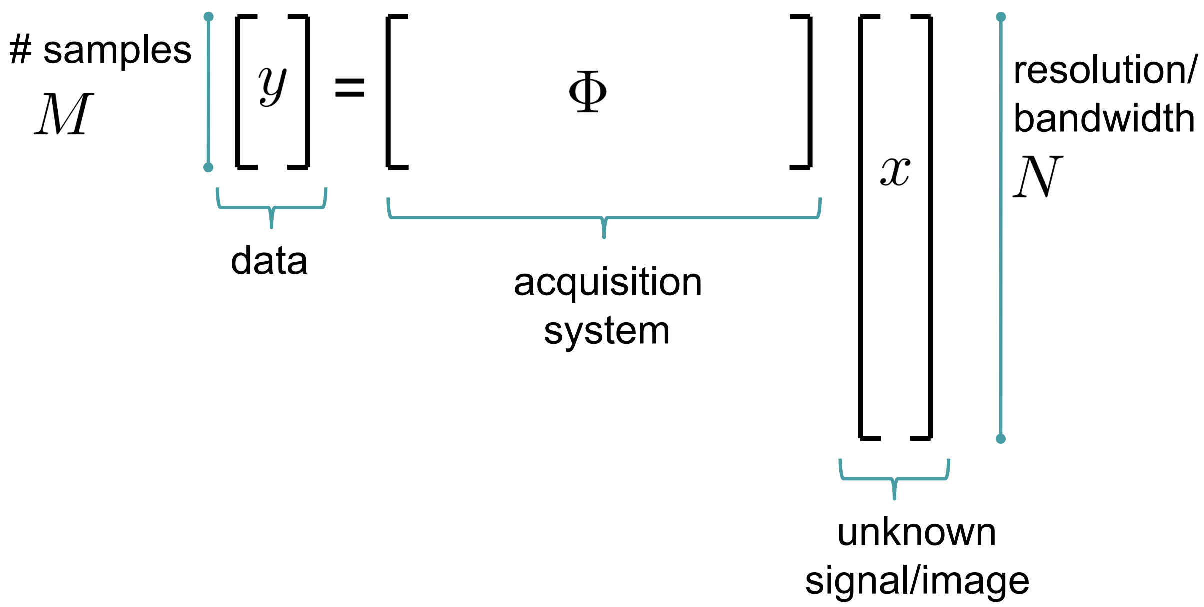
Graphical intuition for ℓ_1

$\min_x \|x\|_2$ s.t. $\Phi x = y$ $\qquad$ $\min_x \|x\|_1$ s.t. $\Phi x = y$





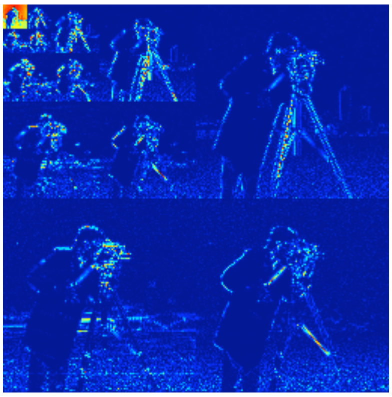
Acquisition as linear algebra



- Small number of samples $=$ underdetermined system
  Impossible to solve in general
- If x is *sparse* and Φ is *diverse*, then these systems can be "inverted"

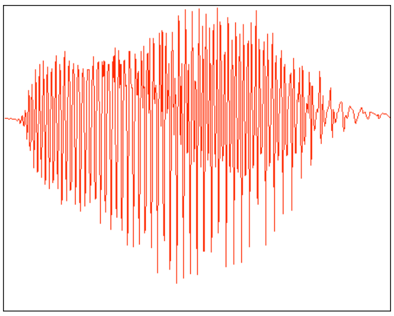
Sparsity/Compressibility

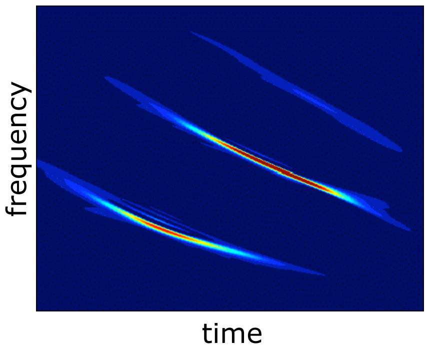
N
pixels



$S \ll N$
large
wavelet
coefficients

N
wideband
signal
samples





$S \ll N$
large
Gabor
coefficients

Wavelet approximation

Take 1% of *largest* coefficients, set the rest to zero (adaptive)

original

approximated

rel. error = 0.031

Linear measurements

- Instead of samples, take *linear measurements* of signal/image x_0

$$y_1 = \langle x_0, \phi_1 \rangle, \quad y_2 = \langle x_0, \phi_2 \rangle, \quad \ldots, y_M = \langle x_0, \phi_K \rangle$$

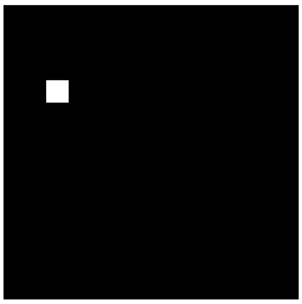
$$y = \Phi x_0$$

- Equivalent to transform-domain sampling,
  $\{\phi_m\}$ = basis functions

- Example: pixels

$$y_m = \Bigg\langle$$

,



$$\Bigg\rangle$$

Linear measurements

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- Example: line integrals (tomography)

$y_m = \Big\langle$

,

$\Big\rangle$

Linear measurements

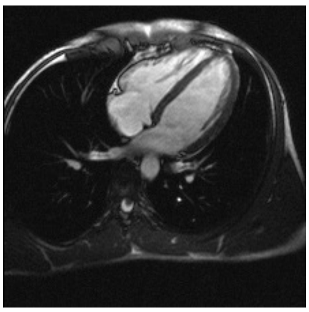
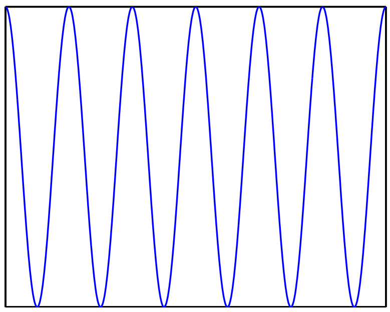
- Instead of samples, take *linear measurements* of signal/image x_0

$$y_1 = \langle x_0, \phi_1 \rangle, \quad y_2 = \langle x_0, \phi_2 \rangle, \quad \ldots, y_M = \langle x_0, \phi_K \rangle$$

$$y = \Phi x_0$$

- Equivalent to transform-domain sampling, $\{\phi_m\}$ = basis functions

- Example: sinusoids (MRI)

$y_m = \Big\langle$  ,  $\Big\rangle$