Chapter 10

Polynomiality of Linear Programming

In the previous section, we presented the Simplex Method. This method turns out to be very efficient for solving linear programmes in practice. While it is known that Gaussian elimination can be implemented in polynomial time, the number of pivot rules used throughout the Simplex Method may be exponential. More precisely, Klee and Minty gave an example of a linear programme such that the Simplex Method goes through each of the 2^n extreme points of the corresponding polytope [5].

In this chapter, we survey two methods for solving linear programmes in polynomial time. On the one hand, the *Ellipsoid Method* [4, 2] is not competitive with the Simplex Method in practice but it has important theoretical side-effects. On the other hand, the *Interior Point Methods* compete with the Simplex Method in practice.

First of all, we define the *input size* of a linear programme. Recall that an integer $i \in \mathbb{Z}$ can be encoded using $< i >= \lceil \log_2(|i|+1) \rceil + 1$ bits. For a rational number $r = p/q \in \mathbb{Q}$, the size of r is $< r >=< p > + < q >$. Similarly, any rational matrix can be encoded using $< A >= \sum_{i=1}^{m} \sum_{j=1}^{n} < a_{i,j} >$ bits. Also, multiplying two integers a and b runs in time $O(< a > + < b >)$.

In what follows, we consider the linear programme L defined by:

$$\begin{array}{rl} \text{Maximize} & \mathbf{c}^T\mathbf{x} \\ \text{Subject to:} & \mathbf{A}\mathbf{x} \leq \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0} \end{array} \qquad (10.1)$$

We restrict ourselves to the case when $\mathbf{A} \in \mathbb{Q}^{m \times n}$, $\mathbf{b} \in \mathbb{Q}^m$ and $\mathbf{c} \in \mathbb{Q}^n$ have rational coefficients. Therefore, the input size of L is $< L >=< A > + < b > + < c >$. In other words, $< L >$ is a polynomial in n, m and $< B >$ where B is the largest coefficient in $\mathbf{A}, \mathbf{b}$ and $\mathbf{c}$.

The two methods presented in this chapter are polynomial in $< L >$. It is a long-standing open problem to know whether linear programming is strongly polynomial, i.e., whether there exists an algorithm that solves a linear programme in time polynomial in n and m.

The interest of the Ellipsoid Method comes from the fact that, in particular cases, it works independently from the number m of constraints. More precisely, if we are given a *separation oracle* that, given a vector $\mathbf{x}$, answers that $\mathbf{x}$ satisfies $\mathbf{A}\mathbf{x} \leq \mathbf{b}$ or returns an inequality not satisfied by $\mathbf{x}$, then the Ellipsoid Method works in time polynomial in n and $< B >$.

10.1 The Ellipsoid Method

The Ellipsoid Method has been proposed in the 70's by Shor, Judin and Nemirovski for solving some non-linear optimization problems. In 1979, Khachyian [4] showed how to use it for solving linear programmes.

10.1.1 Optimization versus feasibility

We first recall (or state) some definitions and results of linear algebra.

A *convex set* $C \subseteq \mathbb{R}^n$ is such that for all $\mathbf{x}, \mathbf{y} \in C$ and for all $0 \leq \lambda \leq 1$, $\lambda \mathbf{x} + (1-\lambda)\mathbf{y} \in C$. An *half space* H is a (convex) set $\{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{c}^T x \leq \delta\}$ with $\mathbf{c} \in \mathbb{R}^n$, $\delta \in \mathbb{R}$. A closed convex set is the intersection of a family of half spaces. A *polyhedron* $\mathcal{K}$ is a closed convex set that is the intersection of a finite family of half spaces, i.e., $\mathcal{K} = \{\mathbf{x} \in \mathbb{R}^n \ : \ \mathbf{Ax} \leq \mathbf{b}\}$, $\mathbf{A} \in \mathbb{R}^{m \times n}$, $\mathbf{b} \in \mathbb{R}^m$. A *polytope* is the convex hull of a finite set $X \subseteq \mathbb{R}^n$, i.e., $\{\mathbf{x} \in \mathbb{R}^n \ : \ \sum \lambda_i \mathbf{x}_i, \ \sum_i \lambda_i \leq 1, \mathbf{x}_i \in X\}$.

