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# BOUNDARY INTEGRAL METHODS FOR VISCOUS INTERNAL WAVES

**Summary** : The theory of internal waves has a long and rich history dating back at least to Rayleigh (1883). Internal waves play a critical role in atmospheric and ocean dynamics. Laboratory experiments are naturally on a smaller scale than geophysical situations and viscosity is therefore expected to play a more important role. However, viscosity has not yet been treated fully consistently in theoretical work, even for the simplest linear generation problem. The bounday integral method is a well-developed tool to solve flows outside of oscillating objects. The goal of my internship was to produce numerical solutions for this problem, taking into account viscosity, for spherical or elliptical geometries.

**Key words** : viscous stratified fluid, boundary layers, viscosity, internal waves.

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# Introduction

Natural bodies of fluid subject to gravity such as the atmosphere, the oceans and lakes are characteristically stably stratified. Their density decreases as one goes upwards. When they are disturbed in any way, internal waves are generated. These motions can explain several phenomenas, like the temperature fluctuations in the deep ocean or the formation of clouds in the lee of a mountain. Progress in this field has permitted to solve very pratical problems in meteorology or in hydraulic engineering.

The solutions for this problem are know for very simple geometries such as cylinders or ellipses in the inviscid case, and only for a disk in the viscous case. Experimental studies have shown that viscous effetcs must be taken into account. The boudary-integral method has been established as a powerful numerical technique for tackling a variety of problems in science involving partial differential equations. This method involves Green functions of the flow. Examples can be drawn from the fields of elasticity, electromagnetism, acoustics, hydraulics or Stokes flows. During my internship, I had to develop this method for the problem of internal waves, created by oscillating objects. I developed this technique, both theorically and numerically only for spherical and elliptical geometries.

In the first part, I will introduce the problem of internal waves created by oscillating objects and remind some fundamental equations. Then in the second part, I will expose the work I have done to compute the Green functions of the problem, both theorically and numerically. In the final part, we will see how to use my results for the boundary-integral method.

# Chapter 1

# Fundamental equations

I will begin the report by describing internal gravity waves in stratified fluids and giving the fundamental equations. All the phenomenas I have studied depend on gravity acting on small density differences in a non-rotating fluid. The fluid has a density distribution which varies in the vertical but is constant in horizontal planes: this is a stratified system.

We consider the flow produced by a rigid body that vibrates at the frequency $\omega$, along the $z$ axis, in an infinite ambient fluid. The fluid will be assumed incompressible and non-diffusive, this means that

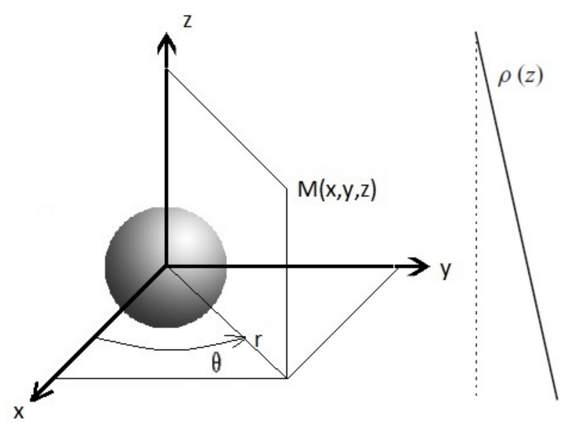


Figure 1.1: Cartoon of the geometry of this problem.

$$\frac{D\rho}{Dt} = 0, \tag{1.1}$$

where $\frac{D}{Dt}$ denotes differentiation following the motion, while the continuity equation in vector notation is

$$\nabla.\mathbf{u} = 0, \tag{1.2}$$

where $\mathbf{u} = (u, v, w)$ is the velocity. The Navier-Stokes equation, with the force of gravity included (with $\mathbf{g} = (0, 0, -g)$) and the $x$ and $y$ axes being in the horizontal plane (as shown on figure (1)), can be written as:

$$\rho\frac{D\mathbf{u}}{Dt} = \rho\left(\frac{\partial\mathbf{u}}{\partial t} + (\mathbf{u}.\nabla\mathbf{u})\right) = -\nabla p + \rho\mathbf{g} + \mu\nabla^2\mathbf{u}. \tag{1.3}$$

The last term is the result of the molecular viscosity $\mu$, assumed constant here. The only external force field considered is that of gravity, which exerts a body force $\rho\mathbf{g}$ per unit volum on each element of fluid (where $\rho$ is the local density and $\mathbf{g}$ is the acceleration due to gravity). The last term is the result of the viscosity $\mu$, assumed to be constant here. The nature of the fluid is not important, most of the ideas may be applied to liquids, in which density variations are due to differences of temperature or concentrations of solutes, or to gases in which there may be differences of temperature.

We analyse now small disturbances to an equilibrium distribution of density $\rho_0(z)$ in an atmosphere or in an ocean for instance. During my internship, I have considered that $\rho_0(z)$ is a continuously decreasing function of the height $z$. If p and $\rho$ are expanded around the values $p_0$ and $\rho_0$ in a reference state of hydrostatic equilibrium (i.e. one sets $p = p_0 + p^{'}$ and $\rho = \rho_0 + \rho^{'}$) we can rewrite (1.3) as

$$(1+\frac{\rho^{'}}{\rho_0})\frac{D\mathbf{u}}{Dt} = -\frac{1}{\rho_0}\nabla p + (1+\frac{\rho^{'}}{\rho_0})\mathbf{g} + \frac{\mu}{\rho_0}\nabla^2\mathbf{u} \tag{1.4}$$

Two approximations for the Navier–Stokes equation must be introduced. The first simplification is that of linearization, which consists in neglecting the non linear terms like $u\frac{\partial u}{\partial x}$, compared to the temporal derivation $\frac{\partial u}{\partial t}$. We can do this because we consider small oscillations. The second approximation is that the density perturbation $\rho^{'}$ is small compared to $\rho_0$.

