Action and duality in supergravity

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0.1 Introduction

The idea of duality is that a same physical system can be equivalently described by different mathematical objects, in our case a p-form and its dual, a $D-p-2$ form. At this point, it is not yet a symmetry of the system, but it does become one when the system is described by the so-called self-twisted duality equations relating a form and its dual. The point is that $G_{duality}$ now has Noether conserved currents, which we will focus on in the sigma model framework. Then, in the second part, we will dwell on the gauging of a subgroup of $G_{duality}$. This idea is also very fundamental and a key to supergravity since the later is naturally introduced when supersymmetry is raised to a local symmetry. Moreover, forms of different rank appear when gauging a non abelian subgroup.

0.2 Action for twisted self-duality

0.2.1 Maxwell equations and differential forms

The electric and magnetic fields[1] are a lagrangian dynamical system, whose phase space can be parametrized by the *electric potential* and its time derivatives $(A_\mu, \dot{A}_\mu)$. However the co-ordinate map between physical states and the phase space is not one-to-one since the *electric field strength* defined as $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$ is sufficient to determine a physical state. A is called a *gauge* potential and the redundancy of the description will lead to gauge symmetries.

Maxwell's equations in vacuum

$$\begin{cases} \partial_\mu F^{\mu\nu} = 0 \\ \epsilon^{\mu\nu\rho\sigma}\, \partial_\nu F_{\rho\sigma} = 0 \end{cases} \tag{1}$$

can be obtained from the following action

$$\mathcal{S}[A_\mu] = \int \mathrm{d}^4x \left(-\frac{1}{4}\, F_{\mu\nu} F^{\mu\nu} \right) \tag{2}$$

A^μ is a four-vector, it can also been seen as a 1-form[2] and $F = \mathrm{d}A$, as a 2-form, in which case the equations (1) read:

$$\begin{cases} \mathrm{d}^\dagger F = 0 \\ \mathrm{d}F = 0 \end{cases} \tag{3}$$

The first one is the actual Euler-lagrange equation which states that ${}^*F$ is a closed form. The second one is also an integrability equation but is rather known as Bianchi's identity. It is an automatic consequence of the definition of the field strength.

This can be generalized for a p-differential form A in D space time dimensions

$$A \;=\; \frac{1}{p!}\, A_{\lambda_1 \cdots \lambda_p}\, \mathrm{d}x^{\lambda_1} \wedge \cdots \wedge \mathrm{d}x^{\lambda_p} \tag{4}$$

Define its field strength as[3]

$$F \;\equiv\; dA \tag{5}$$

$$\Longleftrightarrow \quad \frac{1}{(p+1)!}\, F_{\lambda_1 \cdots \lambda_{p+1}}\, \mathrm{d}x^{\lambda_1} \wedge \cdots \wedge \mathrm{d}x^{\lambda_{p+1}} = \frac{1}{p!}\, \partial_\mu A_{\lambda_1 \cdots \lambda_p}\, \mathrm{d}x^\mu \wedge \mathrm{d}x^{\lambda_1} \wedge \cdots \wedge \mathrm{d}x^{\lambda_p}$$

$$\Longleftrightarrow \quad F_{\lambda_1 \cdots \lambda_{p+1}} \;=\; (p+1)\, \partial_{[\lambda_1} A_{\lambda_2 \cdots \lambda_{p+1}]}$$

The electric and magnetic fields are then defined as its time components and spatial hodge

$$\mathcal{E}^{j_1 \cdots j_p} \;\equiv\; -\, F^{0 j_1 \cdots j_p} \tag{6}$$

$$\mathcal{B}^{j_1 \cdots j_{D-p-2}} \;\equiv\; \frac{(-1)^{(D-p-2)(p+1)}}{(p+1)!}\, \epsilon^{l_1 \cdots l_{p+1}\, j_1 \cdots j_{D-p-2}}\, F_{l_1 \cdots l_{p+1}} \tag{7}$$

[1] Equivalently, the photon, massless spin one boson

[2] Differential form, exterior derivative, Hodge star, adjoint derivative, integrability equation cf. Appendix **subsection ??** (118), (119) & (145)

[3] Complete antisymmetrisation bracket $[\cdots]$, cf. (115), Appendix

The system is still a massless boson, but the idea of spin has to be generalized[4]. Its dynamic is described by the following action

$$\mathcal{S}[A_{\lambda_1\cdots\lambda_p}] = \int \mathrm{d}^D x \left(-\frac{1}{2(p+1)!} F_{\lambda_1\cdots\lambda_{p+1}} F^{\lambda_1\cdots\lambda_{p+1}} \right) \tag{8}$$

which yields the generalization of the homogeneous Maxwell equations

$$\frac{1}{p!}\,\partial_\mu F^{\mu\lambda_2\cdots\lambda_{p+1}} = 0 \qquad \Longleftrightarrow \qquad {}^*\mathrm{d}^*F = 0 \qquad \Longleftrightarrow \qquad \mathrm{d}^*F = 0 \tag{9}$$

The other part of Maxwell equations stems from the definition (5) and properties of the exterior derivative

$$\mathrm{d}F = \mathrm{dd}A = 0 \tag{10}$$

0.2.2 Twisted self-duality

The point in rewritting Maxwell's equation with exterior derivatives of forms is that the general solution of (9) is given by Poincaré's theorem; in a simply connexe space,

$$\mathrm{d}^*F = 0 \quad \Longleftrightarrow \quad \exists\, B \text{ , a } (D-p-2)\text{-form such that } \quad {}^*F = \mathrm{d}B \tag{11}$$

Introducing $H = \mathrm{d}B$, the curvature of the *magnetic potential*[5] B, and using the hodge identity (119) we have:

$$\mathrm{d}B = {}^*\mathrm{d}A \qquad \Longleftrightarrow \qquad H = {}^*F \qquad \Longleftrightarrow \qquad F = (-1)^{(p+1)(D-1)-1\,*}H \tag{12}$$

Rewriting previous equations in a redundant way, we get the so-called *twisted*[6] self-dual equations :

$$\begin{pmatrix} F \\ H \end{pmatrix} = \begin{pmatrix} 0 & (-1)^{(p+1)(D-1)-1} \\ 1 & 0 \end{pmatrix} \cdot {}^* \begin{pmatrix} F \\ H \end{pmatrix} \tag{13}$$

This formulation of Maxwell equations gives rise to the *duality* symmetries, subset of the transformations preserving the 2×2 twist matrix[7]:

$$\begin{aligned} &\forall\, M \in G_{duality} \\ &{}^tM \cdot \begin{pmatrix} 0 & (-1)^{(p+1)(D-1)-1} \\ 1 & 0 \end{pmatrix} \cdot M \;=\; \begin{pmatrix} 0 & (-1)^{(p+1)(D-1)-1} \\ 1 & 0 \end{pmatrix} \end{aligned} \tag{14}$$

Indeed, in general $G_{duality}$ does not even mix the gauge potential and its dual, because it requires the two forms to be of the same rank, that is to say when $D = 2p+2$. However in the σ-model framwork, discussed in the next section, that the duality group will acquire a non trivial structure[8].

Note also that our construction only holds in vacuum. For a coupling with matter cf. [4]. [9]

Yet it is a further equivalent equation that will hold our interest: the spatial part[10] of (13)

$$\begin{pmatrix} F_S \\ H_S \end{pmatrix} = \begin{pmatrix} 0 & (-1)^{(p+1)(D-1)-1} \\ 1 & 0 \end{pmatrix} \cdot {}^* \begin{pmatrix} F \\ H \end{pmatrix}\Bigg|_S = \begin{pmatrix} (-1)^{(p+1)(D-1)-1 *s}\breve{H}_t \\ {}^{*s}\breve{F}_t \end{pmatrix} \tag{15}$$

0.2.3 Action for twisted self duality

This last set of equations follows from a new action which is actually the exact same quantity as (8) but expressed in terms of the spatial components of A and B.

[4]As the set of numbers that define the representation of the "litte group" S0(D-2).

[5]Dual gauge potential, cf. Appendix subsection .1.3

[6]*twisted* self-dual because we have (13) instead of simply $\begin{pmatrix} F \\ H \end{pmatrix} = {}^*\begin{pmatrix} F \\ H \end{pmatrix}$.

[7]Symplectic group $Sp(2)$ if $(p+1)(D-1)-1$ is odd, $SO(2)$ otherwise

[8]Actually, $G_{duality}$ implicitly contains gauge symmetries [4]

[9]Gaillard & Zumino in [] define $\tilde{F}_{\mu\nu} = 2\frac{\partial\mathcal{L}}{\partial F^{\mu\nu}} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\tilde{H}^{\rho\sigma}$ in the $D=4$ case when coupling with other particles are introduced.

[10]cf. Appendix, paragraph .1.6 and ...

Construction of the new lagrangian The first step is to isolate the lagrangian's dependance in $\dot{A}_S$. As in a legendre transformation, define the conjugate momenta:

$$\pi^{\lambda_1 \dots \lambda_p} \equiv \frac{\partial \mathcal{L}}{\partial\left(\partial_0 A_{\lambda_1 \dots \lambda_p}\right)} = -\frac{F^{0\,\lambda_1 \dots \lambda_p}}{p!} \tag{16}$$

However the system $\pi^{\lambda_1 \dots \lambda_p} = \Psi^{\lambda_1 \dots \lambda_p}_{A_{\mu_1 \dots \mu_p}}\left(\partial_0 A_{\rho_1 \dots \rho_p}\right)$ is not invertible; Legendre transformations are carried out only for the $\pi^{spacial\ indices}$, introducing the *hamiltonian* $\mathcal{H}$ and *constraint*[11] terms $\mathcal{G}^{k_1 \dots k_{p-1}}$:

$$\mathcal{S}[A_{j_1 \dots j_p}, \pi_{l_1 \dots l_p}, A_{0\,k_1 \dots k_{p-1}}] = \int \mathrm{d}^D x \left(\pi^{l_1 \dots l_p}\,\dot{A}_{l_1 \dots l_p} - \mathcal{H} - A_{0\,k_1 \dots k_{p-1}}\,\mathcal{G}^{k_1 \dots k_{p-1}}\right) \tag{17}$$

$$\begin{aligned} &\mathcal{H}(A_{j_1 \dots j_p}, \pi_{l_1 \dots l_p}, \partial_k A_{j_1 \dots j_p}, \partial_k \pi_{l_1 \dots l_p}) && (18)\\ &\quad = \frac{1}{2}\left(\frac{1}{p!}\,\pi_{j_1 \cdots j_p}\,\pi^{j_1 \cdots j_p} + \frac{1}{(p+1)!}\,F_{j_1 \cdots j_{p+1}}\,F^{j_1 \cdots j_{p+1}}\right)\\ &\quad = \frac{1}{2}\left(\frac{1}{p!}\,\mathcal{E}_{j_1 \cdots j_p}\mathcal{E}^{j_1 \cdots j_p} + \frac{1}{(D-p-2)!}\,\mathcal{B}_{j_1 \cdots j_{D-p-2}}\mathcal{B}^{j_1 \cdots j_{D-p-2}}\right)\\ &\mathcal{G}^{k_1 \dots k_{p-1}} = -\,\partial_i\,\pi^{i\,k_1 \cdots k_{p-1}} && (19) \end{aligned}$$

The spatial part of the $(D-p-2)$-form B is now introduced by solving $\mathcal{G}^{k_1 \dots k_{p-1}} = 0$, which happens to be the time part of ${}^*\mathrm{d}{}^*F = 0$ [12]

$$\begin{aligned} \pi &\equiv \frac{-1}{p!}\,(-1)^{D-p-1}\,(-1)^{(p+1)(D-1)-1}\ {}^{*s}\mathrm{d}_S B_S = \frac{(-1)^{pD}}{p!}\ {}^{*s}H_S\\ \iff \quad \pi^{l_1 \dots l_p} &= \frac{(-1)^{pD}\ \epsilon^{k_{p+1} \dots k_{D-1}\,l_1 \dots l_p}}{p!\,(D-p-2)!}\,\partial_{k_{p+1}} B_{k_{p+2} \dots k_{D-1}}\\ &= \frac{\epsilon^{l_1 \dots l_p\,k_{p+1} \dots k_{D-1}}}{p!\,(D-p-1)!}\,H_{k_{p+1} \dots k_{D-1}} \end{aligned} \tag{20}$$