Theorem 10.1. *A set is polytope if and only if it is a bounded polyhedron.*

Given $\mathbf{A} \in \mathbb{R}^{m \times n}$ and $\mathbf{b} \in \mathbb{R}^m$, the system of inequalities $\mathbf{Ax} \leq \mathbf{b}$ is *feasible* if there is $\mathbf{x} \in \mathbb{R}^n$ that satisfies it, i.e., if the polyhedron $\{\mathbf{x} \in \mathbb{R}^n \ : \ \mathbf{Ax} \leq \mathbf{b}\}$ is not empty. We say that the system is *bounded* if there is $R \in \mathbb{R}$ such that the set of solutions of the system is included in a ball of radius $\leq R$ in $\mathbb{R}^n$.

The Ellispoid Method aims at deciding whether a polytope is not empty and, if possible, at finding some vector in it. We first show that it is sufficient to solve linear programmes. In other words, the next theorem shows that solving a linear programme can be reduced to the feasibility of a system of linear inequalities.

Theorem 10.2. *If it can be decided in polynomial time whether a system of linear inequalities is feasible, then linear programmes can be solved in polynomial time.*

Proof. Consider the linear programme L described in 10.1. By the Duality Theorem (Theorem 9.10), L admits an optimal solution if and only if the system $\{\mathbf{x} \geq 0, \ \mathbf{y} \geq 0, \ \mathbf{Ax} \leq \mathbf{b}, \mathbf{A^T y} \geq \mathbf{c}, \mathbf{cx} \geq \mathbf{b^T y}\}$ is feasible and bounded, i.e., it is a non-empty polytope. □

Previous Theorem shows that the set $\mathcal{P}$ of optimal vector-solutions of $\{\max \mathbf{c}^T \mathbf{x} \ : \ \mathbf{Ax} \leq \mathbf{b}, \ \mathbf{x} \geq 0\}$ is a polytope (possibly empty) in $\mathbb{R}^n$. Moreover, given $\mathbf{x} \in \mathbb{R}^n$, deciding if $\mathbf{x} \in \mathcal{P}$ can be done in polynomial time. It is sufficient to check alternatively each constraint and if all of them are satisfied, to use the complementary slackness conditions to check the optimality.

Therefore, from now on, we focus on the feasibility of the bounded system

$$\begin{array}{c} \mathbf{Ax} \leq \mathbf{b} \\ \mathbf{A} \in \mathbb{Q}^{m \times n}, \mathbf{b} \in \mathbb{Q}^m \end{array} \tag{10.2}$$

which represents the polytope of the optimal solutions of a linear programme.

10.1.2 The method

In this section, we describe the Ellipsoid Method. Given a polytope $\mathcal{K}$ and $\mathcal{V} \in \mathbb{R}$, this method either returns a vector $\mathbf{x} \in \mathcal{K}$ or states that $\mathcal{K}$ has volume less than $\mathcal{V}$.

Definitions and notations

A matrix $\mathbf{A} \in \mathbb{R}^{m\times n}$ is *positive definite* if and only if $\mathbf{x}^T\mathbf{A}\mathbf{x} > 0$ for any $\mathbf{x} \in \mathbb{R}^n \setminus \{0\}$, or equivalently, $\mathbf{A} = \mathbf{Q}^T diag(\lambda_1, \cdots, \lambda_n)\mathbf{Q}$ where $\mathbf{Q} \in \mathbb{R}^{n\times n}$ is an orthogonal matrix and $\lambda_i > 0$ for any $i \le n$. Another equivalent definition is that $\mathbf{A}$ is positive definite if $\mathbf{A} = \mathbf{B}^T\mathbf{B}$ where $\mathbf{B}$ is triangular with positive diagonal elements.

The *unit ball* $\mathcal{B}(0,1)$ is $\{\mathbf{x} \in \mathbb{R}^n \ : \ ||\mathbf{x}|| \le 1\}$ with volume V_n.

An *ellipsoid*, denoted by $\varepsilon(\mathbf{M},\mathbf{c})$, is the image of $\mathcal{B}(0,1)$ under a linear map $t : \mathbb{R}^n \to \mathbb{R}^n$ such that $t(\mathbf{x}) = \mathbf{M}\mathbf{x} + \mathbf{c}$ where $\mathbf{M} \in \mathbb{R}^{n\times n}$ is an invertible matrix and $\mathbf{c} \in \mathbb{R}^n$. That is, $\varepsilon(\mathbf{M},\mathbf{c}) = \{\mathbf{x} \in \mathbb{R}^n \ : \ ||\mathbf{M}^{-1}\mathbf{x} - \mathbf{M}^{-1}\mathbf{c}|| \le 1\}$. Alternatively, the ellipsoid $\varepsilon(\mathbf{M},\mathbf{c})$ can be defined as $\{\mathbf{x} \in \mathbb{R}^n \ : \ (\mathbf{x}-\mathbf{c})^T\mathbf{N}(\mathbf{x}-\mathbf{c}) \le 1\}$ where $\mathbf{N}$ is the positive definite matrix $(\mathbf{M}^{-1})^T\mathbf{M}^{-1}$.