Using $\nabla p_0 = \rho_0\mathbf{g}$, we obtain the *linearized* Boussinesq equations for a viscous liquid:

$$\frac{\partial\mathbf{u}}{\partial t} = -\frac{1}{\rho_0}\nabla p^{'} + \frac{\rho^{'}}{\rho_0}\mathbf{g} + \frac{\mu}{\rho_0}\nabla^2\mathbf{u}. \tag{1.5}$$

The linear form of (1.1) is just

$$\frac{\partial\rho^{'}}{\partial t} + w\frac{d\rho_0}{dz} = 0 \tag{1.6}$$

We will now introduce a quantity in an elementary way. Consider the motion of an element of fluid displaced a small distance $\xi$ vertically from its equilibrium position in a stable environment. By using (1.5) and (1.6), we can write the vertical component of the motion equation as:

$$\frac{\partial^2\xi}{\partial t^2} = \frac{g}{\rho}\frac{\partial\rho_0}{\partial z}\xi \tag{1.7}$$

The fluid particle will oscillate in harmonioc motion with the frequency : $N = (-\frac{g}{\rho}\frac{\partial\rho_0}{\partial z})^{\frac{1}{2}}$. This is the frequency is named *buoyancy frequency*. The corresponding periods $\frac{2\pi}{N}$ are typically a few minutes in the atmosphere and a couple of hours in deep ocean.

Introducing $b = -\frac{\rho^{'}}{\rho_0}g$ and the buoyancy frequency N, we can rewrite the Boussinesq equations as:

$$\frac{\partial \mathbf{u}}{\partial t} = -\frac{1}{\rho_0}\nabla p^{'} + b\mathbf{e}_z + \nu\nabla^2\mathbf{u}, \tag{1.8}$$

$$\frac{\partial b}{\partial t} + N^2 w = 0 \tag{1.9}$$

where $\nu = \frac{\mu}{\rho_0}$ is the kinematic viscosity. The problem is totally characterized by these two equations. Where are now going to see how this problem can be formulated using Green Functions. The computation of these functions are essential to use the boundary-integral method.

# Chapter 2

# Green functions

The Green's functions are solutions of the continuity equation and the Navier-Stokes equation with a singular force:

$$\frac{\partial \mathbf{u}}{\partial t} = -\frac{1}{\rho_0}\nabla p^{'} + b\mathbf{e}_z + \nu \nabla^2 \mathbf{u} + \mathbf{g}\delta(\mathbf{x} - \mathbf{x}_0) \tag{2.1}$$

where $\mathbf{g}$ is an arbitrary constant vector, $\mathbf{x}_0$ is the pole of the source point, $\mathbf{x}$ is the observation point and $\delta$ is the three-dimensional delta function.

For simplicity, in the ensuig discussion, I will drop the primes on $p$, but we should remember that we are dealing with perturbations around hydrostatic equilibrium.

The Navier-Stokes equation can also be written as:

$$\frac{\partial \mathbf{u}}{\partial t} = \nabla \cdot \sigma + b\mathbf{e}_z + \mathbf{g}\delta(\mathbf{x} - \mathbf{x}_0), \tag{2.2}$$

where $\sigma$ is the stress tensor defined as follows:

$$\sigma_{ij} = -p\delta_{ij} + \nu \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right). \tag{2.3}$$

We can write the solution of (2.1) introducing the Green functions $\mathbf{G}$ and $\mathbf{T}$, associated to the velocity field and the stress tensor, as:

$$u_i(\mathbf{x}) = \frac{1}{8\pi\mu} G_{ij}(\mathbf{x}, \mathbf{x}_0) g_j, \qquad \sigma_{ik}(\mathbf{x}) = \frac{1}{8\pi} T_{ijk}(\mathbf{x}, \mathbf{x}_0) g_j. \tag{2.4}$$

Thus, the velocity field $\mathbf{u}$ represents the solution due to a concentrated point force of strength $\mathbf{g}$ and located at the point $\mathbf{x}_0$. We need the expression of $G_{ij}$, and the easiest way to compute these coefficients is to work in the Fourier Space. The complex Fourier Transform will be definied as

$$\hat{f}(\mathbf{k}, w) = \frac{1}{(2\pi)^3} \int_{wholespace} \int_{time} f(\mathbf{x}, t) e^{i(\omega t - \mathbf{k} \cdot \mathbf{x})} d\mathbf{x} dt \tag{2.5}$$

The purpose of my calculation is to find, in the Fourier space, the relations $\hat{u}_i = \widehat{G_{ij}} g_j$ and $\widehat{\sigma_{ik}} = \widehat{T_{ijk}} g_j$, which will give us $\widehat{G_{ij}}$ and $\widehat{T_{ijk}}$. By Inverse Fourier Transform, we will be able to get $G_{ij}$ and $T_{ijk}$.

## 2.1 Green function for the velocity field

Let's take the Fourier Transform of (1.9) and (2.1).

$$-i\omega\hat{b} = -N^2\hat{\mathbf{w}} \tag{2.6}$$

$$-i\omega\hat{\mathbf{u}} = -\frac{1}{\rho_0}i\mathbf{k}\hat{p} - \nu\left|\mathbf{k}\right|^2\hat{\mathbf{u}} + \mathbf{g} + \hat{b}\mathbf{e}_z. \tag{2.7}$$

We can eliminate $\hat{p}$ by dotting (2.7) with $\mathbf{e}_z$, and also (2.1) with $\cdot\mathbf{k}$ , remembering that the continuity equation gives $\mathbf{k}\cdot\mathbf{u} = 0$:

$$-i\omega\hat{w} = -\frac{1}{\rho_0}ik_z\hat{p} - \nu\left|\mathbf{k}\right|^2\hat{w} + g_z + \frac{N^2}{i\omega}\hat{w}, \tag{2.8}$$

$$0 = -\frac{1}{\rho_0}i\left|\mathbf{k}\right|^2\hat{p} + \mathbf{k}\cdot\mathbf{g} + \frac{N^2}{i\omega}k_z\hat{w}. \tag{2.9}$$

We can eliminate $\hat{p}$ to obtain an equation connecting $\hat{w}$ and $g_z$:

$$\left[-i\omega\left|\mathbf{k}\right|^2 + \nu\left|\mathbf{k}\right|^4 + \frac{N^2}{i\omega}(k_z^2 - \left|\mathbf{k}\right|^2)\right]\hat{w} = \alpha\hat{w} = -k_z\mathbf{k}\cdot\mathbf{g} + \left|\mathbf{k}\right|^2 g_z \tag{2.10}$$

with $\alpha = -i\omega\left|\mathbf{k}\right|^2 + \nu\left|\mathbf{k}\right|^4 + \frac{N^2}{i\omega}(k_z^2 - \left|\mathbf{k}\right|^2)$