We finally obtain[13]

$$\mathcal{S}[A_{k_1 \cdots k_p}, B_{j_1 \cdots j_{D-p-2}}] = \int \mathrm{d}^D x \left(\frac{\epsilon^{k_1 \cdots k_p\,j_1 \cdots j_{D-p-1}}}{p!\,(D-p-1)!} H_{j_1 \cdots j_{D-p-1}} \dot{A}_{k_1 \cdots k_p} - \mathcal{H}\right) \tag{21}$$

with now

$$\begin{aligned} &\mathcal{H}(A_{j_1 \dots j_p}, B_{j_1 \dots j_p}, \partial_k A_{j_1 \dots j_p}, \partial_k B_{j_1 \dots j_p}) && (22)\\ &= \frac{1}{2}\left(\frac{1}{(D-p-1)!}\,H_{j_1 \cdots j_{D-p-1}}H^{j_1 \cdots j_{D-p-1}} + \frac{1}{(D-p-2)!}\,\mathcal{B}_{j_1 \cdots j_{D-p-2}}\mathcal{B}^{j_1 \cdots j_{D-p-2}}\right) \end{aligned}$$

Euler-lagrange equations Minimizing the action

$$\begin{aligned} \delta S \overset{!}{=} 0 = \int \Bigg(&\underbrace{\left[\frac{\partial \mathcal{L}}{\partial A_{l_1 \dots l_p}} - \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_{l_1 \dots l_p})}\right)\right]}_{I} \delta A_{l_1 \dots l_p}(x)\\ &+ \underbrace{\left[\frac{\partial \mathcal{L}}{\partial B_{l_1 \cdots l_{D-p-2}}} - \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu B_{l_1 \cdots l_{D-p-2}})}\right)\right]}_{II} \delta B_{l_1 \cdots l_{D-p-2}}(x)\Bigg)\,\mathrm{d}^D x \end{aligned} \tag{23}$$

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[11] cf. Appendix subsection .1.4 & (147), idea and explicit calculus

[12] Watch out for the factors that take into account the definition of π in (16) as $-\frac{1}{p!}\tilde{F}_t$, (143) and (13).

[13] a gauge invariant expression can be obtained by adding a divergence $\partial_{k_1}\left(\frac{\epsilon^{k_1 \dots k_p\,j_1 \dots j_{D-p-1}}}{(p-1)!\,(D-p-1)!} H_{j_1 \dots j_{D-p-1}}\,A_{k_2 \dots k_p}\right)$

yields the following equations of motion:

$$\begin{aligned} I &= -\,\partial_0 \left(\frac{\epsilon^{l_1 \dots l_p\, j_{p+1} \dots j_{D-1}}}{p!\,(D-p-1)!}\, H_{j_{p+1}\dots j_{D-1}} \right) \;-\; \partial_i \left(-\,\frac{2}{2\,p!}\, F^{\,i\, l_1 \dots l_p} \right) \\ &= \frac{1}{p!}\,\partial_i \left(F^{\,i\, l_1 \dots l_p} \;-\; \frac{\epsilon^{l_1 \dots l_p\, i\, j_{p+2} \dots j_{D-1}}}{(D-p-2)!}\, \partial_0\, B_{j_{p+2}\dots j_{D-1}} \right) \;\overset{!}{=}\; 0 \end{aligned} \tag{24}$$

$$II \;=\; \frac{-1}{(D-p-2)!}\,\partial_i \left(\frac{\epsilon^{j_1 \dots j_p\, i\, l_1 \dots l_{D-p-2}}}{p!}\, \dot{A}_{j_1 \dots j_p} \;-\; H^{\,i\, l_1 \dots l_{D-p-2}} \right) \;\overset{!}{=}\; 0 \tag{25}$$

For $0 < p < D-2$, those equations take the form of integrability equations, Poincaré's lemma will introduce A and B's time part:

$$\begin{aligned} & F^{\,i\, l_1 \dots l_p} - (-1)^{p+(p+1)(D-p-2)}\, \frac{\epsilon^{j_{p+2}\dots j_{D-1}\, i\, l_1 \dots l_p}}{(D-p-2)!}\, \partial_0\, B_{j_{p+2}\dots j_{D-1}} \\ & \qquad \equiv \; (-1)^{(p+1)(D-1)}\, \frac{(D-p-2)\,\epsilon^{j_{p+2}\dots j_{D-1}\, i\, l_1 \dots l_p}}{(D-p-2)!}\, \partial_{j_{p+2}}\, B_{0\, j_{p+3}\dots j_{D-1}} \\ \Longleftrightarrow \quad & F^{\,i\, l_1 \dots l_p} \;=\; (-1)^{(p+1)(D-1)-1}\, \frac{\epsilon^{j_{p+2}\dots j_{D-1}\, i\, l_1 \dots l_p}}{(D-p-2)!} \left(\underbrace{\dot{B}_{j_{p+2}\dots j_{D-1}} - (D-p-2)\,\partial_{j_{p+2}}\, B_{0\, j_{p+3}\dots j_{D-1}}}_{\equiv H_{0\, j_{p+2}\dots j_{D-1}}} \right) \end{aligned} \tag{26}$$

$$H^{\,i\, l_1 \dots l_{D-p-2}} \;=\; \frac{\epsilon^{k_1 \dots k_p\, i\, l_1 \dots l_{D-p-2}}}{p!} \left(\underbrace{\dot{A}_{k_1 \dots k_p} - p\, \partial_{k_1} A_{0\, k_2 \dots k_p}}_{\equiv F_{0\, k_1 \dots k_p}} \right) \tag{27}$$

Those are indeed the equations of (15).

0.3 Sigma model

The 0-form case of the previous model is the massless Klein Gordon action. The sigma model generalizes it for N scalar fields ϕ^a introducing a non trivial metric G_{ab} on the "field manifold" $\mathcal{M}$ where a point $\boldsymbol{\varphi}$ has coordinates $(\phi^a)_{a=1\dots N}$.

$$S[\phi^a] = \int \mathrm{d}^D x \left(-\frac{1}{2} (\partial_\mu \phi^a)(\partial^\mu \phi^b) G_{ab}(\boldsymbol{\varphi}) \right) \tag{28}$$

with the following equation of motion

$$\partial_\mu \left(\partial^\mu \phi^b G_{cb} \right) - \frac{1}{2} (\partial_\mu \phi^a)(\partial^\mu \phi^b) \frac{\partial G_{ab}}{\partial \phi^c} = 0 \tag{29}$$

The duality symmetry came up because the equation of motion could be expressed as a closed form condition, which is not the case anymore. Fortunately an equivalent equation with the appropriate form can be written.

The sigma model lagrangian is very similar to that of the geodesic equation, but the coordinates ϕ^a are now functions of the x^μ instead of being functions of a single parameter λ (field vs parametric curve). Nevertheless generators of possible symmetries will be Killing fields[14] K_α in the same way:

$$\delta\phi^c = \lambda^\alpha K^c{}_\alpha \tag{30}$$

$$\begin{aligned} \delta\mathcal{S} &\equiv \int \mathrm{d}^D x \left(\left[\frac{\partial \mathcal{L}}{\partial \phi^c} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^c)} \right) \right] \delta\phi^c \; - \partial_\mu \underbrace{\left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^c)} \delta\phi^c \right]}_{\lambda^\alpha J^\mu_{\alpha,\, Noether}} \right) \;\overset{!}{=}\; 0 \\ &= \int \mathrm{d}^D x \left(\left[\partial_\mu \left(\partial^\mu \phi^b G_{cb} \right) - \frac{1}{2} (\partial_\mu \phi^a)(\partial^\mu \phi^b) \frac{\partial G_{ab}}{\partial \phi^c} \right] \lambda^\alpha K^c{}_\alpha - \partial_\mu \left[(\partial^\mu \phi^b)\, G_{cb}\, \lambda^\alpha K^c{}_\alpha \right] \right) \\ &= -\frac{\lambda^\alpha}{2} \int \mathrm{d}^D x\, (\partial_\mu \phi^a)(\partial^\mu \phi^b) \underbrace{\left(G_{cb} \frac{\partial K^c{}_\alpha}{\partial \phi^a} + G_{ac} \frac{\partial K^c{}_\alpha}{\partial \phi^b} + \frac{\partial G_{ab}}{\partial \phi^c} K^c{}_\alpha \right)}_{\equiv \mathcal{L}_{K_\alpha} G_{ab} = 0} \end{aligned} \tag{31}$$

[14]cf. Appendix subsection .1.5, Lie derivative and Killing fields.

The current defined by Killing fields is actually Noether's current:

$$\mathcal{J}^\mu{}_\alpha \;\equiv\; K^c{}_\alpha(\Phi)\,(\partial^\mu\phi^b)\,G_{cb}(\Phi) \;=\; \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^c)}\,\delta_\alpha\phi^c \;=\; J^\mu_{\alpha,\,Noether} \tag{32}$$

Its conservation is equivalent to the Euler-lagrange equation (29):

$$\begin{aligned}\partial_\mu\,\mathcal{J}^\mu{}_\alpha = \;& \underbrace{(\partial_\mu\phi^a)(\partial^\mu\phi^b)\,\frac{\partial K^c{}_\alpha}{\partial\phi^a}\,G_{cb}}_{\text{Symmetry of } G_{ab}\text{, def. Killing fields}} \;+ \underbrace{\Box\,\phi^b\,K^c{}_\alpha\,G_{cb} + (\partial_\mu\phi^a)(\partial^\mu\phi^b)\,K^c{}_\alpha\,\frac{\partial G_{cb}}{\partial\phi^a}} \;=\; 0\\ =\;& -\frac{1}{2}(\partial_\mu\phi^a)(\partial^\mu\phi^b)\,\frac{\partial G_{ab}}{\partial\phi^c}\,K^c{}_\alpha \;+\; \partial_\mu\left(\partial^\mu\phi^b G_{cb}\right)K^c{}_\alpha\end{aligned} \tag{33}$$

Solving the latter introduces a magnetic potentials $B_{\boldsymbol{\alpha}}$ for each Killing field

$$\mathcal{J}^\mu{}_{\boldsymbol{\alpha}} \;=\; \frac{(-1)^D}{(D-2)!}\,\epsilon^{\mu\,\nu_1\cdots\nu_{D-1}}\,\partial_{\nu_1}B_{\nu_2\cdots\nu_{D-1}\boldsymbol{\alpha}} \;=\; \frac{(-1)^D}{(D-1)!}\,\epsilon^{\mu\,\nu_1\cdots\nu_{D-1}}\,H_{\nu_1\cdots\nu_{D-1}\boldsymbol{\alpha}} \tag{34}$$

We see that noether's currents play the role of the potentials' field strengths, but they are not in the same number.

0.3.1 Action for twisted self-dual equations

We now have to figure out the lagrangian that will yield the twisted self-dual equations

$$\begin{cases} \mathcal{J}_{k\boldsymbol{\alpha}} \;=\; \dfrac{(-1)^D\,\epsilon^{j_1\ldots j_{D-2}}{}_k}{(D-2)!}\,H_{0\,j_1\ldots j_{D-2}\,\boldsymbol{\alpha}} \\ H_{l_1\ldots l_{D-1}\,\boldsymbol{\alpha}} \;=\; \epsilon_{l_1\ldots l_{D-1}}\,\mathcal{J}_{0\boldsymbol{\alpha}} \end{cases} \tag{35}$$

Unfortunately, our previous reasoning fails in the $p=0$ case since (35) are not solutions of integrability equations.