Proposition 10.3. *The volume $vol(\varepsilon(\mathbf{M},\mathbf{c}))$ of $\varepsilon(\mathbf{M},\mathbf{c})$ is $|det(M)|.V_n$.*

The Ellipsoid Method.

Let $\mathcal{K} = \{\mathbf{x} \in \mathbb{R}^n \ : \ \mathbf{A}\mathbf{x} \le \mathbf{b}\}$ be a polytope with $\mathbf{A} \in \mathbb{R}^{m\times n}$, $\mathbf{b} \in \mathbb{R}^m$, and let $\mathcal{V} \in \mathbb{R}$. Assume we are given $\mathbf{M_0} \in \mathbb{R}^{n\times n}$ and $\mathbf{c_0} \in \mathbb{R}^n$ such that $\mathcal{K} \subseteq \varepsilon_0(\mathbf{M_0},\mathbf{c_0})$. The Ellipsoid Method proceeds as follows.

Algorithm 10.1 (Ellipsoid Method)**.**

1. Set $k := 0$. Note that $\mathcal{K} \subseteq \varepsilon_k(\mathbf{M}_k,\mathbf{c}_k)$;
2. If $vol(\varepsilon_k(\mathbf{M}_k,\mathbf{c}_k)) < \mathcal{V}$, then stop;
3. Otherwise, if $\mathbf{c}_k \in \mathcal{K}$, then return $\mathbf{c}_k$;
4. Else let $\mathbf{a}_i^T\mathbf{x} \le b_i$ be an inequality defining $\mathcal{K}$, i.e., $\mathbf{a}_i^T$ is a row of $\mathbf{A}$, such that $\mathbf{a}_i^T\mathbf{c}_k > b_i$;

   Find an ellipsoid $\varepsilon_{k+1}(\mathbf{M}_{k+1},\mathbf{c}_{k+1})$ with volume at most $\le e^{-\frac{1}{2(n+1)}} \cdot vol(\varepsilon_k(\mathbf{M}_k,\mathbf{c}_k))$ such that
   $$\varepsilon_k(\mathbf{M}_k,\mathbf{c}_k) \cap \{\mathbf{x} \in \mathbb{R}^n \ | \ \mathbf{a}_i^T\mathbf{x} \le \mathbf{a}_i^T\mathbf{c}_k\} \subseteq \varepsilon_{k+1}(\mathbf{M}_{k+1},\mathbf{c}_{k+1});$$
5. $k := k+1$ and go to step 2.

Theorem 10.4. *The Ellipsoid Method computes a point in $\mathcal{K}$ or asserts that $vol(\mathcal{K}) < \mathcal{V}$ in at most $2 \cdot n \cdot \ln \frac{vol(\varepsilon_0(\mathbf{M_0},\mathbf{c_0}))}{\mathcal{V}}$ iterations.*

Proof. After k iterations, $vol(\varepsilon_{k+1}(\mathbf{M}_{k+1},\mathbf{c}_{k+1})/vol(\varepsilon_0(\mathbf{M_0},\mathbf{c_0})) \leq e^{-\frac{k}{2(n+1)}}$. Since we stop as soon as $vol(\varepsilon_{k+1}(\mathbf{M}_{k+1},\mathbf{c}_{k+1}) < \mathcal{V}$, there are at most $2 \cdot n \cdot \ln \frac{vol(\varepsilon_0(\mathbf{M_0},\mathbf{c_0}))}{\mathcal{V}}$ iterations. □

Discussion about the complexity and separation oracle.

Let $\mathcal{K}$ be the rational polytope defined in (10.2). Let B be the largest absolute value of the coefficients of $\mathbf{A}$ and $\mathbf{b}$. We now show that the Ellipsoid Method can be implemented in time polynomial in $< \mathcal{K} >$, i.e., in n,m and B.