We can do exactly the same thing for $\hat{u}$ and $\hat{v}$:

$$\left[-i\omega\left|\mathbf{k}\right|^2 + \nu\left|\mathbf{k}\right|^4\right]\hat{u} = \beta\hat{u} = \left|\mathbf{k}\right|^2 g_x - k_x\mathbf{k}\cdot\mathbf{g} - \frac{N^2k_xk_z}{i\omega\alpha}\left(\left|\mathbf{k}\right|^2 g_z - k_z\mathbf{k}\cdot\mathbf{g}\right) \tag{2.11}$$

$$\left[-i\omega\left|\mathbf{k}\right|^2 + \nu\left|\mathbf{k}\right|^4\right]\hat{v} = \beta\hat{v} = \left|\mathbf{k}\right|^2 g_y - k_y\mathbf{k}\cdot\mathbf{g} - \frac{N^2k_yk_z}{i\omega\alpha}\left(\left|\mathbf{k}\right|^2 g_z - k_z\mathbf{k}\cdot\mathbf{g}\right) \tag{2.12}$$

These three relations can be written as:

$$\hat{\mathbf{u}} = \frac{1}{8\pi\mu}\widehat{\mathbf{G}}(\mathbf{k},\omega)\mathbf{g} \tag{2.13}$$

where $\widehat{\mathbf{G}}$ is the Fourier Transform of the Green function:

$$\widehat{\mathbf{G}}(\mathbf{k},\omega) = 8\pi\mu\begin{pmatrix} \frac{|\mathbf{k}|^2-k_x^2}{\beta} + \frac{N^2k_x^2k_z^2}{i\omega\alpha\beta} & -\frac{k_xk_y}{\beta} + \frac{N^2k_xk_yk_z^2}{i\omega\alpha\beta} & -\frac{k_xk_z}{\beta} + \frac{N^2k_xk_z(k_z^2-|\mathbf{k}|^2)}{i\omega\alpha\beta} \\ -\frac{k_xk_y}{\beta} + \frac{N^2k_xk_yk_z^2}{i\omega\alpha\beta} & \frac{|\mathbf{k}|^2-k_y^2}{\beta} + \frac{N^2k_y^2k_z^2}{i\omega\alpha\beta} & -\frac{k_yk_z}{\beta} + \frac{N^2k_yk_z(k_z^2-|\mathbf{k}|^2)}{i\omega\alpha\beta} \\ -\frac{k_xk_z}{\alpha} & -\frac{k_yk_z}{\alpha} & \frac{|\mathbf{k}|^2-k_z^2}{\alpha} \end{pmatrix}$$

If N=0, i.e. if there is no density gradient, then $\alpha = \beta = -i\omega\left|\mathbf{k}\right|^2 + \nu\left|\mathbf{k}\right|^4$.

## 2.2 Green function for the stress tensor

The equation (2.9) provides a link between the pressure $p$ and the vertical component of the velocity $w$:

$$\hat{p} = -\frac{N^2\rho_0}{\omega|\mathbf{k}|^2}k_z\hat{w} + \frac{\rho_0}{i\,|\mathbf{k}|^2}\mathbf{k}\cdot\mathbf{g} \tag{2.14}$$

We can obtain $\hat{\sigma}_{ij}$ thanks to this relation:

$$\hat{\sigma}_{ij} = -\hat{p}\delta_{ij} + i\nu(k_j\hat{u}_i + k_i\hat{u}_j) = \frac{N^2\rho_0}{\omega|\mathbf{k}|^2}k_z\hat{G_{zk}}g_k\delta_{ij} - \frac{\rho_0}{i\,|\mathbf{k}|^2}\mathbf{k}\cdot\mathbf{g}\delta_{ij} + \frac{i}{8\pi\mu}(k_j\hat{G_{ik}} + k_i\hat{G_{jk}})g_k \tag{2.15}$$

which finally gives:

$$\hat{T}_{ijk}(\mathbf{k},\omega) = 8\pi\left(\frac{N^2\rho_0}{\omega|\mathbf{k}|^2}k_z\hat{G_{zk}}\delta_{ij} - \frac{\rho_0}{i\,|\mathbf{k}|^2}k_k\delta_{ij} + \frac{i}{8\pi}(k_j\hat{G_{ik}} + k_i\hat{G_{jk}})\right) \tag{2.16}$$

## 2.3 Calculation of the Green function coefficients for the velocity field

Taking the Inverse Fourier transform of $\widehat{\mathbf{G}}$, only for the space dependance, we obtain

$$G_{ij}(\mathbf{x},\omega) = \frac{1}{(2\pi)^3}\int_{\mathbf{k}}\widehat{G_{ij}}(k_x,k_y,k_z)e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}_0)}d^3\mathbf{k} = \frac{1}{(2\pi)^3}\int_{\mathbf{k}}\widehat{G_{ij}}(k_x,k_y,k_z)e^{i\mathbf{k}\cdot\hat{\mathbf{x}}}d^3\mathbf{k} \tag{2.17}$$

where $\hat{\mathbf{x}} = \mathbf{x} - \mathbf{x}_0$. Because the objet has the circular symmetry, we will make the variable change:

$$\begin{cases} k_x = \kappa\cos(\phi) \\ k_y = \kappa\sin(\phi) \end{cases}, \qquad \begin{cases} \hat{x} = r\cos(\theta) \\ \hat{y} = r\sin(\theta) \end{cases}$$

(2.18)

We can decompose $\mathbf{k}$ as $\mathbf{k} = \mathbf{k}_h + k_z$ (where $\mathbf{k}_h = k_x\mathbf{e}_x + k_y\mathbf{e}_y$), and rewrite $G_{ij}$ as:

$$\begin{aligned} G_{ij} =& \frac{1}{(2\pi)^3}\int_{\kappa=0}^{\infty}\int_{\phi=0}^{2\pi}\int_{k_z=-\infty}^{\infty}\widehat{G_{ij}}(k_x,k_y,k_z)e^{i\kappa r(\cos(\theta)\cos(\phi)+\sin(\theta)\sin(\phi))}e^{ik_z\hat{z}}\kappa d\kappa d\phi dk_z \\ =& \frac{1}{(2\pi)^3}\int_{\kappa=0}^{\infty}\int_{\phi=0}^{2\pi}\int_{k_z=-\infty}^{\infty}\widehat{G_{ij}}(k_x,k_y,k_z)e^{i\kappa r\cos(\theta-\phi)}e^{ik_z\hat{z}}\kappa d\kappa d\phi dk_z \end{aligned} \tag{2.19}$$

The variable change will be very useful to compute the integrals, over $k_z$, $\kappa$ and $\phi$.