We simply proceed by trial, with

$$\mathcal{L} = \frac{\epsilon^{k_1\cdots k_{D-1}}}{(D-1)!}\,\hat{L}^\alpha{}_a(\boldsymbol{\varphi})\,H_{k_1\cdots k_{D-1}\alpha}\,\dot{\phi}^a - \frac{1}{2}\left(\frac{1}{(D-1)!}\,H^{k_1\cdots k_{D-1}}{}_\alpha\,H_{k_1\cdots k_{D-1}\,\beta}\,\hat{\eta}^{\alpha\beta}(\boldsymbol{\varphi}) + (\partial_i\phi^a)(\partial^i\phi^b)\,\hat{\mathcal{G}}_{ab}(\boldsymbol{\varphi})\right) \tag{36}$$

where $\hat{L}^\alpha{}_a(\boldsymbol{\varphi})$, $\hat{\eta}^{\alpha\beta}(\boldsymbol{\varphi})$ and $\hat{\mathcal{G}}_{ab}(\boldsymbol{\varphi})$ are undetermined functions. This lead to the following Euler-lagrange equations:

$$\begin{aligned}&\frac{\partial\mathcal{L}}{\partial A_{l_1\ldots l_p}} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu A_{l_1\ldots l_p})}\right) \;\stackrel{!}{=}\; 0\\ &=\;\left(\frac{\epsilon^{k_1\ldots k_{D-1}}}{(D-1)!}\,H_{k_1\ldots k_{D-1}\,\alpha}\,\dot{\phi}^a\,\frac{\partial\,\hat{L}^\alpha{}_a}{\partial\phi^c} - \frac{H^{k_1\cdots k_{D-1}}{}_\alpha\,H_{k_1\cdots k_{D-1}\,\beta}}{2\,(D-1)!}\,\frac{\partial\,\hat{\eta}^{\alpha\beta}}{\partial\phi^c} - \frac{(\partial_i\phi^a)(\partial^i\phi^b)}{2}\,\frac{\partial\,\hat{\mathcal{G}}_{ab}}{\partial\phi^c}\right)\\ &\quad-\left(\frac{\epsilon^{k_1\ldots k_{D-1}}}{(D-1)!}\,\partial_0\left(\hat{L}^\alpha{}_c\,H_{k_1\ldots k_{D-1}\,\alpha}\right) - \partial_i\left(\partial^i\phi^b\,\hat{G}_{cb}\right)\right)\end{aligned} \tag{37}$$

$$\begin{aligned}&\frac{\partial\mathcal{L}}{\partial B_{l_1\cdots l_{D-p-2}}} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu B_{l_1\cdots l_{D-p-2}})}\right) \;\stackrel{!}{=}\; 0\\ &=\; -\frac{(D-1)}{(D-1)!}\,\partial_i\left(\epsilon^{i\,k_2\ldots k_{D-1}}\,\hat{L}^\gamma{}_a\,\phi^a - H^{i\,k_2\ldots k_{D-1}}{}_\beta\,\hat{\eta}^{\gamma\beta}\right)\end{aligned} \tag{38}$$

Those are not the equations we are looking for but they will look more alike after contracting (37) with $K^c{}_\delta$ and solving (38):

$$
\begin{aligned}
&\partial_i \left(K^c{}_\delta \, (\partial^i \phi^b) \, \hat{G}_{cb} \; - \; \frac{\epsilon^{i\,k_2 \dots k_{D-1}}}{(D-2)!} \, K^c{}_\delta \, \hat{L}^\alpha{}_c \, (\partial_0 \, B_{k_2 \dots k_{D-1}\,\alpha}) \right) \\
&\quad + ({}^{*s}H_{S\,\alpha}) \, K^c{}_\delta \, \Big(\underbrace{\dot{\phi}^a \, (\partial_c \, \hat{L}^\alpha{}_a) - \dot{\phi}^d \, (\partial_d \, \hat{L}^\alpha{}_c)}_{III} \Big) - \; \frac{(\partial_i \phi^a)(\partial^i \phi^b)}{2} \, \Big(\overbrace{K^c{}_\delta \, \partial_c \, \hat{\mathcal{G}}_{ab} + 2 \, \hat{G}_{cb} \, \partial_a \, K^c{}_\delta}^{\mathcal{L}_{K_\delta} \hat{\mathcal{G}}_{ab}} \Big) \\
&\quad + \frac{\epsilon^{i\,k_2 \dots k_{D-1}}}{(D-2)!} \, \Big(\partial_i \phi^d \, \partial_d \, (K^c{}_\delta \, \hat{L}^\alpha{}_c) \Big) \, (\partial_0 \, B_{k_2 \dots k_{D-1}\,\alpha}) - \; \frac{K^c{}_\delta}{2} \, ({}^{*s}H_{S\,\alpha})({}^{*s}H_{S\,\beta}) \, (\partial_c \, \hat{\eta}^{\alpha\beta}) \\
= &\quad 0
\end{aligned}
\tag{39}
$$

$$
\begin{aligned}
H^{k_1 \dots k_{D-1}}{}_\beta \, \hat{\eta}^{\gamma\beta} - \epsilon^{k_1 \dots k_{D-1}} \, \hat{L}^\gamma{}_a \, \dot{\phi}^a \quad &= \quad \epsilon^{k_1 \dots k_{D-1}} f^\gamma(t) \\
\iff \qquad ({}^{*s}H_{S\,\beta}) \, \hat{\eta}^{\gamma\beta} - \, \hat{L}^\gamma{}_a \, \dot{\phi}^a \quad &= \quad f^\gamma(t)
\end{aligned}
\tag{40}
$$

Instead of being an integrability equation, (38) is simply the cancellation of $\vec{1} \cdot \vec{\mathrm{grad}}$. Redifining the scalar field in order to absorb this disrupting f^γ might be done by devising an action of the time dependant function group on the $\mathcal{M}$, $\boldsymbol{\varphi} \to g(t)^\gamma \cdot \boldsymbol{\varphi}$ such that

$$
\hat{L}^\gamma{}_a \left(g(t)^\gamma \cdot \boldsymbol{\varphi} \right) \, \partial_0 \, \left(g(t)^\gamma \cdot \boldsymbol{\varphi} \right)^a \; = \; f(t)^\gamma + g(t)^\gamma \tag{41}
$$

This is a non-trivial problem and we just put f^γ at zero, assuming ϕ is null at infinity..

0.3.2 Group manifold

Nevertheless, when assuming the manifold of scalar fields $\mathcal{M}$ has an additional group structure, previous equation (39) greatly simplify .

Algebraic structures $G \equiv \mathcal{M}$ is now a Lie group, with generators X_α spanning a Lie algebra $\mathfrak{g}$ fitted with a Cartan-Killing form $\eta^{\alpha\beta} = \mathrm{Tr}(X_\alpha X_\beta)$[15].

This algebra acts on the group[16] either on the left or on the right:

$$
\forall V \, \in \, G \quad \begin{cases} \delta_\alpha V \equiv \, \delta_\alpha \phi^a \, \partial_a \, V \; \equiv \; X_\alpha \cdot V & : \text{left action} \\ \tilde{\delta}_\alpha V \equiv \, \tilde{\delta}_\alpha \phi^a \, \partial_a \, V \; \equiv \; V \cdot X_\alpha & : \text{right action} \end{cases} \tag{42}
$$

and by definition of an action, $\delta_\alpha V \in G$ so we can write:

$$
\begin{cases} X_\alpha = \; \delta_\alpha V \cdot V^{-1} \\ X_\alpha = \; V^{-1} \cdot \tilde{\delta}_\alpha V \end{cases} \tag{43}
$$

We define a left and right current

$$
\mathbf{J}_\mu \;\; \equiv \;\; \partial_\mu \, V \cdot V^{-1} \; = \; (\partial_\mu \phi^a) \, (\partial_a \, V \cdot V^{-1}) \; \equiv \; (\partial_\mu \phi^a) \, L_a{}^\alpha \, X_\alpha \tag{44}
$$

$$
\tilde{\mathbf{J}}_\mu \;\; \equiv \;\; V^{-1} \cdot \partial_\mu \, V \; = \; (\partial_\mu \phi^a) \, (V^{-1} \cdot \partial_a \, V) \; \equiv \; (\partial_\mu \phi^a) \, \tilde{L}_a{}^\alpha \, X_\alpha \tag{45}
$$

such that σ-model Lagrangian can be written

$$
\mathcal{L} \;\; = \;\; -\frac{1}{2} \, \mathrm{Tr} \left(\mathbf{J}_\mu \cdot \mathbf{J}^\mu \right) \;\; = \;\; -\frac{1}{2} \, \mathrm{Tr} \left(\tilde{\mathbf{J}}_\mu \cdot \tilde{\mathbf{J}}^\mu \right) \tag{46}
$$

The two quantitites $(\partial_a \, V \cdot V^{-1})$ and $(V^{-1} \cdot \partial_a \, V)$ belong to $\mathfrak{g}$[17] and are conjugated:

$$
\mathbf{J}_\mu = V \cdot \tilde{\mathbf{J}}_\mu \cdot V^{-1} \tag{47}
$$

[15]Cartan-Killing form is non degenerate only for semi-simple groups (Cartan) and it depends on the representation, here it is the adjoint representation of the algebra on itself. $\eta^{\alpha\beta}$ is proportional to $f^{\alpha\delta}{}_\gamma \, f^{\beta\gamma}{}_\delta$ where f are the structure constants

[16]Equivalently, the group acts on itself or on the algebra. Also equivalent to the algebra acts on the group. Or that the lie algebra is embedded in a bigger algebra

[17]For any G-valued parametrized curve $V(t)$, $\frac{d[V(t) \cdot V^{-1}(0)]}{dt}$ and $\frac{d[V^{-1}(0) \cdot V(t)]}{dt} \in T_e G \equiv \mathfrak{g}$

They have been expanded in the generators basis in the right hand side of (44) and (45) with

$$L^\alpha{}_a \;\equiv\; \mathrm{Tr}\Big((\partial_a V\cdot V^{-1})\cdot X_\beta\Big)\,\eta^{\alpha\beta} \;=\; \Big((\partial_a V\cdot V^{-1})^\gamma\,\eta_{\gamma\delta}\,X^\delta_\beta\Big)\,\eta^{\alpha\beta} \tag{48}$$

$$\tilde L^\alpha{}_a \;\equiv\; \mathrm{Tr}\Big((V^{-1}\cdot\partial_a V)\cdot X^\alpha\Big) \;=\; \Big((V^{-1}\cdot\partial_a V)^\gamma\,\eta_{\gamma\delta}\,X^\delta_\beta\Big)\,\eta^{\alpha\beta} \tag{49}$$

Identifying (46) with the lagrangian of (28), we have:

$$\begin{aligned} G_{ab} &= \mathrm{Tr}\Big((\partial_a V\cdot V^{-1})\cdot(\partial_b V\cdot V^{-1})\Big) \;=\; \mathrm{Tr}\Big((V^{-1}\cdot\partial_a V)\cdot(V^{-1}\cdot\partial_b V)\Big)\\ &= L^\alpha{}_a\,L^\beta{}_b\,\mathrm{Tr}\Big(X_\alpha\cdot X_\beta\Big) \;=\; L^\alpha{}_a\,L^\beta{}_b\,\eta_{\alpha\beta} \;=\; \tilde L^\alpha{}_a\,\tilde L^\beta{}_b\,\eta_{\alpha\beta} \end{aligned} \tag{50}$$

and hence[18]

$$\begin{cases} (L^{-1})^c{}_\alpha \;=\; L^\beta{}_b\,\eta_{\alpha\beta}\,G^{bc}\\ (\tilde L^{-1})^c{}_\alpha \;=\; \tilde L^\beta{}_b\,\eta_{\alpha\beta}\,G^{bc} \end{cases} \tag{51}$$

