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Lecture 2. Analytic functions, Runge phenomenon, barycentric interpolation

Outline of lectures (tentative)

1
- Chebyshev points, interpolants, polynomials, series

2
- Chebyshev series of analytic functions
- Runge phenomenon (equispaced interpolation)
- Barycentric interpolation formula

3
- Rootfinding
- Optimization
- Quadrature

4
- Spectral methods for ODEs
- Chebfun2 (two dimensions)
- Gaussian elimination as an iterative algorithm

1. Exploring the Chebfun web site and software

2. Chebyshev series of analytic functions [*ATAP* chap 8]

The following can be proved by contour integrals in x or z (Bernstein 1912).

The Bernstein ellipse E_ρ is the image of the circle $|z| = \rho$ under the Joukowski map $x = \frac{1}{2}(z + z^{-1})$.

Theorem 8.1. Chebyshev coefficients of analytic functions. *Let a function f analytic in $[-1,1]$ be analytically continuable to the open Bernstein ellipse E_ρ, where it satisfies $|f(x)| \le M$ for some M. Then its Chebyshev coefficients satisfy $|a_0| \le M$ and*

$$|a_k| \le 2M\rho^{-k}, \quad k \ge 1. \tag{8.1}$$

Theorem 8.2. Convergence for analytic functions. *If f has the properties of Theorem 8.1, then for each $n \ge 0$ its Chebyshev projections satisfy*

$$\|f - f_n\| \le \frac{2M\rho^{-n}}{\rho - 1} \tag{8.2}$$

and its Chebyshev interpolants satisfy

$$\|f - p_n\| \le \frac{4M\rho^{-n}}{\rho - 1}. \tag{8.3}$$

3 Runge phenomenon (equispaced interpolation) [*ATAP* chaps 13-14]

Since Runge (1900) it has been known that polynomial interpolants in equispaced points diverge exponentially as $n \to \infty$, even if the function is analytic, and even in exact arithmetic. The explanation involves potential theory, though we will not go into it here. Here is a code for comparing Chebyshev and equispaced points.

```
f = chebfun('tanh(10*x)');
while 1
    n = input('n? ');
    s = linspace(-1,1,n); p1 = chebfun.interp1(s,f(s));
    subplot(1,2,1), plot(p1), title equispaced
    hold on, plot(s,f(s),'.'), hold off
    s = chebpts(n); p2 = chebfun(f,n);
    subplot(1,2,2), plot(p2), title Chebyshev
    hold on, plot(s,f(s),'.'), hold off
end
```

4 Barycentric interpolation formula [*ATAP* chap 5; see also my essay "Six myths…."]

Some obvious algorithms for Chebyshev interpolation are exponentially unstable, like the solution of a linear system of equations involving a Vandermonde matrix. This fact, with the Runge phenomenon, have led to a widespread misconception since the 1950s that high-order polynomial interpolation is dangerous and should be avoided.

In fact, polynomial interpolation in Chebyshev points is perfectly well-behaved if you do it right. Indeed, by the change of variables $x = \cos(\theta)$, it is equivalent to trigonometric interpolation in equispaced points, which nobody worries about. A suitable algorithm is the barycentric interpolation formula (Salzer 1972, proved stable by Higham 2004). More generally there is a barycentric formula for any set of interpolation points.

Theorem 5.2. Barycentric interpolation in Chebyshev points. *The polynomial interpolant through data $\{f_j\}$ at the Chebyshev points (2.2) is*

$$p(x) = \sum_{j=0}^{n}{}' \frac{(-1)^j f_j}{x - x_j} \Bigg/ \sum_{j=0}^{n}{}' \frac{(-1)^j}{x - x_j}, \tag{5.13}$$

with the special case $p(x) = f_j$ if $x = x_j$. The primes on the summation signs signify that the terms $j = 0$ and $j = n$ are multiplied by $1/2$.