### 2.3.1 Integration over $k_z$

We can compute by hand all the integrations over $k_z$. Hence, I will develop the compution method for the first coefficient $G_{xx}$. I will first take the case $N = 0$ to expose the method. We have:

$$G_{xx} = \frac{1}{(2\pi)^3} \int_{\kappa=0}^{\infty} \int_{\phi=0}^{2\pi} \int_{k_z=-\infty}^{\infty} \frac{k_y^2 + k_z^2}{-i\omega \left|\mathbf{k}\right|^2 + \nu \left|\mathbf{k}\right|^4} e^{i\kappa r \cos(\theta-\phi)} e^{ik_z \hat{z}} \kappa d\kappa d\phi dk_z \tag{2.20}$$

To evaluate the integral over $k_z$, we can use the residue theorem. Since the function $(k_y^2 + k_z^2)e^{ik_z\hat{z}}$ has no singularities at any point in the complex plane, the integrand has singularities only where the denominator $-i\omega \left|\mathbf{k}\right|^2 + \nu \left|\mathbf{k}\right|^4$ is equal to zero. We have two possibilities, $\left|\mathbf{k}\right|^2 = 0$ or $\left|\mathbf{k}\right|^2 = \frac{i\omega}{\nu}$. The case $\left|\mathbf{k}\right|^2 = 0$ leads to $k_z = 0$ and $\mathbf{k}_h^2 = 0$, that is why we will ignore it. The second equality gives $k_z^2 = \frac{i\omega}{\nu} - \left|\mathbf{k}_h\right|^2 = \frac{i\omega}{\nu} - \kappa^2$, leading to two possibilities: $k_z = \sqrt{\frac{i\omega}{\nu} - \left|\mathbf{k}_h\right|^2}$ or $k_z = -\sqrt{\frac{i\omega}{\nu} - \left|\mathbf{k}_h\right|^2}$. We need to find the value of the imaginary part of $k_z$. By posing $k_z = x + iy$, we have:

$$\begin{cases} x^2 - y^2 = -\left|\mathbf{k}_h\right|^2 \\ 2xy = \frac{\omega}{\nu} \\ x^2 + y^2 = \sqrt{(\frac{\omega}{\nu})^2 + \left|\mathbf{k}_h\right|^4} \end{cases}$$

(2.21)

Solving this system, we easily find that the imaginary part of $k_z = \sqrt{\frac{i\omega}{\nu} - \left|\mathbf{k}_h\right|^2}$ is equal to $\left(\frac{1}{2}\sqrt{\frac{\omega}{\nu}^2 + \left|\mathbf{k}_h\right|^2} + \left|\mathbf{k}_h\right|^2\right)^{\frac{1}{2}}$. For $k_z = -\sqrt{\frac{i\omega}{\nu} - \left|\mathbf{k}_h\right|^2}$, the imaginary part is the opposite.

Supposing that $z > 0$, we define the contour $C$ that goes along the real line from -$a$ to $a$ making a semicircle centered at 0 from $a$ to -$a$. Taking $a$ as an arbitrary great value ,so that the imaginary part of $k_z$ is enclosed within the curve, we obtain that:

$$G_{xx} = \frac{1}{(2\pi)^3} \int_{\kappa=0}^{\infty} \int_{\phi=0}^{2\pi} 2i\pi res_{k_z=\sqrt{\frac{i\omega}{\nu}-\kappa^2}} \left( \frac{k_y^2 + k_z^2}{-i\omega \left|\mathbf{k}\right|^2 + \nu \left|\mathbf{k}\right|^4} e^{ik_z\hat{z}} \right) e^{i\kappa r \cos(\theta-\phi)} \kappa d\kappa d\phi \tag{2.22}$$

We finally obtain, for $\hat{z} > 0$:

$$G_{xx} = \frac{1}{(2\pi)^3} \int_{\kappa=0}^{\infty} \int_{\phi=0}^{2\pi} 2i\pi \frac{\frac{i\omega}{\nu} - \kappa^2 \cos(\phi)^2}{2i\omega\sqrt{\frac{i\omega}{\nu} - \kappa^2}} e^{-\sqrt{\kappa^2 - \frac{i\omega}{\nu}}\hat{z}} e^{i\kappa r \cos(\theta-\phi)} \kappa d\kappa d\phi \tag{2.23}$$

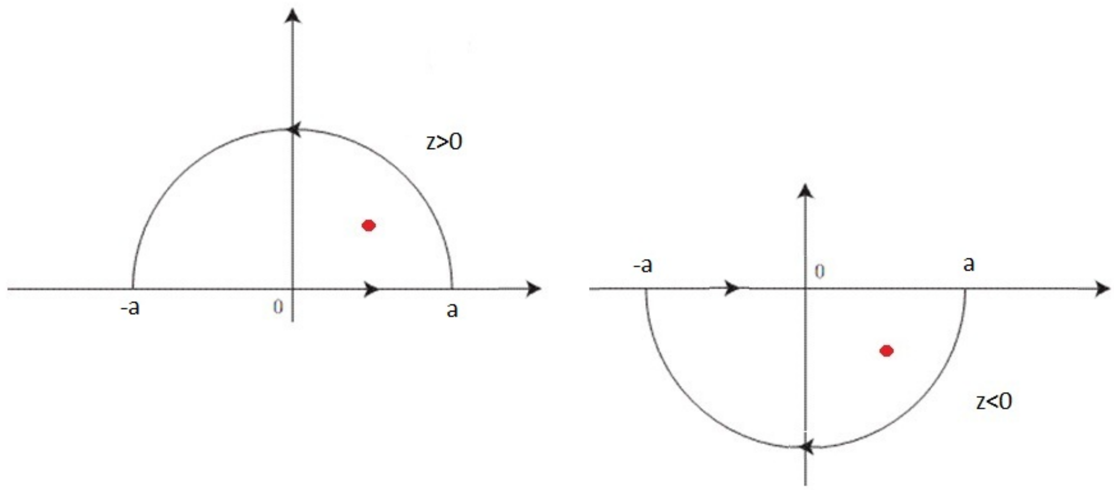


Figure 2.1: Contour $C$ used to apply the residue theorem.