The system's symmetry group is the product of the field manifold itself $G\times G$[19]; the generators are Killing fields: for $\delta S = \lambda^\alpha\,S\cdot X_\alpha$ and $\tilde\delta S = \lambda^\alpha\,X_\alpha\cdot S$ we have

$$\delta\,\mathbf{J}_\mu \;=\; \lambda^\alpha\,\delta_\alpha\Big(\partial_\mu V\cdot V^{-1}\Big) \;=\; \lambda^\alpha\Big(\partial_\mu X_\alpha + [X_\alpha,\mathbf{J}_\mu]\Big) \;=\; \lambda^\alpha\,[X_\alpha,\mathbf{J}_\mu] \tag{52}$$

$$\tilde\delta\,\tilde{\mathbf{J}}_\mu \;=\; \lambda^\alpha\,\tilde\delta_\alpha\Big(V^{-1}\cdot\partial_\mu V\Big) \;=\; \lambda^\alpha\Big(\partial_\mu\tilde X_\alpha + [\tilde{\mathbf{J}}_\mu,X_\alpha]\Big) \;=\; \lambda^\alpha\,[\tilde{\mathbf{J}}_\mu,X_\alpha] \tag{53}$$

$$\Longrightarrow\quad \delta\mathcal{L} \;=\; -\,\mathrm{Tr}\Big((\delta\,\mathbf{J}_\mu)\cdot\mathbf{J}^\mu\Big) \;=\; -\,\mathrm{Tr}\Big(\lambda^\alpha\,[X_\alpha,\mathbf{J}_\mu]\cdot\mathbf{J}^\mu\Big) \;=\; 0 \tag{54}$$

$$=\; -\,\mathrm{Tr}\Big((\delta\,\tilde{\mathbf{J}}_\mu)\cdot\tilde{\mathbf{J}}^\mu\Big) \;=\; -\,\mathrm{Tr}\Big(\lambda^\alpha\,[\tilde{\mathbf{J}}_\mu,X_\alpha]\cdot\tilde{\mathbf{J}}^\mu\Big) \;=\; 0 \tag{55}$$

Writing K_α the action of a generator on the coordinates we have:

$$\begin{array}{ccc} \begin{cases} K^a{}_\alpha\,\partial_a V = X_\alpha\cdot V\\ \tilde K^a{}_\alpha\,\partial_a V = S\cdot X_\alpha \end{cases} & \Longleftrightarrow & \begin{cases} K^a{}_\alpha\,\partial_a V\cdot V^{-1} = X_\alpha\\ K^a{}_\alpha\,V^{-1}\cdot\partial_a V = X_\alpha \end{cases}\\ \begin{cases} K^a{}_\alpha\,L^\beta{}_a\,X_\beta = X_\alpha\\ \tilde K^a{}_\alpha\,\tilde L^\beta{}_a\,X_\beta = X_\alpha \end{cases} & \Longleftrightarrow & \begin{cases} K^a{}_\alpha = (L^{-1})^a{}_\alpha\\ \tilde K^a{}_\alpha = (\tilde L^{-1})^a{}_\alpha \end{cases} \end{array} \tag{56}$$

This is the link between the different currents we have defined:

$$\begin{aligned} \mathcal{J}_{\mu\,\alpha} &= K^a{}_\alpha\,(\partial_\mu\phi^c)\,G_{ac}\\ \mathbf{J}_\mu &= V^{-1}\cdot\partial_\mu V \;=\; L^\beta{}_b\,(\partial_\mu\phi^b)\,X_\beta\\ K^a{}_\alpha &= L^\beta{}_b\,\eta_{\alpha\beta}\,G^{ba},\quad \text{with (51)}\\ \Longrightarrow\quad \mathcal{J}_{\mu\,\alpha} &= L^\beta{}_b\,\eta_{\alpha\beta}\,G^{ba}\,(\partial_\mu\phi^c)\,G_{ac}\\ &= \mathrm{Tr}\Big(\mathbf{J}_\mu\cdot X_\alpha\Big) \end{aligned} \tag{57}$$

Going back to (39) We now have

$$\hat L^\alpha{}_a \;=\; (K^{-1})^\alpha{}_a = L^\alpha{}_a \quad\Longrightarrow\quad \partial\hat L^\alpha{}_a = -L^\alpha{}_b\,(\partial\,K^b{}_\beta)\,L^\beta{}_a \tag{58}$$

$$\hat G_{ab} \;=\; G_{ab} \;=\; L_{\alpha\,a}\,L^\alpha{}_b \tag{59}$$

$$\hat\eta^{\alpha\beta} \;=\; \eta^{\alpha\beta} \qquad \text{constant and invertible} \tag{60}$$

[18] Left and right inverse are equal when they exist, with $\det(A) = \det({}^tA)$.

[19] One group is acting on the left, the other one on the right. A right action is equivalent to a left action of the inverse group

Most of the terms of (39) disappear and we are left with III, in which we replace $({}^{*s}H_{S\,\alpha})$ using (40)

$$\begin{aligned}
&\eta_{\alpha\gamma}\left(L^{\gamma}{}_{a}\,\dot\phi^{a}+f^{\gamma}(t)\right)\;K^{c}{}_{\delta}\left(\dot\phi^{a}\,(\partial_c\,L^{\alpha}{}_{a})-\dot\phi^{d}\,(\partial_d\,L^{\alpha}{}_{c})\right)\\
=\;&\underbrace{\dot\phi^{a}\dot\phi^{b}}_{symmetric}\;K^{c}{}_{\delta}\left(\underbrace{L_{\alpha\,a}\,(\partial_c\,L^{\alpha}{}_{b})}_{\frac12\,\partial_c G_{ab}}\;-\;L_{\alpha\,a}\quad\underbrace{(\partial_b\,L^{\alpha}{}_{c})}_{-\,L^{\alpha}{}_{d}\,(\partial_b K^{d}{}_{\beta})\,L^{\beta}{}_{c},(58)}\right)+K^{c}{}_{\delta}\,f_{\alpha}\,\dot\phi^{b}\left((\partial_c\,L^{\alpha}{}_{b})-(\partial_b\,L^{\alpha}{}_{c})\right)\\
=\;&\dot\phi^{a}\dot\phi^{b}\left(\underbrace{\frac12\,K^{c}{}_{\delta}\,(\partial_c\,G_{ab})\;+\;G_{ad}\,(\partial_b K^{d}{}_{\beta})\,\delta^{\beta}_{\delta}}_{\mathcal{L}_{K_\delta}\,G_{ab}\,=\,0}\right)+K^{c}{}_{\delta}\,f_{\alpha}\,\dot\phi^{b}\left((\partial_c\,L^{\alpha}{}_{b})-(\partial_b\,L^{\alpha}{}_{c})\right)
\end{aligned}\tag{61}$$

With (58), (40) can be written:

$$\dot\phi^{b}=K^{b}{}_{\gamma}\left(({}^{*s}H_{S\,\beta})\,\eta^{\gamma\beta}-f^{\gamma}\right)\tag{62}$$

then (61) becomes:

$$\begin{aligned}
&K^{c}{}_{\delta}\,f_{\alpha}\,K^{b}{}_{\gamma}\left(({}^{*s}H_{S\,\beta})\,\eta^{\gamma\beta}-f^{\gamma}\right)\left(2!\;\partial_{[c}\,L^{\alpha}{}_{b]}\right)\\
=\;&2!\;f_{\alpha}\,\left(({}^{*s}H_{S\,\beta})\,\eta^{\gamma\beta}-f^{\gamma}\right)K^{c}{}_{\delta}\,K^{b}{}_{\gamma}\;\partial_{[c}\,\mathrm{Tr}\left(\partial_{b]}S\cdot S^{-1}\cdot X^{\alpha}\right)\\
=\;&f_{\alpha}\,\left(({}^{*s}H_{S\,\beta})\,\eta^{\gamma\beta}-f^{\gamma}\right)\;\mathrm{Tr}\left(\;\underbrace{[X_{\delta},X_{\gamma}]\cdot X^{\alpha}}_{f_{\delta\gamma}{}^{\alpha},\ \alpha-\gamma\ \text{antisymmetry}}\;\right)
\end{aligned}\tag{63}$$

We finally get

$$\partial_i\left[(\partial^{i}\phi^{b})\,K^{c}{}_{\delta}\,G_{cb}-\frac{\epsilon^{i\,j_2\cdots\,j_{D-1}}}{(D-2)!}\,K^{c}{}_{\delta}\,L^{\alpha}{}_{c}(\partial_0\,B_{j_2\cdots j_{D-1}\,\alpha})+\frac{\epsilon^{i\,j_2\cdots\,j_{D-1}}}{(D-2)!}\,B_{j_2\cdots\,j_{D-1}\,\beta}\,\eta^{\gamma\beta}\,f^{\alpha}\,f_{\delta\gamma}{}^{\alpha}\right]=0\tag{64}$$

Thus, under certain assumptions, we have an equivalent first order equation.

Algebraic calculus Making the most of the Lie group structure, previous calculus can be carried out much more easily. Let's define

$$\begin{aligned}
({}^{*s}\mathbf{H}_S)\equiv\;&({}^{*s}H_{S\,\alpha})\,X^{\alpha}\;=\;\frac{\epsilon^{j_1\,\dots\,j_{D-1}}}{(D-1)!}\,H_{j_1\,\dots\,j_{D-1}\,\alpha}\,X^{\alpha}\\
\mathbf{f}\equiv\;&f^{\gamma}\,X_{\gamma}
\end{aligned}\tag{65}$$

Lagrangian (36) now reads for a right current[20]:

$$\mathcal{L}=\mathrm{Tr}\left[\tilde{\mathbf{J}}_0\cdot({}^{*s}\mathbf{H_S})-\frac12\left(\frac{\mathbf{H}^{j_1\dots\,j_{D-1}}\cdot\mathbf{H}_{j_1\dots\,j_{D-1}}}{(D-1)!}+\tilde{\mathbf{J}}^{i}\cdot\tilde{\mathbf{J}}_i\right)\right]\tag{66}$$

Instead of taking the variation of each scalar field ϕ^a separately, a variation of a group element V can be directly written. Furthermore, since the field strengths belong to a vector space with a non degenerate metric, namely the Cartan Killing form, there is no need to write a more general variation than

$$\delta V(x)\;=\;V\cdot\Sigma(x)\qquad\Longleftrightarrow\qquad\Sigma\;=\;V^{-1}\cdot\delta V\qquad\text{with }\Sigma\in\mathfrak{g}\tag{67}$$

to obtain the Euler-lagrange equations[21]:

$$\begin{aligned}
\frac{\delta\mathcal{S}}{\delta V}\,\delta V\;&=\;\mathrm{Tr}\left[\left([\tilde{\mathbf{J}}_0,\Sigma]+\partial_0\,\Sigma\right)\cdot({}^{*s}\mathbf{H}_S)-\left([\tilde{\mathbf{J}}_i,\Sigma]+\partial_i\,\Sigma\right)\cdot\tilde{\mathbf{J}}^{i}\right]\\
&=\;\mathrm{Tr}\left[\left(\left([({}^{*s}\mathbf{H}_S),\tilde{\mathbf{J}}_0]-\partial_0({}^{*s}\mathbf{H}_S)\right)+\partial_i\tilde{\mathbf{J}}^{i}\right)\cdot\Sigma\right]\\
\Longrightarrow\quad\partial_i\,\mathbf{J}^{i}-&\left[\partial_i\left(\frac{\epsilon^{i\,j_2\dots\,j_{D-1}}}{(D-2)!}\,\partial_0\,\mathbf{B}_{j_2\dots\,j_{D-1}}\right)-[({}^{*s}\mathbf{H}_S),\tilde{\mathbf{J}}_0]\right]\;=\;0
\end{aligned}\tag{68}$$

[20]Recall that the trace/Cartan-Killing form is defined for the adjoint of the Lie algebra on itself.