Here, we first discuss the main steps of the proof, the technical results needed for this purpose are postponed in next subsection.

(i) First, a lower bound $\mathcal{V}$ on the volume of $\mathcal{K}$ must be computed. Theorem 10.5 defines such a $\mathcal{V}$ in the case $\mathcal{K}$ is full dimensional. Moreover, $\mathcal{V}$ is only exponential in n and B.

Otherwise, there is a first difficulty. Theorem 10.5 shows that there is $t > 0$, such that $\mathcal{K}$ is empty if and only if $\mathcal{K}' = \{\mathbf{x} \in \mathbb{R}^n \ : \ \mathbf{Ax} \leq \mathbf{b}+\mathbf{t}\}$ is empty, where $< t >$ is polynomial in n,B and $\mathbf{t} \in \mathbb{R}^m$ is the vector with all coordinates equal to ε. Moreover, $\mathcal{K}'$ is full dimensional (see Exercise 10.3).

Therefore, the Ellipsoid Method actually applies on the polytope $\mathcal{K}'$. If $\mathcal{K}'$ is empty, then $\mathcal{K}$ is empty as well. Otherwise, the Ellipsoid Method returns $\mathbf{x}' \in \mathcal{K}'$ and the solution $\mathbf{x}'$ is rounded to a solution $\mathbf{x} \in \mathcal{K}$. We do not detail this latter operation in this note.

(ii) Then, an initial ellipsoid $\varepsilon_0(\mathbf{M_0},\mathbf{c_0})$ containing $\mathcal{K}$ is required. Theorem 10.6 describes how to define it in such a way that $< \varepsilon_0(\mathbf{M_0},\mathbf{c_0}) >$ is polynomial in n and B. Moreover, its volume is only exponential in n and B.

(iii) The crucial step of the Ellipsoid Method is Step 4. Theorem 10.8 proves that the desired ellipsoid $\varepsilon_{k+1}(\mathbf{M}_{k+1},\mathbf{c}_{k+1})$ always exists. Moreover, it can be computed from $\varepsilon_k(\mathbf{M}_k,\mathbf{c}_k)$ and the vector $\mathbf{a}_i$, with a number of operations that is polynomial in $< \varepsilon_k(\mathbf{M}_k,\mathbf{c}_k) >$ and $< \mathbf{a}_i >$.

Another technicality appears here. Indeed, following Theorem 10.8, some square-roots are needed when defining $\varepsilon_{k+1}(\mathbf{M}_{k+1},\mathbf{c}_{k+1})$. Therefore, its encoding might not be polynomial in $< \varepsilon_k(\mathbf{M}_k,\mathbf{c}_k) >$ and $< \mathbf{a}_i >$. It is actually possible to ensure that $\varepsilon_{k+1}(\mathbf{M}_{k+1},\mathbf{c}_{k+1})$ satisfies the desired properties and can be encoded polynomially in $< \varepsilon_k(\mathbf{M}_k,\mathbf{c}_k) >$ and $< \mathbf{a}_i >$. We do not give more details in this note.

Since, by (ii), $< \varepsilon_0(\mathbf{M_0},\mathbf{c_0}) >$ is polynomial in n and B, therefore, for any $k > 0$, $< \varepsilon_k(\mathbf{M_k},\mathbf{c_k}) >$ and the computation of the ellipsoid are polynomial in n and B.

(iv) Now, using (iii) and Proposition 10.3, Step 2 is clearly polynomial in n and B because $\mathcal{V}$ and $vol(\varepsilon_0(\mathbf{M_0},\mathbf{c_0}))$ are only exponential in n and B.

(v) Finally, let us consider the following question, that must be dne in Step 3 and 4. Given a system $\mathcal{K} = \{\mathbf{x} \in \mathbb{Q}^n \ : \ \mathbf{Ax} \leq \mathbf{b}\}$ with $\mathbf{A} \in \mathbb{Q}^{m \times n}$, $\mathbf{b} \in \mathbb{Q}^m$ and a vector $\mathbf{c} \in \mathbb{Q}^n$, decide if $c \in \mathcal{K}$ or return an index $i \leq m$ such that $\mathbf{a}_i^T\mathbf{c} > b_i$. Clearly, this can be decided in time polynomial in m, n and B (the largest absolute value of the coefficients) by simply checking the m row inequalities one by one.