If $\hat{z} < 0$, we have to take the solution $k_z = -\sqrt{\frac{i\omega}{\nu} - |\mathbf{k}_h|^2}$, and we obtain:

$$G_{xx} = -\frac{1}{(2\pi)^3} \int_{\kappa=0}^{\infty} \int_{\phi=0}^{2\pi} 2i\pi \frac{\frac{i\omega}{\nu} - \kappa^2 \cos(\phi)^2}{2i\omega\sqrt{\frac{i\omega}{\nu} - \kappa^2}} e^{\sqrt{\kappa^2 - \frac{i\omega}{\nu}}\hat{z}} e^{i\kappa r \cos(\theta-\phi)} \kappa d\kappa d\phi \qquad (2.24)$$

In both cases, there are no convergence problems. The only problem is for $\hat{z} = 0$. In this particular case, the integral is divergent. Applying this method to all $G_{ij}$ coefficients, we can compute all the integrations over $k_z$ by hand.

### 2.3.2 Integration over $\phi$

We will now compute analytically all the integrals over $\phi$. The purpose is to transform the integrals to recognize Bessel functions. With $\phi = \theta + \alpha$ and $\beta = \alpha + \frac{\pi}{2}$ we obtain:

$$\int_{\phi=0}^{2\pi} e^{i\kappa r cos(\theta-\phi)} d\phi = \int_{\alpha=0}^{2\pi} e^{i\kappa r \cos(\alpha)} d\alpha = \int_{\beta=-\pi}^{\pi} e^{i\kappa r \sin(\beta)} d\beta = 2\pi J_0(\kappa r) \qquad (2.25)$$

with $J_0$ the Bessel function of order 0. In the same way, we obtain:

$$\begin{aligned}\int_0^{2\pi} cos(\phi) e^{i\kappa r \cos(\theta-\phi)} d\phi = &\int_{-\pi}^{\pi} (\cos(\theta)\sin(\beta) - \sin(\theta)\cos(\beta) e^{i\kappa r \sin(\beta)} d\beta \\ =&2i\cos(\theta)\int_0^{\pi} \sin(\beta)\sin(\kappa r \sin(\beta)) d\beta - 2\cos(\theta)\int_0^{\pi} \cos(\beta)\cos(\kappa r \sin(\beta)) d\beta \\ =&2i\pi\cos(\theta) J_1(\kappa r)\end{aligned} \qquad (2.26)$$

with $J_1$ the Bessel function of order 1.

Applying this method to all integrals over $\phi$, we finally obtain:

$$G_{xx} = \frac{1}{(8\pi\omega)} \int_{\kappa=0}^{\infty} \frac{\kappa e^{-\sqrt{\kappa^2 - \frac{i\omega}{\nu}}|\hat{z}|}}{\sqrt{\frac{i\omega}{\nu} - \kappa^2}} \left( J_0(\kappa r)(\frac{2i\omega}{\nu} - \kappa^2) + \kappa^2 J_2(\kappa r)\cos(2\theta) \right) d\kappa \tag{2.27}$$

$$G_{xy} = G_{yx} = \frac{\sin(2\theta)}{(8\pi\omega)} \int_{\kappa=0}^{\infty} \frac{\kappa^3}{\sqrt{\frac{i\omega}{\nu} - \kappa^2}} J_2(\kappa r) e^{-\sqrt{\frac{\kappa^2 - i\omega}{\nu}}|\hat{z}|} d\kappa \tag{2.28}$$

$$G_{xz} = G_{zx} = -\frac{\cos(\theta)}{(4\pi\omega)} \int_{\kappa=0}^{\infty} i\kappa^2 J_1(\kappa r) e^{-\sqrt{\frac{\kappa^2 i\omega}{\nu}}|\hat{z}|} d\kappa \tag{2.29}$$

$$G_{yy} = \frac{1}{(8\pi\omega)} \int_{\kappa=0}^{\infty} \frac{\kappa e^{-\sqrt{\kappa^2 - \frac{i\omega}{\nu}}|\hat{z}|}}{\sqrt{\frac{i\omega}{\nu} - \kappa^2}} \left( J_0(\kappa r)(\frac{2i\omega}{\nu} - \kappa^2) - \kappa^2 J_2(\kappa r)\cos(2\theta) \right) d\kappa \tag{2.30}$$

$$G_{yz} = G_{zy} = \frac{-\sin(\theta)}{(4\pi\omega)} \int_{\kappa=0}^{\infty} i\kappa^2 J_1(\kappa r) e^{-\sqrt{\kappa^2 - \frac{i\omega}{\nu}}|\hat{z}|} d\kappa \tag{2.31}$$

$$G_{zz} = \frac{1}{(4\pi\omega)} \int_{\kappa=0}^{\infty} \frac{\kappa^3}{\sqrt{\frac{i\omega}{\nu} - \kappa^2}} J_0(\kappa r) e^{-\sqrt{\kappa^2 - \frac{i\omega}{\nu}}|\hat{z}|} d\kappa \tag{2.32}$$