[21]Variation of $\mathbf{J}$, cf. Appendix **subsection** .1.6

$$\frac{\delta \mathcal{S}}{\delta B_{k_1\ldots k_{D-2}\,\alpha}}\,\delta B_{k_1\ldots k_{D-2}\,\alpha} \;=\; \mathrm{Tr}\left[\left(-\frac{\epsilon^{i\,k_1\ldots k_{D-2}}}{(D-2)!}\,\partial_i\tilde{\mathbf{J}}_0+\frac{1}{(D-2)!}\,\partial_i\,\mathbf{H}^{i\,k_1\ldots k_{D-2}}\right)\cdot\delta B_{k_1\ldots k_{D-2}\,\alpha}\,X^\alpha\right]$$

$$\Longrightarrow\quad \partial_i\left(-\,\epsilon^{i\,k_1\ldots k_{D-2}}\,\tilde{\mathbf{J}}_0+\mathbf{H}^{i\,k_1\ldots k_{D-2}}\right)\;=\;0 \tag{69}$$

We rederive (41) and (64) as expected, or more precisely, the former are the trace of these

$$\begin{cases}\partial_i\left[\tilde{\mathbf{J}}^i-\dfrac{\epsilon^{i\,j_2\ldots j_{D-1}}}{(D-2)!}\,(\partial_0\,\mathbf{B}_{j_2\ldots j_{D-1}})\;-\;f^\gamma\left[\dfrac{\epsilon^{i\,j_2\ldots j_{D-1}}}{(D-2)!}\,\mathbf{B}_{j_2\ldots j_{D-1}}\,,\;X_\gamma\right]\right]=0\\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad -\,\tilde{\mathbf{J}}_0+{}^{*s}\mathbf{H}_S=f^\gamma(t)\,X_\gamma\end{cases} \tag{70}$$

Solving the first equation introduces $B_{\boldsymbol{\alpha}}$'s time components

$$\tilde{\mathbf{J}}^i-\frac{\epsilon^{i\,j_2\ldots j_{D-1}}}{(D-2)!}\,(\partial_0\,\mathbf{B}_{j_2\ldots j_{D-1}})-\left[\frac{\epsilon^{i\,j_2\ldots j_{D-1}}}{(D-2)!}\,\mathbf{B}_{j_2\ldots j_{D-1}}\,,\;\mathbf{f}\right]\equiv-\frac{\epsilon^{i\,j_2\ldots j_{D-1}}}{(D-3)!}\,(\partial_{j_2}\,\mathbf{B}_{0\,j_3\ldots j_{D-1}}) \tag{71}$$

Putting $\mathbf{f}=0$ gives the sought equations, but let's see what modifications $\mathbf{f}^\gamma$ brings to the original equations.

"Second order equations" Equation (29) is a second order equations for the 0-form fields ϕ, but a first order equation in terms of its field strength. Since it only involves one field and not its dual, it is also referred to as a one potential equation.

Equation for J Inject (70) and (71) in Bianchi's identity for $\mathbf{H}$:

$$\mathrm{d}\mathbf{H}\;=\;0\;=\;{}^*\mathrm{d}\mathbf{H}\qquad\Longleftrightarrow\qquad \partial_0({}^{*s}\mathbf{H}_S)\;=\;\frac{(-1)^D}{(D-2)!}\,\epsilon^{k_1\ldots k_{D-1}}\,\partial_{k_1}\,\mathbf{H}_{0\,k_2\ldots k_{D-1}}\;=\;(-1)^D\,\partial_i\,({}^{*s}\breve{\mathbf{H}}_t)^i$$

$${}^{*s}\mathbf{H}_S\;=\;\mathbf{f}(t)+\tilde{\mathbf{J}}_0$$

$$({}^{*s}\breve{\mathbf{H}}_t)^i\;=\;(-1)^{D-2}\left(\tilde{\mathbf{J}}^i\;-\;\left[\frac{\epsilon^{i\,j_2\ldots j_{D-1}}}{(D-2)!}\,\mathbf{B}_{j_2\ldots j_{D-1}}\,,\;\mathbf{f}\right]\right)$$

$$\Longrightarrow\qquad \partial^0\left(\mathbf{f}+\tilde{\mathbf{J}}_0\right)\;=\;\partial_i\left(\tilde{\mathbf{J}}^i\;-\;\left[\frac{\epsilon^{i\,j_2\ldots j_{D-1}}}{(D-2)!}\,\mathbf{B}_{j_2\ldots j_{D-1}}\,,\;\mathbf{f}\right]\right)$$

$$\Longrightarrow\qquad \partial_\mu\,\tilde{\mathbf{J}}^\mu\;=\;\dot{\mathbf{f}}\;+\;\left[({}^{*s}\mathbf{H}_S)\,,\;\mathbf{f}\right]\;=\;\dot{\mathbf{f}}\;+\;\left[\tilde{\mathbf{J}}_0\,,\;\mathbf{f}\right] \tag{72}$$

This is not an integrability equation.

Equation for H Let's now calculate the generalization of Bianchi's identity for the field strength $\tilde{\mathbf{J}}_\mu\,\in\,\mathfrak{g}$

$$\begin{aligned}\partial_{[\mu}\,\tilde{\mathbf{J}}_{\nu]}=\;\partial_{[\mu}\left(V^{-1}\cdot\partial_{\nu]}V\right)\;&=\;-\,V^{-1}\cdot(\partial_{[\mu}V)\cdot V^{-1}\cdot(\partial_{\nu]}V)\;+\;V^{-1}\cdot(\partial_{[\mu}\partial_{\nu]}V)\\ =\;-\;\frac{1}{2}\left[\tilde{\mathbf{J}}_\mu,\,\tilde{\mathbf{J}}_\nu\right]&\end{aligned} \tag{73}$$

After some calculation – cf. Appendix 166 & 168 – we can write a similar equation for a redefined field strength:

$$\mathsf{H}^{\mu_1\ldots\mu_{D-1}}\;=\;\begin{cases}\mathbf{H}^{0\,m_2\ldots m_{D-1}}\;+\;[\,\mathbf{B}^{m_2\ldots m_{D-1}}\,,\,\mathbf{f}\,]\\ \mathbf{H}^{m_1\ldots m_{D-1}}\;+\;\epsilon^{m_1\ldots m_{D-1}}\,\mathbf{f}\end{cases} \tag{74}$$

In both case, $\mathbf{f}$ appears and not even as a current that could be interpreted as an interaction with matter.

$$\partial_\mu\,\mathsf{H}^{\mu\,\lambda_2\ldots\lambda_{D-1}}\;=\;-\;\frac{1}{2}\left[({}^*\mathsf{H})_\nu\,,\,({}^*\mathsf{H})_\rho\right]\,\epsilon^{\nu\,\rho\,\lambda_2\ldots\lambda_{D-1}} \tag{75}$$

0.3.3 Coset manifold

In a more general case, the manifold has a coset structure G/H where H is the maximal compact subgroup[22]. This means that our previous description with G as the phase space is redundant; two points $p_1,\ p_2\ \in G$ such that $p_1 \cdot (p_2)^{-1}\ \in\ H$ now correspond to the same state.

We assume[23] G's Lie algebra can be decomposed as $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$ with $\mathfrak{h} \equiv \mathrm{Lie}(H)$, such that

$$\forall\, \mathbf{h}_1, \mathbf{h}_2\ \in \mathfrak{h} \text{ and } \mathbf{p}_1, \mathbf{p}_2\ \in \mathfrak{p} \qquad \begin{array}{l} [\mathbf{h}_1, \mathbf{h}_2] \in \mathfrak{h} \\ [\mathbf{p}_1, \mathbf{h}_1] \in \mathfrak{p} \\ [\mathbf{p}_1, \mathbf{p}_2] \in \mathfrak{h} \end{array} \tag{76}$$

The lagrangian of the right or left coset only involves the projection of the right or left group current on $\mathfrak{p}$ but there are now indeed, two different lagrangians:

$$\forall\, V \in G \qquad \mathbf{J}_\mu \ = \ V^{-1} \cdot \partial_\mu V \ = \ \underset{\in \mathfrak{h}}{\mathbf{Q}_\mu} + \underset{\in \mathfrak{p}}{\mathbf{P}_\mu} \tag{77}$$

One potential formulation

$$\mathcal{L} \ = \ -\ \frac{1}{2}\,\mathrm{Tr}\left(\mathbf{P}_\mu \cdot \mathbf{P}^\mu\right) \tag{78}$$

For a general transformation $\delta V \equiv V \cdot \Sigma(x)$ with $\Sigma \in \mathfrak{p}$ we have

$$\delta \mathbf{J}_\mu \ = \ [\mathbf{J}_\mu, \Sigma(x)] \ + \ \partial_\mu \Sigma(x) \tag{79}$$

$$\delta \mathbf{P}_\mu \ \equiv \ \delta \mathbf{J}_\mu\big|_{\mathfrak{p}} \ = \ [\mathbf{Q}_\mu, \Sigma(x)] \ + \ \partial_\mu \Sigma(x) \tag{80}$$

$$\begin{aligned} \frac{\delta \mathcal{S}}{\delta \mathbf{P}_\nu}\, \delta \mathbf{P}_\nu \ &= \ \mathrm{Tr}\left[\left([\mathbf{Q}_\nu, \Sigma(x)] \ + \ \partial_\nu \Sigma(x)\right) \cdot \mathbf{P}^\nu\right] \\ &= \ \mathrm{Tr}\left[\left([\mathbf{P}^\nu, \mathbf{Q}_\nu] \ - \ \partial_\nu \mathbf{P}^\nu\right) \cdot \Sigma(x) \ + \ \partial_\nu \left(\Sigma(x) \cdot \mathbf{P}^\nu\right)\right] \end{aligned} \tag{81}$$

$$\Longrightarrow \qquad \mathcal{D}_\mu \mathbf{P}^\mu \ \equiv \ \partial_\mu \mathbf{P}^\mu + [\mathbf{Q}_\mu, \mathbf{P}^\mu] \ = \ 0 \tag{82}$$

[24]

This last equation can be re-written with an exterior derivative

$$\partial_\mu \left(V \cdot \mathbf{P}^\mu \cdot V^{-1}\right) \ = \ V \cdot \left(\partial_\mu \mathbf{P}^\mu \ + \ [\mathbf{J}_\mu, \mathbf{P}^\mu]\right) \cdot V^{-1} \ = \ V \cdot \mathcal{D}_\mu \mathbf{P}^\mu \cdot V^{-1} \ = \ 0 \tag{83}$$

Upon solving it, we get

$$V \cdot \mathbf{P}^\mu \cdot V^{-1} \ = \ -\ \frac{\epsilon^{\mu\,\lambda_1 \dots \lambda_{D-1}}}{(D-2)!}\, \partial_{\lambda_1} B_{\lambda_2 \dots \lambda_{D-1}} \ \equiv \ (-1)^D\, ({}^*\mathbf{H})^\mu \tag{84}$$

$$\mathbf{P}^\mu \ = \ (-1)^D\, V^{-1} \cdot ({}^*\mathbf{H})^\mu \cdot V \ \in \ \mathfrak{p} \tag{85}$$

Two potentials formulation

$$\mathcal{L} = \mathrm{Tr}\left[\mathbf{P}_0 \cdot ({}^{*s}\mathbf{H}_\mathbf{S}) - \frac{1}{2}\left(\frac{\mathbf{H}^{j_1 \dots j_{D-1}} \cdot \mathbf{H}_{j_1 \dots j_{D-1}}}{(D-1)!} + \mathbf{P}^i \cdot \mathbf{P}_i\right)\right] \tag{86}$$

[22] Note that if H is normal than the coset is a group. A group is said to be simple if it has no trivial normal subgroup. In this case it is also semisimple, which means that is has no abelian normal subgroup.