The above discussion proves that the Ellipsoid Method runs in time polynomial in m, n and B. Moreover, note that, m appears in the complexity only when solving the question of (v) (Steps 3 and 4). The main interest of the Ellipsoid Method is the following: if we are given an oracle for answering the question of (v) independently of m, then the Ellipsoid Method runs in time polynomial in n and B (independently of m). Furthermore, it is not necessary to explicitly know all constraints (see Exercise 10.4).

10.1.3 Complexity of the Ellipsoid Method

In this section, we formally state (and prove some of) the theorems used in the analysis of the Ellipsoid Method we did above.

In the following, for any $\mathbf{x} \in \mathbb{R}^n$, $x(i)$ denotes the i^{th} coordinate of $\mathbf{x}$.

Initialization: finding the lower bound

The next theorem allows to find a lower bound for the volume of the considered polytope, if it is not empty.

Theorem 10.5. *Let $\mathcal{K} = \{\mathbf{x} \in \mathbb{R}^n \ : \ \mathbf{Ax} \leq \mathbf{b}\}$ be a polytope and let B be the largest absolute value of the coefficients of $\mathbf{A}$ and $\mathbf{b}$.*

- *If $\mathcal{K}$ is full dimensional, then its volume is at least $1/(nB)^{3n^2}$;*
- *Let $t = 1/((n+1).(nB)^n)$ and let $\mathcal{K}' = \{\mathbf{x} \in \mathbb{R}^n \ : \ \mathbf{Ax} \leq \mathbf{b} + \mathbf{t}\}$ where $\mathbf{t}$ is the vector with all components equal to t. Then, $\mathcal{K}'$ is full dimensional and is non-empty if and only if $\mathcal{K}$ is non-empty.*

Proof. We sketch the proof.

If $\mathcal{K}$ is full dimensional, let us consider $v_0, \cdots, v_n \in \mathcal{K}$ such that the simplex defined by these n vectors spans $\mathbb{R}^n$. The volume of this simplex is lower bounded by $\frac{1}{n!} \cdot |det \begin{pmatrix} 1 & 1 & \cdots & 1 \\ v_0 & v_1 & \cdots & v_n \end{pmatrix}|$. Using Cramer's formula and the Hadamard inequality, the denominator of this volume can be upper bounded and then the result holds.

If $\mathcal{K}'$ is empty then $\mathcal{K}$ is clearly empty. Now, assume that $\mathcal{K}$ is empty. Farkas' Lemma states that $\{\mathbf{x} \ : \ \mathbf{Ax} \leq \mathbf{b}\}$ is empty iff there is $\lambda \in \mathbb{R}^m_{\geq 0}$ such that $\lambda^T \mathbf{A} = 0$ and $\lambda^T \mathbf{b} = -1$. By Cramer's formula and Hadamard's inequality, λ can be chosen such that $\lambda_i \leq (n \cdot B)^n$. Taking $t = 1/((n+1)(b \cdot B)^n)$, we get that $\lambda^T(\mathbf{b} + \mathbf{t}) < 0$. Therefore, the same λ satisfies the Farkas' conditions for $\mathcal{K}'$ which is not feasible. □

Initialization: finding the initial ellipsoid

The next theorem allows to define the initial ellipsoid $\varepsilon_0(\mathbf{A_0}, \mathbf{a_0})$.

Theorem 10.6. *Let $\mathcal{K} = \{\mathbf{x} \in \mathbb{R}^n \ : \ \mathbf{Ax} \leq \mathbf{b}\}$ be a polytope and let B be the largest absolute value of the coefficients of* $\mathbf{A}$ *and* $\mathbf{b}$. *Then $\mathcal{K}$ is contained into the ball $\mathcal{B}(0, n^n B^n)$.*

Proof. The *extreme points* of a convex set are those points of it that are not a linear combination of any other vectors in this convex set. Equivalently, an extreme point of $\mathcal{K}$ is determined as the unique solution $\mathbf{v}$ of a linear system $\mathbf{A'x} = \mathbf{b'}$ where $\mathbf{A'x} \leq \mathbf{b'}$ is a subsystem of $\mathbf{Ax} \leq \mathbf{b}$ and $\mathbf{A'}$ is non-singular.