### 2.3.3 Case N$\neq$ 0

We now take into account the buoyancy frequency N. The $G_{ij}$ coefficients are more complicated, but the strategy for their computation is still the same. I will still use the residue theorem for the integration over $k_z$. For the part $N \neq 0$, the integrand has singularities in the complex plane only where the denominator $\alpha.\beta = (-i\omega |\mathbf{k}|^2 + \nu |\mathbf{k}|^4 + \frac{N^2}{i\omega}(k_z^2 - |\mathbf{k}|^2).(-i\omega |\mathbf{k}|^2 + \nu |\mathbf{k}|^4) = 0$. Solving $\alpha = \nu |\mathbf{k}|^4 - i\omega |\mathbf{k}|^2 - \frac{N^2\kappa^2}{i\omega} = 0$, we find that $|\mathbf{k}|_0^2 = \frac{i\omega + \sqrt{\frac{4\nu\kappa^2 N^2}{i\omega} - \omega^2}}{2\nu}$. We will only take this solution among the two possible, because this one gives us back the good coefficient for $N = 0$. Writing $\frac{1}{\alpha\beta} = \frac{a}{\alpha} + \frac{b}{\beta}$, we finf that $a = -b = \frac{i\omega}{\kappa^2 N^2}$. Using the residue theorem for the integration over $k_z$ and the same strategy for the integration over $\phi$, we can finally compute all the $Gij$ coefficients. Defining $A(\kappa) = \sqrt{|\mathbf{k}|_0^2 - \kappa^2} e^{-\sqrt{\kappa^2 - |\mathbf{k}|_0^2}|\hat{z}|}$ and $B(\kappa r) = \sqrt{\frac{i\omega}{\nu} - \kappa^2} e^{-\sqrt{\kappa^2 - \frac{i\omega}{\nu}}|\hat{z}|}$, we find:

$$G_{xx} = G_{xx}(N=0) + \int_{\kappa=0}^{\infty} \frac{\kappa^3}{8\pi} \left( J_0(\kappa r) - \cos(2\theta) J_2(\kappa r) \right) \left( \frac{iA(\kappa r)}{2\nu |\mathbf{k}|_0^2 - i\omega} - \frac{B(\kappa r)}{\omega} \right) d\kappa \tag{2.33}$$

$$G_{xy} = G_{yx} = G_{xy}(N=0) + \int_{\kappa=0}^{\infty} \kappa\pi^2 \sin(2\theta) J_2(\kappa r) \left( \frac{B(\kappa r)}{\omega} - \frac{iA(\kappa r)}{2\nu\, |\mathbf{k}|_0^2 - i\omega} \right) d\kappa \tag{2.34}$$

$$G_{xz} = G_{xz}(N=0) + \int_{\kappa=0}^{\infty} \frac{i\kappa^2 \cos(\theta) J_1(\kappa r)}{8\pi^2} \left( \frac{e^{-\sqrt{\kappa^2 - |\mathbf{k}|_0^2}|\hat{z}|}}{2\nu\, |\mathbf{k}|_0^2 - i\omega} - \frac{e^{-\sqrt{\kappa^2 - \frac{i\omega}{\nu}}|\hat{z}|}}{i\omega} \right) d\kappa \tag{2.35}$$

$$G_{yy} = G_{yy}(N=0) + \int_{\kappa=0}^{\infty} \frac{\kappa^3}{8\pi} \left( J_0(\kappa r) + \cos(2\theta) J_2(\kappa r) \right) \left( \frac{iA(\kappa r)}{2\nu\, |\mathbf{k}|_0^2 - i\omega} - \frac{B(\kappa r)}{\omega} \right) d\kappa \tag{2.36}$$

$$G_{yz} = G_{yz}(N=0) + \int_{\kappa=0}^{\infty} \frac{i\kappa^2 \sin(\theta) J_1(\kappa r)}{8\pi^2} \left( \frac{e^{-\sqrt{\kappa^2 - |\mathbf{k}|_0^2}|\hat{z}|}}{2\nu\, |\mathbf{k}|_0^2 - i\omega} - \frac{e^{-\sqrt{\kappa^2 - \frac{i\omega}{\nu}}|\hat{z}|}}{i\omega} \right) d\kappa \tag{2.37}$$

$G_{zx}$,$G_{zy}$ and $G_{zz}$ are independant of N. We easily see that if $N = 0$, $|\mathbf{k}|_0^2 = \frac{i\omega}{\nu}$ so that $A(\kappa) = B(\kappa)$. We refind all the $G_{ij}$ coefficients for the case $N = 0$.

### 2.3.4 Integration over $\kappa$

We can't compute the integrals over $\kappa$ by hand, so I used Matlab to do the calculations. I wrote a program in which you can enter a value for $x$, $y$, $z$, $\omega$, $\nu$, and $N$ and the program gives us all the $G_{ij}$ coefficients. Several results can be deduce from this program.

The first one is that as $z$ become smaller, the $G_{ij}$ coefficients get larger. This is a consequence of the divergence of the integrals in $z = 0$. Values for the matrix coefficients are given in the tabular.

| | | |
|---|---|---|
| z=1 | $1.10^{-21}$ | $\begin{pmatrix} -0.0002+0.2555i & -0.0000+0.0001i & 0.0000+0.0000i \\ -0.0000+0.0001i & -0.0000+0.0004i & 0.0000+0.0000i \\ -0.0000+0.0000i & -0.0000+0.0000i & -0.0000+0.0000i \end{pmatrix}$ |
| z=0.1 | $1.10^{2}$ | $\begin{pmatrix} 0.0364-6.0781i & -1.8483+7.2093i & 0.0022-0.0060i \\ -1.8483+7.2093i & 0.0077-0.0606i & 0.0022-0.0060i \\ 0.0012-0.0047i & 0.0012-0.0047i & -0.0006+0.0355i \end{pmatrix}$ |
| z=0.01 | $1.10^{9}$ | $\begin{pmatrix} 0.0163-4.5566i & 0.0002-0.1119i & -0.0001-0.0007i \\ 0.0002-0.1119i & 0.0002-0.0455i & -0.0001-0.0007i \\ 0.0000-0.0007i & 0.0000-0.0007i & 0.0000-0.0009i \end{pmatrix}$ |

Table 2.1: Green Function values for different $z$. All other parameters are equal to 1

The second one is the influence of the frequency $\omega$. As $\omega$ become larger, the $G_{ij}$ coefficients get larger too.