[23] [2, **4.10** Semisimple and abelian Lie algebra] When G is reductive, $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}] + \mathfrak{z}$, where $e^{\mathfrak{z}}$ commutes with all element of the group, or else we only have a semi direct sum of two algebra, Levi decomposition of $\mathfrak{g}$

[24] Covariant derivative, cf. next section

0.4 Gauge

Gauging a subgroup H of the global symmetry group is to impose the system to be invariant under an action that has been made local. This requires to modify the derivation, introducting *gauge fields* A_μ^α[25]

$$\mathcal{D}_\mu \cdot \equiv \partial_\mu \cdot - A_\mu^\alpha(x)\, t_\alpha \cdot \tag{87}$$

[26] such that for any vector Ψ^a and its *covariant derivative* $\mathcal{D}_\mu \Psi^a$ transform in the same way under H[27]. The t_α are H's generators in the representation of the field on which the derivatives act. This condition imposes the gauge fields to transform in a very particular representation of H:

$$\begin{aligned} \delta\Psi^a &\equiv \lambda^\alpha(x)\,(t_\alpha)^a{}_b\,\Psi^b \\ \delta\left(\mathcal{D}_\mu\Psi^a\right) &\overset{!}{=} \lambda^\alpha(x)\,(t_\alpha)^a{}_b\left(\mathcal{D}_\mu\Psi^b\right) \qquad (88)\\ &= \delta\big(\partial_\mu\Psi^a\big) - g\,\delta\big(A_\mu^\alpha\big)\,(t_\alpha)^a{}_b\,\Psi^b - g\,A_\mu^\alpha\,(t_\alpha)^a{}_b\,\delta\big(\Psi^b\big) \\ \Longrightarrow \qquad \delta\big(A_\mu^\alpha\big) &= \frac{1}{g}\,\mathcal{D}_\mu\lambda^\alpha \qquad (89) \end{aligned}$$

In the last line λ^α lives in the algebra's adjoint representation. Hint: Important ingredient since Supergravity arises when supersymmetry is made local.

However gauging a subgroup of the global symmetry group has different consequences according to the structure of the phase space.

0.4.1 Gauging a subgroup

Abelian gauge group Defining gauge invariant quantities, we have formally identical relations[28]

$$\tilde{\mathbf{J}}_\mu \equiv V^{-1}\cdot\mathcal{D}_\mu V = V^{-1}\cdot\partial_\mu V - A_\mu^\alpha\, t_\alpha \tag{90}$$

$$\mathbf{H}_{\lambda_1\ldots\lambda_{D-1}} \equiv (D-1)\,\mathcal{D}_{[\lambda_1}\mathbf{B}_{\lambda_2\ldots\lambda_{D-1}]} = (D-1)\left(\partial_{[\lambda_1}\mathbf{B}_{\lambda_2\ldots\lambda_{D-1}]} - A^\alpha_{[\lambda_1}\,\mathbf{B}_{\lambda_2\ldots\lambda_{D-1}]}{}^\beta\left[t_\alpha\,,\,X_\beta\right]\right) \tag{91}$$

$$\mathcal{L} = \mathrm{Tr}\left[\mathbf{J}_0\cdot({}^{*s}\mathbf{H_S}) - \frac{1}{2}\left(\frac{\mathbf{H}^{j_1\ldots j_{D-1}}\cdot\mathbf{H}_{j_1\ldots j_{D-1}}}{(D-1)!} + \mathbf{J}^i\cdot\mathbf{J}_i\right)\right] \tag{92}$$

Since $h(x) \in \mathfrak{h}$ now generates symmetries – Appendix, (156) – a general variation of V can be written $\delta V = V\cdot\Sigma(x)$, with $\Sigma \in \mathfrak{p}$. Calculations are the same as in (68) and (69), but with covariant derivatives and a projection on $\mathfrak{p}$

$$\frac{\delta\mathcal{S}}{\delta V} = \left[\mathcal{D}_i\left(\mathbf{J}^i - \frac{\epsilon^{i\,j_2\ldots j_{D-1}}}{(D-2)!}\,\mathcal{D}_0\,\mathbf{B}_{j_2\ldots j_{D-1}}\right) + \left[({}^{*s}\mathbf{H}_S)\,,\,\mathbf{J}_0 + A_0^\alpha t_\alpha\right]\right]_{\mathfrak{p}} \overset{!}{=} 0 \tag{93}$$

$$\frac{\delta\mathcal{S}}{\delta\mathbf{B}_{k_1\ldots k_{D-2}}} = \left[\frac{1}{(D-2)!}\,\mathcal{D}_i\left(\epsilon^{i\,k_1\ldots k_{D-2}}\,\tilde{\mathbf{J}}_0 + \mathbf{H}^{i\,k_1\ldots k_{D-2}}\right)\right]_{\mathfrak{p}} \overset{!}{=} 0 \tag{94}$$

But there are also additional Euler-lagrange equations coming from varying the action with respect to A_μ

$$\frac{\delta\mathcal{S}}{\delta A_0} = \left[\mathbf{J}^i + \frac{\epsilon^{i\,j_2\ldots j_{D-1}}}{(D-2)!}\left(\left[\mathbf{J}_0, \mathbf{B}_{j_2\ldots j_{D-1}}\right] + \left[\mathbf{B}_{j_2\ldots j_{D-1}}, ({}^{*s}\mathbf{H}_S)\right]\right)\right]_{\mathfrak{h}} \overset{!}{=} 0 \tag{95}$$

$$\frac{\delta\mathcal{S}}{\delta A_i} = ({}^{*s}\mathbf{H})\Big|_{\mathfrak{h}} \overset{!}{=} 0 \tag{96}$$

(97)

[25] Not to get confused with gauge potential. Idea of connection?

[26] When the t_α are normalized, a multiplying constant g, a.k.a. the coupling constant is added in the second term.

[27] $\mathcal{D}_\mu\cdot$ looks like a "representation morphism"

[28] $A_\mu^\alpha\, t_\alpha$ is invariant under conjugation, is it valid if H is not the center of G?

Embedding tensors for non abelian gauge subgroup In order to have a systematic gauging procedure for any subgroup H, we introduce a projector $\Theta : \mathfrak{g} \to \mathfrak{g}$ such that $\mathrm{Im}(\Theta) = \mathfrak{h}$ is the gauge group's algebra.

The image of G's $N = \dim(\mathfrak{g})$ generators are N linearly dependant vectors of $\mathfrak{h}$:

$$t_{\mathcal{M}} \equiv (\Theta_{\mathcal{M}}{}^{\alpha} X_{\alpha})_{\mathcal{M}=1\ldots N} \tag{98}$$

By analogy with any Lie algebra's adjoint representation[29], we define $t_{\mathcal{MN}}{}^{\mathcal{P}}$ and its Lie bracket, a.k.a. *closure constraints*[30] since $\mathfrak{h}$ is a subalgebra.

$$[\, t_{\mathcal{M}} \,,\, t_{\mathcal{N}} \,] = -t_{[\mathcal{MN}]}{}^{\mathcal{P}}\, t_{\mathcal{P}} \tag{99}$$

Indeed $t_{\mathcal{MN}}{}^{\mathcal{P}}$ is not necessarily antisymmetric, but its symmetric part is projected out

$$\Theta_{\mathcal{P}}{}^{\alpha}\, t_{(\mathcal{MN})}{}^{\mathcal{P}} = 0 \tag{100}$$

As a consequence of their linear dependance, Jacobi's identity isn't satisfied

$$t_{\mathcal{MQ}}{}^{\mathcal{S}}\, t_{\mathcal{NP}}{}^{\mathcal{Q}} + t_{\mathcal{NQ}}{}^{\mathcal{S}}\, t_{\mathcal{PM}}{}^{\mathcal{Q}} + t_{\mathcal{PQ}}{}^{\mathcal{S}}\, t_{\mathcal{MN}}{}^{\mathcal{Q}} = 3\, t_{[\mathcal{MN}}{}^{\mathcal{Q}}\, t_{\mathcal{P}]\mathcal{Q}}{}^{\mathcal{S}} = 2\, t_{(\mathcal{Q}[\mathcal{M})}{}^{\mathcal{S}}\, t_{\mathcal{NP}]}{}^{\mathcal{Q}} \tag{101}$$

unless projected on $\mathfrak{h}$. Moreover the field strength and the action of the gauge group on the gauge field are modified, introducing a 2-form $B_{\mu\nu}$ and a 1-form Ξ_{μ}:

$$\begin{aligned} \mathcal{F}_{\mu\nu}{}^{\mathcal{M}} &\equiv \partial_{\mu} A_{\nu}{}^{\mathcal{M}} - \partial_{\nu} A_{\mu}{}^{\mathcal{M}} + t_{[\mathcal{NP}]}{}^{\mathcal{M}} A_{\mu}{}^{\mathcal{N}} A_{\nu}{}^{\mathcal{P}} + t_{[\mathcal{NP}]}{}^{\mathcal{M}} B_{\mu\nu}{}^{\mathcal{NP}} && (102) \\ \delta A_{\mu}{}^{\mathcal{M}} &\equiv \mathcal{D}_{\mu}\lambda^{\mathcal{M}} - t_{(\mathcal{NP})}{}^{\mathcal{M}}\, \Xi_{\mu}{}^{\mathcal{NP}} && (103) \end{aligned}$$

$\mathcal{F}$ now transforms covariantly under $\lambda^{\mathcal{P}} t_{\mathcal{P}} \in \mathfrak{h}$ and $t_{[\mathcal{NP}]}{}^{\mathcal{M}}\, \Xi_{\mu}{}^{\mathcal{NP}}$:

$$\delta\, \mathcal{F}_{\mu\nu}{}^{\mathcal{M}} = -\lambda^{\mathcal{P}}\, t_{\mathcal{PN}}{}^{\mathcal{M}}\, \mathcal{F}_{\mu\nu}{}^{\mathcal{N}} \tag{104}$$

Since under a combined gauge transformation of $\mathcal{F}$, $\delta\, B_{\mu\nu}{}^{\mathcal{MN}}$ and $2\, A_{\mu}{}^{\mathcal{M}} \delta\, A_{\nu}{}^{\mathcal{N}}$ contribute a term in $t_{(\mathcal{PN})}{}^{\mathcal{M}}$ a "generic covariant"/combined transformation for B is defined

$$\Delta B_{\mu\nu}{}^{\mathcal{MN}} \equiv \delta\, B_{\mu\nu}{}^{\mathcal{MN}} - 2\, A_{\mu}{}^{\mathcal{M}} \delta\, A_{\nu}{}^{\mathcal{N}} = 2\, \mathcal{D}_{[\mu}\, \Xi_{\nu]}{}^{\mathcal{MN}} - 2\, \lambda^{\mathcal{M}}\, \mathcal{F}_{\mu\nu}{}^{\mathcal{N}} + \cdots \tag{105}$$

And in the ellipses is added a higher rank field associated to another gauge transformation. $B_{\mu\nu}$'s field strength is at its turn modified to be covariant under the new gauge, by adding a higher rank form $C_{\mu\nu\rho}$ giving rise to a whole *hierarchy of tensors.*

0.4.2 Gauging in a coset

This structure is built such that H generates symmetries, even when acting locally.

One potential formulation

$$\mathcal{L} = -\frac{1}{2}\,\mathrm{Tr}\left[\mathbf{P}_{\mu} \cdot \mathbf{P}^{\mu}\right] \tag{106}$$

$h(x)$ generates symmetries

$$\begin{aligned} \delta\mathbf{P}_{\mu} &\equiv \delta\mathbf{J}_{\mu}|_{\mathfrak{p}} = [\mathbf{P}_{\mu}, h(x)] && (107) \\ \Longrightarrow \quad \frac{\delta\mathcal{S}}{\delta V}\,\delta V &= -\mathrm{Tr}\left([\mathbf{P}_{\mu}, h(x)] \cdot \mathbf{P}^{\mu}\right) && (108) \end{aligned}$$

Euler-Lagrange equations:

There is no additional degrees of freedom, the covariant derivation is defined with[31]

$$\begin{aligned} \mathcal{D}_{\mu} &\equiv \partial_{\mu} - \mathbf{Q}_{\mu} \\ \delta\mathbf{Q}_{\mu} &= -\partial_{\mu}\, h(x) + [\, h(x)\,,\, \mathbf{Q}_{\mu}] \,, \quad \text{with } h \in \mathfrak{h} \end{aligned} \tag{109}$$

[29] Explicit matrices, cf. Appendix (171)

[30] This is obtained by imposing that Θ be invariant under H or its algebra. $\Theta_{\mathcal{M}}{}^{\alpha}$ is naturally a 2-tensor transforming under changes of basis of its domain and codomain, the $GL(\mathfrak{g}) \times GL(\mathfrak{g})$ group. However, in our case there are many restrictions. To begin with, the group involved is $G \times H$ in their adjoint representation, but there are further requirements coming from supersymmetry or else. Now, it seems that imposing a transformation matrix to be invariant under a well chosen subgroup of $GL(\mathfrak{g})$ will force it to stablelize $\mathfrak{h}$. Does Θ look like idendity on a subspace, and 0 elsewhere, as a projector should?