By Cramer's rule, the ith coordinate of such a solution $\mathbf{v}$ is $v_i = det(\mathbf{A}'_i)/det(\mathbf{A}')$ where $\mathbf{A}'_i$ is the matrix obtained from $\mathbf{A}'$ by replacing its ith column by the vector $\mathbf{b}$. Hadamard's inequality states that $|det(\mathbf{M})| \leq \Pi_{i=1}^n ||\mathbf{m}_i||$ for any $\mathbf{M} \in \mathbb{R}^{n\times n}$ with columns $\mathbf{m}_i$, $i \leq n$. Therefore, $|det(\mathbf{A})|$, $|det(\mathbf{A}')|$, $|det(\mathbf{A}'_i)|$ and $|v_i|$ are all upper bounded by $n^{n/2}B^n$.

Hence, any extreme point of $\mathcal{K}$ lies in $\{\mathbf{x} \in \mathbb{R}^n \ : \ ||\mathbf{x}|| \leq n^{n/2}B^n\}$. To conclude it is sufficient to recall that a polytope equals the convex hull of its extreme points. □

Computing the next ellipsoid containing $\mathcal{K}$ (Step 4)

Lemma 10.7. *The half-unit ball $\mathcal{B}_{1/2} = \{\mathbf{x} \in \mathbb{R}^n \mid ||\mathbf{x}|| \leq 1, x(1) \geq 0\}$ is contained in the ellipsoid $E = \varepsilon(\mathbf{A}, \mathbf{b})$ with volume at most $V_n \cdot e^{-\frac{1}{2(n+1)}}$ where $\mathbf{A} = diag\left(\frac{n}{n+1}, \sqrt{\frac{n^2}{n^2-1}}, \cdots, \sqrt{\frac{n^2}{n^2-1}}\right)$ and $\mathbf{b} = (1/(n+1), 0, \cdots, 0)$.*

Proof. First, $E = \{\mathbf{x} \in \mathbb{R}^n \mid ||\mathbf{A}^{-1}\mathbf{x} - \mathbf{A}^{-1}\mathbf{b}||^2 = (\frac{n+1}{n})^2(x(1) - \frac{1}{n+1})^2 + \frac{n^2-1}{n^2}\sum_{i=2}^n x(i)^2 \leq 1\}$.

Let $\mathbf{y} \in \mathcal{B}_{1/2}$, we show that $\mathbf{y} \in E$ and then $\mathcal{B}_{1/2} \subseteq E$.

$$\begin{aligned}
& (\tfrac{n+1}{n})^2(y(1) - \tfrac{1}{n+1})^2 + \tfrac{n^2-1}{n^2}\textstyle\sum_{i=2}^n y(i)^2 \\
= \; & \tfrac{2n+2}{n^2} y(1)(y(1)-1) + \tfrac{1}{n^2} + \tfrac{n^2-1}{n^2}\textstyle\sum_{i=2}^n y(i)^2 \\
\leq \; & \tfrac{1}{n^2} + \tfrac{n^2-1}{n^2} \quad \leq 1
\end{aligned}$$

Moreover, the volume of E is $|det(A)| \cdot V_n$ and $det(A) = \frac{n}{n+1}(\frac{n^2}{n^2-1})^{(n-1)/2}$. Using the fact that $1 + x \leq e^x$, we obtain that $det(A) \leq e^{-1/(n+1)}e^{(n-1)/(2(n^2-1))} = e^{-\frac{1}{2(n+1)}}$. □

Theorem 10.8. *The half-ellipsoid $\varepsilon(\mathbf{A}, \mathbf{a}) \cap \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{c}^T\mathbf{x} \leq \mathbf{c}^T\mathbf{a}\}$ is contained in the ellipsoid $\varepsilon(\mathbf{A}', \mathbf{a}')$ where*

$$\mathbf{a}' = \mathbf{a} - \frac{1}{n+1}\mathbf{b} \quad \textit{and} \quad \mathbf{A}' = \frac{n^2}{n^2-1}(\mathbf{A} - \frac{2}{n+1}\mathbf{b}\mathbf{b}^T) \quad \textit{where} \quad \mathbf{b} = \mathbf{Ac}/\sqrt{\mathbf{c^T A c}}$$

Moreover, the ratio $vol(\varepsilon(\mathbf{A}', \mathbf{a}'))/vol(\varepsilon(\mathbf{A}, \mathbf{a})) \leq e^{-1/(2(n+1))}$.

