Notation

We will use $L_2(\mathbb{R})$ to denote the space of finite-energy signals:

$$x(t) \in L_2(\mathbb{R}) \quad \Leftrightarrow \quad \int_{-\infty}^{\infty} |x(t)|^2 \ \mathrm{d}t < \infty.$$

Similarly, $L_2([a,b])$ is the space of finite-energy signals that are either supported on (i.e. zero outside of) $[a,b]$ or are periodic with period $b-a$:

$$x(t) \in L_2([a,b]) \quad \Leftrightarrow \quad \int_{a}^{b} |x(t)|^2 \ \mathrm{d}t < \infty.$$

We will use $\ell_2(\mathbb{Z})$ to denote the space of discrete signals which are square-summable:

$$x[n] \in \ell_2 \quad \Leftrightarrow \quad \sum_{n=-\infty}^{\infty} |x[n]|^2 < \infty.$$

The space of discrete signals which have finite length N are simply vectors in $\mathbb{R}^N$.

We will encounter many different discrete spaces in our travels, and it is sometime convenient to index them in different ways (although the index set will always can always be put in 1-to-1 correspondence with the integers). We will often use the general index set Γ for discrete spaces, and the corresponding set of square-summable sequences is $\ell_2(\Gamma)$.

We will occasionally use the term **Hilbert space** to mean a collection of signals that is closed under addition, and for which we have an inner product defined. In this course, we are interested only in the "standard inner products" defined as follows. If x and y are continuous-time signals, then

$$\langle x, y \rangle = \int x(t) y(t)^* \ \mathrm{d}t,$$

where here $y^*(t)$ is the complex-conjugate of $y(t)$. If $x[n]$ and $y[n]$ are discrete signals indexed by Γ, then

$$\langle x, y\rangle = \sum_{n\in\Gamma} x[n]y[n]^*.$$

As often as possible, we will treat all four cases above using the unified language of functional analysis and Hilbert spaces. In the finite case, this reduces to basic linear algebra. We will often use $x(t) \in L_2(\mathbb{R})$ as the default case, but it should be recognized that unless explicitly stated the results translate in a straightforward way to the other cases. When we are talking about two different spaces (say they are H and G) with different inner products, we will use $< \cdots, \cdot\rangle_H$ and $\langle\cdot,\cdot\rangle_G$ to distinguish the inner products when necessary.

We will use the generic notation $\|\cdot\|_2$ for the (standard Euclidean) norm induced by the standard inner product. If $x(t)$ is a continuous-time signal, then

$$\|x\|_2^2 = \int |x(t)|^2 \,\mathrm{d}t,$$

and if $x[n]$ is a sequence/vector, then

$$\|x\|_2^2 = \sum_{\gamma\in\Gamma} |x[\gamma]|^2.$$

We will also make extensive use of the ℓ_1 and ℓ_∞ norm of a finite vector over Γ. These norms are defined as

$$\|x\|_1 = \sum_{\gamma\in\Gamma} |x[\gamma]|$$

$$\|x\|_\infty = \max_{\gamma\in\Gamma} |x[\gamma]|$$

When A is a matrix, we will also use A^* for the conjugate-transpose; that is

$$A^*[i,j] = (A[j,i])^*.$$

The **Fourier transform** of a continuos-time signal $x(t)$ is denoted by $\hat{x}(\omega)$, and is given by

$$\hat{x}(\omega) = \int x(t)\, \mathrm{e}^{-\mathrm{j}\omega t}\, \mathrm{d}t.$$

The classical Parseval/Plancherel theorem for the continuous-time Fourier transform tells us that Fourier transform preserves inner products:

$$\langle x(t), y(t)\rangle = \int x(t)y(t)^*\mathrm{d}t = \frac{1}{2\pi}\int \hat{x}(\omega)\hat{y}(\omega)^*\mathrm{d}\omega = \frac{1}{2\pi}\langle \hat{x}(\omega), \hat{y}(\omega)\rangle,$$

and hence also energy:

$$\|x(t)\|_2^2 = \frac{1}{2\pi}\|\hat{x}(\omega)\|_2^2.$$