[31] [5]

0.5 $D = 3$ case

Corresponds to a particular solution of a higher dimensionnal equation, dimensional reduction. Forms and vectors are dual, $D-1-1=1$. Notations:

Chern Simons term Topological term because does require a metric.

$$\mathcal{L} \;=\; \mathrm{Tr}\left[\frac{\epsilon^{ij}}{2}\,\mathbf{J}_0\cdot\mathbf{F}_{ij} + \frac{\epsilon^{\mu\nu\rho}}{2}\,A_\mu{}^{\mathcal{M}}\,\Theta_{\mathcal{M}\mathcal{N}}\cdot\left(\partial_\nu\,A_\rho{}^{\mathcal{N}} + \frac{1}{3}\,X_{[RS]}{}^{N}A_\nu{}^{S}\,A_\rho{}^{R}\right)\right] \tag{110}$$

The equations of motion are obtained by adding to previous equations (110) Euler-lagrange's equations for the Chern-Simon's action . Chern-Simon's extra contribution to the action is

$$\begin{aligned}\delta\mathcal{S}_{CS} =\; & \epsilon^{\alpha\mu\nu}\left[\Theta_{MN}\,\partial_\mu\,A_\nu{}^{N} + \frac{1}{6}\left(\Theta_{MN}X^{N}{}_{[RS]}\,A_\mu{}^{R}\,A_\nu{}^{S} + 2\Theta_{RN}\,X^{N}{}_{[MS]}A_\mu{}^{S}\,A_\nu{}^{R}\right)\right] \\ =\; & -\,\frac{1}{2}\,\epsilon^{\alpha\mu\nu}\mathcal{F}_{\mu\nu}{}^{\mathcal{N}}\,\Theta_{\mathcal{M}\mathcal{N}}\end{aligned} \tag{111}$$

0.6 Conclusion

The action for self-twisted duality promoted duality rotations to a symmetry for our free massless D dimensional bosons. Then gauging a non-abelian subgroup introduced coupling with a "hierarchy" of different rank antisymmetric tensors.

Acknowledgement I would like to thank Henning Samtleben for giving me the time to build my own understanding of things, and for the precision and clarity of his explanations.

.1 Appendix

.1.1 Differential forms

Signature of permutations ϵ In D dimension, you may see permutation of all D indices, or permutation of the $(D-1)$ spatial indices only

$$\begin{aligned}\epsilon_{\mu_1\cdots\mu_D} &\equiv \epsilon\left(\begin{pmatrix} 1 & \cdots & D \\ \mu_1 & & \mu_D\end{pmatrix}\right)\\ \epsilon_{j_1\cdots j_{D-1}} &\equiv \epsilon\left(\begin{pmatrix} 1 & \cdots & D-1 \\ j_1 & & j_{D-1}\end{pmatrix}\right)\end{aligned} \tag{112}$$

They are related through

$$\epsilon_{k_1\cdots k_q=0\cdots k_D} = (-1)^{q+1}\epsilon_{k_1\cdots\hat{k}_q\cdots k_D} \tag{113}$$

where $\hat{k}_q$ means that this indice is omitted.

Differential forms An *inner product*[32] in a vector space maps vectors and vectors of the dual space. q-contravariant p covariant $p+q$-linear forms are built with tensor product. Finally p-diffenrential forms are obtained by antisymmetrizing a p-covariant form. However we are implicitly talking about p-form fields, the components are functions of space-time. A basis of p-form is given by $(\mathrm{d}x^{\lambda_1}\wedge\cdots\wedge\mathrm{d}x^{\lambda_p})_{0\leq\lambda_1<\lambda_2<\cdots<\lambda_p\leq D}$. An integrability equation for a differential form ω is[33]

$$\mathrm{d}\omega = 0 \tag{114}$$

Antisymmetrisation bracket Let T be a p-covariant tensor,

$$T_{[\lambda_1\cdots\lambda_p]} \equiv \frac{1}{p!}\sum_{\sigma\in\mathcal{S}_p}\epsilon(\sigma)T_{\sigma(\lambda_1)\cdots\sigma(\lambda_p)} = \frac{\epsilon\begin{pmatrix} 1 & \cdots & p \\ \lambda_1 & & \lambda_p\end{pmatrix}}{p!}\sum_{\sigma\in\mathcal{S}_p}\epsilon(\sigma)T_{\sigma(1)\cdots\sigma(p)} \tag{115}$$

It is normalized such that

$$\epsilon^{\lambda_1\cdots\lambda_p\,\mu_{p+1}\cdots\mu_D}T_{[\lambda_1\cdots\lambda_p]} = \epsilon^{\lambda_1\cdots\lambda_p\,\mu_{p+1}\cdots\mu_D}T_{\lambda_1\cdots\lambda_p}$$

ϵ identity

$$\frac{1}{p!}\epsilon^{\mu_1\,\ldots\,\mu_p\,\nu_{p+1}\,\ldots\,\nu_D}\,\epsilon_{\mu_1\,\ldots\,\mu_p\,\rho_{p+1}\,\ldots\,\rho_D} = (D-p)!\,\delta^{\nu_{p+1}\,\ldots\,\nu_D}_{[\rho_{p+1}\,\ldots\,\rho_D]} \tag{116}$$

Left hand side is non zero only if there is a bijective map from $\{\nu_{p+1},\ldots,\nu_D\}$ to $\{\rho_{p+1},\ldots,\rho_D\}$ and when it is non zero, it has the same absolute value, no matter what $(D-p)$-tuplet $(\nu_{p+1},\ldots,\nu_D) = (\rho_{p+1},\ldots,\rho_D)$ is chosen, depending only on the number p of contracted indices. This is expressed by the antisymmetrized delta. We just have to figure out the right factor between the two terms, which is achieved by evaluating both side in the particular case where $\{\nu_{p+1},\ldots,\nu_D\} = \{\rho_{p+1},\ldots,\rho_D\}$, and a sum is performed. Left hand side gives:

$$l.h.s._{\text{summed}} = \frac{D!}{p!}$$

[32] e.g. a metric, symmetric bilinear non degenerate form
[33] Exterior derivative cf. (5)

And right hand side

$$
\begin{aligned}
r.h.s._{\text{summed}} &= (D-p)! \sum_{\nu_{p+1},\dots,\nu_D=0}^{D-1} \frac{1}{(D-p)!} \left(\sum_{\sigma \in \mathcal{S}_{D-P}} \epsilon(\sigma) \prod_{j=1}^{D-p} \delta^{\nu_{p+j}}_{\nu_{p+\sigma(j)}} \right) \\
&= \sum_{f \in [\![0,D-1]\!]^{[\![1,D-p]\!]}} \left(\sum_{\sigma \in \mathcal{S}_{D-P}} (\sigma) \prod_{j=1}^{D-p} \delta^{f(j)}_{f(\sigma(j))} \right) \\
&= \sum_{\substack{X \subset [\![0,D-1]\!] \\ |X|=D-p}} \underbrace{\sum_{\substack{f \in X^{[\![1,D-p]\!]} \\ f \text{ bijectif}}} \det M(f)}_{\forall f,\, \det M(f)=1} \\
&= \binom{D}{D-p} \times (D-p)!
\end{aligned}
$$

In a curved space there's an additional $g = \det(g_{\mu\nu})$ factor:

$$
\frac{1}{g\,p!}\, \epsilon^{\mu_1 \dots \mu_p\, \nu_{p+1} \dots \nu_D}\, \epsilon_{\mu_1 \dots \mu_p\, \rho_{p+1} \dots \rho_D} = (D-p)!\, \delta^{\nu_{p+1} \dots \nu_D}_{[\rho_{p+1} \dots \rho_D]} \tag{117}
$$

.1.2 Hodge star

It is an isomorphism[34] mapping the differential p-forms vector space with that of the $(D-p)$ differential forms, defined by:

$$
\begin{aligned}
{}^*\omega \;&\equiv \frac{\sqrt{|g|}}{(D-p)!}\frac{1}{p!}\, \epsilon^{\lambda_1 \cdots \lambda_p}{}_{\mu_{p+1} \cdots \mu_D}\, \omega_{\lambda_1 \cdots \lambda_p}\, \mathrm{d}x^{\mu_{p+1}} \wedge \cdots \wedge \mathrm{d}x^{\mu_D} \\
\Longleftrightarrow \quad {}^*\omega_{\mu_1 \cdots \mu_{D-p}} &= \frac{\sqrt{|g|}}{p!}\, \epsilon^{\lambda_1 \cdots \lambda_p}{}_{\mu_1 \cdots \mu_{D-p}}\, \omega_{\lambda_1 \cdots \lambda_p}
\end{aligned} \tag{118}
$$

In our flat spacetime case, the determinant of the metric is ± 1. One has to pay attention to the order of the indices.

As a consequence of (117), we have the following identity

$$
\begin{aligned}
{}^{**}\omega &= {}^* \left(\frac{\sqrt{|g|}}{(D-p)!}\frac{1}{p!}\, \epsilon^{\lambda_1 \cdots \lambda_p}{}_{\mu_{p+1} \cdots \mu_D}\, \omega_{\lambda_1 \cdots \lambda_p}\, \mathrm{d}x^{\mu_{p+1}} \wedge \cdots \wedge \mathrm{d}x^{\mu_D} \right) \\
&= \frac{|g|}{p!(D-p)!\,p!}\, \epsilon^{\mu_{p+1} \cdots \mu_D}{}_{\nu_1 \cdots \nu_p}\, \epsilon^{\lambda_1 \cdots \lambda_p}{}_{\mu_{p+1} \cdots \mu_D}\, \omega_{\lambda_1 \cdots \lambda_p}\, \mathrm{d}x^{\nu_1} \wedge \cdots \wedge \mathrm{d}x^{\nu_p} \\
&= \frac{|g|}{(p!)^2} \frac{(-1)^{p(D-p)}}{(D-p)!}\, \epsilon_{\mu_{p+1} \cdots \mu_D \nu_1 \cdots \nu_p}\, \epsilon^{\mu_{p+1} \cdots \mu_D \lambda_1 \cdots \lambda_p}\, \omega_{\lambda_1 \cdots \lambda_p}\, \mathrm{d}x^{\nu_1} \wedge \cdots \wedge \mathrm{d}x^{\nu_p} \\
&= \frac{|g|}{(p!)^2} (-1)^{p(D-p)} \frac{p!}{g}\, \delta^{\lambda_1 \dots \lambda_p}_{[\nu_1 \dots \nu_p]}\, \omega_{\lambda_1 \cdots \lambda_p}\, \mathrm{d}x^{\nu_1} \wedge \cdots \wedge \mathrm{d}x^{\nu_p} \\
&= \frac{(-1)^{p(D-p)+\epsilon}}{p!}\, \omega_{\nu_1 \cdots \nu_p}\, \mathrm{d}x^{\nu_1} \wedge \cdots \wedge \mathrm{d}x^{\nu_p} \\
&= (-1)^{p(D-p)+\epsilon}\, \omega
\end{aligned} \tag{119}
$$

where $g = (-1)^{\epsilon}|g|$

.1.3 Dual potential

The electric and magnetic potentials (and their time derivatives) $(A_{\lambda_1 \dots \lambda_p}, \dot{A}_{\lambda_1 \dots \lambda_p})$, $(B_{\mu_1 \dots \mu_{D-p-2}}, \dot{B}_{\mu_1 \dots \mu_{D-p-2}})$ are a possible coordinate system on the phase space.