Proof. This follows from the fact that the half-ellipsoid is the image of the half-unit ball under a linear transformation and from Lemma 10.7. Also, the ratio of the volumes of the two ellipsoids is invariant under the linear transformation. □

Note that the square root in the definition of $\varepsilon(\mathbf{A}', \mathbf{a}')$ implies that this ellipsoid may not be computed exactly. Nevertheless, it can be modified so that the intermediate results are rounded using a number of bits that is polynomial in $< \mathcal{K} >$. A suitable choice of the rounding constants ensures that the obtained rational ellipsoid still contains $\mathcal{K}$.

10.1.4 Examples of separation oracle

In this section, we show that, in some case, an explicit description of all the (exponential) constraints is not necessary to decide if a solution is feasible or not. In such cases, the Ellipsoid Method implies a complexity independent of the number of constraints.

Maximum Weight matching Problem

Edmonds shown that the convex hull of the matchings of a graph $G = (V, E)$ is given by

$$\begin{aligned} \sum_{e \in E(S)} x_e \leq \frac{|S|-1}{2} & \quad S \subset V, |S| \; odd \\ \sum_{v \in e} x_e \leq 1 & \quad v \in V \\ x_e \geq 0 & \quad e \in E \end{aligned}$$

While there is an exponential number of constraints, it is possible to check if a vertex $\mathbf{x}$ is a realizable solution of the above system in time polynomial in n. Therefore, the ellipsoid method ensures that the maximum weight matching problem can be solved in time polynomial in n (independent of m).

Let $\mathbf{x} \geq 0$. Let us present a separation oracle used to decide if it belongs to the above polytope. W.l.o.g., we may assume that V is even (otherwise, we may add an extra vertex). Let $s_v = 1 - \sum_{v \in e} x_e$ for all $v \in V$. Note that,

$$\sum_{e \in E(S)} x_e \leq \frac{|S|-1}{2} \quad \Leftrightarrow \quad \sum_{v \in S} s_v + \sum_{e \in E(S)} x_e \geq 1$$

Let $H = (V \cup \{r\}, E' = E \cup \{(r, v) \mid v \in V\})$ be the graph with vertex-set V plus one extra universal vertex r. For any $e \in E'$, the capacity w_e of e is x_e if $e \in E$ and equals s_v if $e = (r, v)$. In H, we have

$$\sum_{v \in S} s_v + \sum_{e \in E(S)} x_e \geq 1 \quad \Leftrightarrow \quad \sum_{e \in \delta_H(S)} w_e \geq 1$$

Therefore, $\mathbf{x}$ is feasible if and only if there is no odd set S in H such that the cut $(S, V(H) \setminus S)$ has capacity strictly less than 1. This can be solved polynomially as a subcase of the minimum T-odd cut problem.

10.2 Interior Points Methods

The Interior Points Methods is a family of methods with the same approach. They consists in walking through the interior of the set of feasible solutions (while the simplex method walks along the boundary of this the polytope from extreme point to extreme point, and the Ellipsoid Method encircles this polytope).

Among those Interior Points Methods, Karamakar [3] proposed an algorithm and proved that it solves linear programmes in polynomial time.

It is important to know that Interior Points Methods typically outperforms the Simplex Method on very large problems.

10.3 Exercises

Exercise 10.1. Prove Theorem 10.1.

Exercise 10.2. Prove Theorem 10.3

Exercise 10.3. Let $P = \{\mathbf{x} \in \mathbb{R}^n \ : \ \mathbf{A}\mathbf{x} \leq \mathbf{b}\}$ be a polyhedron and $t > 0$. Let $\mathbf{t} \in \mathbb{R}^m$ be the vector with all coordinates equal to t. Shows that $\{\mathbf{x} \in \mathbb{R}^n \ : \ \mathbf{A}\mathbf{x} \leq \mathbf{b} + \mathbf{t}\}$ is full dimensional.

Exercise 10.4. Minimum cost arborescence. Let D be a directed graph on n vertices each arc a of which has weight c_a and let $r \in V(D)$. Consider the following problem:

$$\begin{array}{rll} \text{Minimize} & \sum_{a \in A(D)} c_a x_a & \\ \text{Subject to:} & \sum_{a \in \delta^-(S)} x_a \geq 1 & \forall S \subseteq V(D) \setminus \{r\} \\ & x_a \geq 0 & \forall a \in A(D) \end{array}$$

Give an algorithm that decides whether $\mathbf{x} \in \mathbb{R}^n$ is a feasible solution or returns a certificate $S \subseteq V(D) \setminus \{r\}$ that $\mathbf{x}$ is not feasible, in time polynomial in n. Conclusion?

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