| | | |
|---|---|---|
| $\omega = 1$ | $1.10^2$ | $\begin{pmatrix} 0.0364 - 6.0781i & -1.8483 + 7.2093i & 0.0022 - 0.0060i \\ -1.8483 + 7.2093i & 0.0077 - 0.0606i & 0.0022 - 0.0060i \\ 0.0012 - 0.0047i & 0.0012 - 0.0047i & -0.0006 + 0.0355i \end{pmatrix}$ |
| $\omega = 10$ | | $\begin{pmatrix} -0.0295 + 0.0041i & 0.0894 + 0.1282i & 0.0043 + 0.0020i \\ 0.0894 + 0.1282i & -0.0253 + 0.0056i & 0.0043 + 0.0020i \\ 0.0043 + 0.0020i & 0.0043 + 0.0020i & -0.0678 - 0.0141i \end{pmatrix}$ |
| $\omega = 10^4$ | | $\begin{pmatrix} -0.0003 - 0.0010i & 0.0001 + 0.0000i & -0.0001 - 0.0002i \\ 0.0001 + 0.0000i & -0.0003 - 0.0010i & -0.0001 - 0.0002i \\ 0.0001 - 0.0002i & -0.0001 - 0.0002i & -0.0002 - 0.0000i \end{pmatrix}$ |

Table 2.2: Green Function values for different $\omega$, with $z = 0.1$. All other parameters are equal to 1

The third one concerns the viscosity. As $\nu$ become larger, the $G_{ij}$ coefficients get larger too. And finally, we can see the influence of the stratification. As $N$ become larger, i.e the fluid is more and more stratified, the $G_{ij}$ coefficients get larger too.

## 2.4 Calculation of the Green function coefficients for the stress tensor

In section 2.2, I have shown that the Fourier Transform of each coefficients for the stress tensor is:

$$\hat{T}_{ijk}(\mathbf{k},\omega) = 8\pi \left( \frac{N^2\rho_0}{\omega|\mathbf{k}|^2} k_z \hat{G_{zk}} \delta_{ij} - \frac{\rho_0}{i\,|\mathbf{k}|^2} k_k \delta_{ij} + \frac{i}{8\pi}(k_j \hat{G_{ik}} + k_i \hat{G_{jk}}) \right) \tag{2.38}$$

To compute the $T_{ijk}$ coefficients, the principle is exactly the same than the one for the $G_{ij}$ coefficients. For instance, for $T_{xxz}$, we have:

$$\hat{T}_{xxz}(\mathbf{k},\omega) = 8\pi \left( \frac{N^2\rho_0}{\omega|\mathbf{k}|^2} k_z \hat{G_{zx}} - \frac{\rho_0}{i\,|\mathbf{k}|^2} k_z + \frac{i}{4\pi} k_x \hat{G_{xz}} \right) \tag{2.39}$$

Using the residue theorem, we find that the integral

$$\int_{z=-\infty}^{\infty} k_z \hat{G_{zx}} dk_z = \int_{z=-\infty}^{\infty} \frac{-k_x k_z^2}{-i\omega\,|\mathbf{k}|^2 + \nu\,|\mathbf{k}|^4} dk_z \tag{2.40}$$

is equal to zero (the integrand has singularities for $|\mathbf{k}|^2$=0). The principal value of the integral over $k_z$ of the second term is also equal to zero. Finally, we have:

$$T_{xxz} = \frac{1}{(2\pi)^3} \int_{\kappa=0}^{\infty} \int_{\phi=0}^{2\pi} \int_{k_z=-\infty}^{\infty} 2ik_x \hat{G_{xz}} \kappa d\kappa d\phi dk_z \tag{2.41}$$

The integrations over $\kappa$ and $\phi$ are made with the principle I have developed before. After the numerical integrations over $\kappa$ for each coefficient, we now have all the coefficients of **G** and **T**. We are now going to see how we can use my results to compute the velocity at any point.

# Chapter 3

# The boundary integral method

## 3.1 The Lorentz reciprocity relation

Let's consider $\mathbf{u}$ and $\mathbf{u'}$, two solutions of the Navier-Stokes equation, associated to the stress tensors $\sigma$ and $\sigma'$. The Lorentz reciprocity relation is:

$$\frac{\partial}{\partial x_j}(u'_i\sigma_{ij} - u_i\sigma'_{ij}) = 0 \quad (3.1)$$

This relation is essential to established the boundary-integral equation. The purpose of my work in this section is to see what becomes this relation for a very general case, i.e for viscous internal gravity waves, in a fluid in rotation and with a force $\mathbf{f}$. Using the continuity equation, we easily show that

$$\frac{\partial}{\partial x_j}(u'_i\sigma_{ij} - u_i\sigma'_{ij}) = u'_i\frac{\partial\sigma_{ij}}{\partial x_j} - u_i\frac{\partial\sigma'_{ij}}{\partial x_j} \quad (3.2)$$

We have for $(u_i, \sigma_{ij})$ and $(u'_i, \sigma'_{ij})$:

$$-i\omega u_i + \epsilon_{ijk} f \delta_{jz} u_k = \frac{\partial\sigma_{ij}}{\partial x_j} + \frac{N^2}{i\omega}\delta_{iz}\delta_{jz}u_j + f_i \quad (3.3)$$

$$-i\omega u'_i + \epsilon_{ijk} f \delta_{jz} u'_k = \frac{\partial\sigma'_{ij}}{\partial x_j} + \frac{N^2}{i\omega}\delta_{iz}\delta_{jz}u'_j + f'_i \quad (3.4)$$

remembering that $b = \frac{N^2}{i\omega}u_z$. $f$ is the rotation velocity. Multiplying (3.3) by $u'_i$ and (3.4) by $u_i$ we obtain by subtraction:

$$\epsilon_{ijk} f(\delta_{jz}u_k u'_i - \delta_{jz}u'_k u_i) = \frac{\partial\sigma_{ij}}{\partial x_j}u'_i - \frac{\partial\sigma'_{ij}}{\partial x_j}u_i + \frac{N^2}{i\omega}\delta_{iz}\delta_{jz}(u'_i u_j - u_i u'_j) + f_i u'_i - f'_i u_i \quad (3.5)$$

This gives after simplifications

$$\frac{\partial\sigma_{ij}}{\partial x_j}u'_i - \frac{\partial\sigma'_{ij}}{\partial x_j}u_i = f'_i u_i - f_i u'_i \quad (3.6)$$

The result I have found works only if we take -f in the equation (3.4). Indeed, we can't simplify the equation (3.5) if the rotation forces have the same signs. Although during my internship, I have considerated a non-rotating fluid.

## 3.2 The boundary integral equation

To derive the boundary-integral representation of a viscous flow, we will apply the Lorentz identity (3.6) that I have shown. We first select a control volume $V$ that is bounded by the closed surface $D$, as illustrated on figure (3.1). We should note that $D$ can be composed of fluid surfaces, or solid surfaces. In addition, we select a point $\mathbf{x}_0$ inside $V$. Using the divergence theorem to convert the volum integral over $V$ into a surface integral over $D$, we obtain

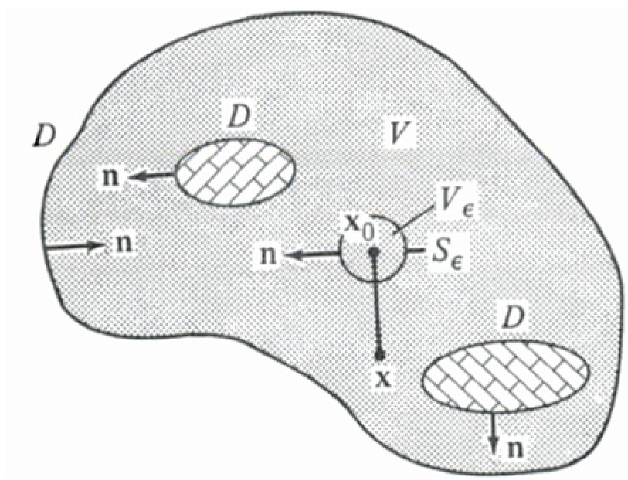


Figure 3.1: A control volume $V$ in the domain of flow

$$0 = \int_S (\sigma_{ik} u'_i - \sigma'_{ik} u_i) n_k dS + \int_V (f_i u'_i - f'_i u_i) dV \tag{3.7}$$

To develop an integral equation, we now identify $\mathbf{u'}$ with the flow due to a point force located at the point $\mathbf{x}_0$ by choosing $\mathbf{f'} = \mathbf{g'}\delta(\mathbf{x} - \mathbf{x}_0)$. We also take $\mathbf{f} = 0$. Remembering that the solutions are given by $u'_i = \frac{1}{8\pi\mu} G_{ij} g_j$ and $\sigma'_{ik} = \frac{1}{8\pi} T_{ijk} g_j$ we obtain

$$0 = \int_S (\sigma_{ik} \frac{1}{8\pi\mu} G_{ij} g_j - \frac{1}{8\pi} T_{ijk} g_j u_i) n_k dS - \int_V g_j u_j \delta(\mathbf{x} - \mathbf{x}_0) dV \tag{3.8}$$

We obtain

$$u_j(\mathbf{x}_0) g_j = \frac{1}{8\pi\eta} \int_S (\sigma_{ik} G_{ij} g_j - \mu T_{ijk} g_j u_i) n_k dS = \left[ \frac{1}{8\pi\eta} \int_S (\sigma_{ik} G_{ij} - \mu T_{ijk} u_i) n_k dS \right] g_j \tag{3.9}$$

Eliminating the arbitrary constant vector $\mathbf{g}$ we finally obtain

$$u_j(\mathbf{x}_0) = \frac{1}{8\pi\eta} \int_S \left( \sigma_{ik}(\mathbf{x}) G_{ij}(\mathbf{x}, \mathbf{x}_0) - \eta T_{ijk}(\mathbf{x}, \mathbf{x}_0) u_i(\mathbf{x}) \right) n_k dS(\mathbf{x}) \tag{3.10}$$

We should note that if the point $\mathbf{x}_0$ is located outside of the volume control $V$ we obtain

$$0 = \int_S \left(\sigma_{ik}(\mathbf{x}) G_{ij}(\mathbf{x}, \mathbf{x}_0) - \eta T_{ijk}(\mathbf{x}, \mathbf{x}_0) u_i(\mathbf{x})\right) n_k dS(\mathbf{x}) \tag{3.11}$$

It will be convenient to introduce the surface force $\mathbf{f} = \sigma \mathbf{n}$ and rewrite (3.10) in the equivalent form

$$u_j(\mathbf{x}_0) = \frac{1}{8\pi\eta} \int_S \left(f_i(\mathbf{x}) G_{ij}(\mathbf{x}, \mathbf{x}_0) - \eta T_{ijk}(\mathbf{x}, \mathbf{x}_0) u_i(\mathbf{x})\right) dS(\mathbf{x}) \tag{3.12}$$

We can finally show that (see the book *Boundary integral and singularity methods for linearized viscous flow* ) if the point $\mathbf{x}_0$ is located right on the boundary, we have

$$u_j(\mathbf{x}_0) = \frac{1}{4\pi\eta} \int_S \left(f_i(\mathbf{x}) G_{ij}(\mathbf{x}, \mathbf{x}_0) - \eta T_{ijk}(\mathbf{x}, \mathbf{x}_0) u_i(\mathbf{x})\right) dS(\mathbf{x}) \tag{3.13}$$

Thanks to the work I have done, we have a numerical expression for $G_{ij}$ and $T_{ijk}$. If we have the expression for the velocity field on the boundary of the body, and the expression of the surface force $\mathbf{f} = \sigma.\mathbf{n}$, we can compute $\mathbf{u}(\mathbf{x}_0)$ for any point $\mathbf{x}_0$. Unfortunately, I didn't have the time to compute the expression of $\mathbf{u}(\mathbf{x}_0$.

# Conclusion

Internal waves in stratified fluids are generated by oscillating objects. The comprehension of this problem became a challenge for engineers in hydrodynamics, oceanography or meteorology. Indeed, the atmosphere and the oceans are stratified fluids. During this internship, I derived a complete solution for the wave field due to a spherical or elliptical object in an incompressible stratified viscous fluid. I adapted the method for the Stokes flow exposed in the book *Boundary integral and singularity methods for linearized viscous flow* to apply it for the problem of internal waves in stratified fluids. This method, the boundary-integral method, needs the computation of the Green Functions of the flow. During these four months, I had to find a way to compute these Functions, both anatically and then numerically. This work took the major part of the internship, and that is why I did not have the time to apply the method for concrete situations. The method developed should now be applied to non-symmetrical objects.

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