Those two descriptions have the same systems of equations:

$$
\text{electric potential: } \begin{cases} \mathrm{d}^*F = 0 \\ \mathrm{d}F = 0 \end{cases}, \text{ magnetic potential: } \begin{cases} \mathrm{d}^*H = 0 \\ \mathrm{d}H = 0 \end{cases} \tag{120}
$$

[34]Intuitively we see that $\binom{D}{p} = \binom{D}{D-p}$

Poincaré's theorem (11) proves the equivalence between the two systems of (120), that is exactly (13):

$$\begin{cases} \mathrm{d}^*F \;=\; 0 & \Longleftrightarrow \quad {}^*F \overset{def.}{=} H \\ \mathrm{d}^*H \;=\; 0 & \Longleftrightarrow \quad {}^*H \overset{def.}{=} (-1)^{(p+1)(D-1)-1}\, F \end{cases} \tag{121}$$

This means that they describe the same physical system.

Validity The spatial part of the self-twisted duality (15) can be related to integrability equations only for $0 < p < D-2$:

$$\begin{pmatrix} F_S \\ H_S \end{pmatrix} = \begin{pmatrix} (-1)^{(p+1)(D-1)-1 *s}\breve{H}_t \\ {}^{*s}\breve{F}_t \end{pmatrix} \quad \Longleftrightarrow \quad \begin{cases} \mathrm{d}_S\, F_S &= 0 \\ \mathrm{d}_S\,({}^{*s}\breve{F}_t) &= 0 \end{cases} \tag{122}$$

Indeed, either the first or the second integrability equation becomes automatically nil when $p = 0$ or $D-2$ since there is no D-form in a $D-1$ space.

Electric and magnetic field Let F be a $p+1$-form electric field strength, the electric and magnetic fields are then defined as its time components and spatial hodge :

$$\mathcal{E}^{j_1 \cdots j_p} \equiv -F^{0 j_1 \cdots j_p} \tag{123}$$

$$\mathcal{B}^{j_1 \cdots j_{D-p-2}} \equiv \frac{(-1)^{(D-p-2)(p+1)}}{(p+1)!}\, \epsilon^{l_1 \cdots l_{p+1}\, j_1 \cdots j_{D-p-2}}\, F_{l_1 \cdots l_{p+1}} \tag{124}$$

such that $\partial_\mu F^{\mu\, \lambda_2 \cdots \lambda_{p+1}}$ can be re-written

$$\begin{cases} \partial_j\, F^{j\, k_2 \ldots 0 \ldots k_{p+1}} \\ \partial_\mu\, F^{\mu\, k_2 \cdots k_{p+1}} \end{cases} = \begin{array}{ll} (-1)^{q+1}\, \partial_j\, \mathcal{E}^{j\, k_2 \ldots \hat{k}_q \ldots k_{p+1}} & = \pm\text{"div"}\mathcal{E} \\ \dfrac{(-1)^{D-p}\, \epsilon^{\mu\, j_1 \cdots j_{D-p-2}\, k_2 \cdots k_{p+1}}}{(D-p-1)!}\, \partial_\mu\, \mathcal{B}_{j_1 \cdots j_{D-p-2}} & = \pm\text{"}\vec{\mathrm{rot}}\text{"}\mathcal{B} \end{array} \tag{125}$$

In terms of the magnetic field strength,

$$\frac{1}{p!}\, \mathcal{E}_{j_1 \cdots j_p}\, \mathrm{d}x^{j_1} \wedge \cdots \wedge \mathrm{d}x^{j_p} = (-1)^{pD-1}\, {}^{*s}H_S \tag{126}$$

$$\frac{1}{(D-p-2)!}\, \mathcal{B}_{j_1 \cdots j_{D-p-2}}\, \mathrm{d}x^{j_1} \wedge \cdots \wedge \mathrm{d}x^{j_{D-p-2}} = (-1)^{(p+1)(D-1)-1}\, \breve{H}_t \tag{127}$$

From (16), the spatial components of the electric and magnetic conjugate momenta are

$$\begin{aligned} \pi^{j_1 \cdots j_p} &= \frac{\mathcal{E}^{j_1 \cdots j_p}}{p!} \\ \rho^{j_1 \cdots j_{D-p-2}} &= (-1)^{(D-1)(p+1)}\, \frac{1}{(D-p-2)!}\, \mathcal{B}_{j_1 \cdots j_{D-p-2}} \end{aligned} \tag{128}$$

.1.4 Gauge

Inverse function theorem This theorem generalizes to functions between Banach spaces, or in our case[35], manifolds, the idea that a function of one variable is locally invertible if its derivative is non zero. It gives a sufficient condition for invertibility.

Let $\mathcal{M}$ and $\mathcal{N}$ be two $\mathcal{C}^k$ manifolds, $\Psi \in \mathcal{C}^k(\mathcal{M},\mathcal{N})$ and $x_0 \in \mathcal{M}$. If $\mathrm{d}\Psi(x_0) : T_{x_0}\mathcal{M} \longrightarrow T_{\Psi(x_0)}\mathcal{N}$ is a linear isomorphism, then there exists $U \in \mathcal{M}$ an open neighbourhood of x_0, and $\Psi^{-1} \in \mathcal{C}^k(\Psi(U), U)$ such that $\Psi \circ \Psi^{-1} = Id_{\Psi(U)}$.

In (16), since $\pi^{\lambda_1 \cdots \lambda_p} = 0$ whenever a λ_k is zero because of the total antisymmetry of F, $\frac{\partial^2 \mathcal{L}}{\partial(\partial_0 A_{\lambda_1 \ldots \lambda_p})\partial(\partial_0 A_{\rho_1 \ldots \rho_p})}\, \mathrm{d}\dot{A}_{\rho_1 \ldots \rho_p}$ is degenerate for all values of $\dot{A}_{\rho_1 \ldots \rho_p}$.

[35] Actually it seems that $\dot{A}_{\lambda_1 \ldots \lambda_p}$ and $\pi^{\lambda_1 \cdots \lambda_p}$ are the components of vectors of the tangent and cotangent space of the manifold spanned by the coordinates $A_{\lambda_1 \ldots \lambda_p}$. Indeed the first is a derivative of parametrized curve and the second is similar to the derivative of a scalar field with respect to the coordinates.

Constraints When Ψ is not invertible, the image space $\Psi(\mathcal{M})$ a.k.a. *constraint surface* is a submanifold of $\mathcal{N}$ and can be intrinsically defined by $r = \dim \mathcal{N} - \dim[\Psi(\mathcal{M})]$ functions $\mathcal{G}_m : N \longrightarrow \mathbb{R}$, known as holonomous constraints. Let π_k be the $n = \dim \mathcal{N}$ coordinates of $\mathcal{N}$,

$$\Psi(M) = \left\{ (\pi_1, \ldots, \pi_n), \text{ such that } \mathcal{G}_m(\pi_1 \ldots \pi_n) = 0 \, , \, 1 \leq m \leq r \right\} \tag{129}$$

The action in the Hamiltonian form – functional of $A_{\lambda_1 \ldots \lambda_p}$ and $\pi^{\lambda_1 \cdots \lambda_p}$– is equivalent to the one in Lagrangian form – functional of $A_{\lambda_1 \ldots \lambda_p}$ and $\partial_0 A_{\lambda_1 \ldots \lambda_p}$– only in the constraint surface.

What's remarkable is that those $\mathcal{G}_m$ can in turn become coordinates on $\mathcal{N}$, spanning the rest of $\mathcal{N}$ And the hamiltonian can be extended to the totality of N. (Kind of factorization theorem, like P(X)=0 en X=XO, alors P s'écrit (X-XO)Q))

Gauge generators Gauge group arises with the redundance of the description. Dirac conjectured that first class constraints are generators of gauge groups [6].

.1.5 Lie derivative and Killing fields

Lie derivative Let $X = X^\mu \partial_\mu$ be a vector field on riemannian manifold.

The Lie derivative of a scalar field f along the X direction is the usual action of a vector on a scalar field

$$\mathcal{L}_X f \;=\; X^\mu \, \partial_\mu \, f \tag{130}$$

In particular when f is a coordinate as in (30), we have

$$\mathcal{L}_X x^\nu \;=\; X^\mu \, \partial_\mu \, x^\nu \;=\; X^\nu \tag{131}$$

However for a vector field $Y = Y^\mu \partial_\mu$ it may be defined as[36]

$$\mathcal{L}_X Y \;=\; \left(X^\mu \, \partial_\mu \, Y^\nu - Y^\mu \, \partial_\mu \, X^\nu \right) \partial_\nu \;=\; [X, Y]^\nu \partial_\nu \tag{132}$$

For tensor fields of any order Lie derivative is defined by imposing Leibniz rule for contraction and tensor product. In a local coordinate system:

$$\mathcal{L}_X T^{\mu_1 \ldots \mu_p}{}_{\nu_1 \ldots \nu_q} = X^\rho \, \partial_\rho \, T^{\mu_1 \ldots \mu_p}{}_{\nu_1 \ldots \nu_q} - \sum_{j=1}^{p} T^{\mu_1 \ldots \, \rho \, \ldots \mu_p}{}_{\nu_1 \ldots \nu_q} \, \partial_\rho \, X^{\mu_j} + \sum_{k=1}^{q} T^{\mu_1 \ldots \, \rho \, \ldots \mu_p}{}_{\nu_1 \ldots \, \rho \, \ldots \nu_q} \, \partial_{\nu_k} \, X^\rho \tag{133}$$

However, Lie derivative doesn't depend on the coordinate system, and for a torsion-free connection, partial derivatives can be replaced by covariant derivatives in the previous definitions. Note also that Lie derivative doesn't change the rank of a tensor, it actually defines an action of vector fields on tensors of any order. Furthermore, when the space of vector fields has an additional Lie bracket operation, Lie derivative is a Lie algebra morphism:

$$\mathcal{L}_{[X,Y]} T = \mathcal{L}_X \mathcal{L}_Y T - \mathcal{L}_Y \mathcal{L}_X T \tag{134}$$

Killing field Killing fields K_α generate the symmetries of the metric, the are properly defined by[37]:

$$\mathcal{L}_{K_\alpha} \, g_{\mu\nu} \;=\; 0 \tag{135}$$

$$\Longleftrightarrow \quad K^\rho{}_\alpha \, \partial_\rho \, g_{\mu\nu} + g_{\rho\nu} \, \partial_\mu \, K^\rho{}_\alpha + g_{\mu\rho} \, \partial_\nu \, K^\rho{}_\alpha \;\underset{civita}{\overset{Levi}{=}}\; \partial_\mu \, K_{\nu\,\alpha} + \partial_\nu \, K_{\mu\,\alpha} \;=\; 0 \tag{136}$$

In general relativity, the contraction of Killing fields with four-velocity yields the evolution constants. This is a consequence of the equation of motion, which is the geodesic equation:

$$\ddot{x}^\mu + \underbrace{\frac{1}{2} g^{\mu\lambda} \left(\partial_\alpha \, g_{\lambda\beta} + \partial_\beta \, g_{\alpha\lambda} - \partial_\lambda \, g_{\alpha\beta} \right)}_{\Gamma^\mu_{\alpha\beta}} \dot{x}^\alpha \dot{x}^\beta \;=\; \dot{x}^\rho \, \mathcal{D}_\rho \, \dot{x}^\mu \;=\; 0 \tag{137}$$

[36] cf. Nakahara **5.3.2** 2nd edition, for a more general definition.
[37] Levi-civita connexion: $\mathcal{D}_\rho \, g_{\mu\nu} = 0$ and $\Gamma^\rho{}_{\mu\nu} = -\Gamma^\rho{}_{\nu\mu}$

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}(K_{\nu\;\alpha}\dot{x}^\nu) = \dot{x}^\mu\,\partial_\mu(K_{\nu\;\alpha}\dot{x}^\nu)$$
$$= \dot{x}^\mu\dot{x}^\nu\,\mathcal{D}_\mu K_{\nu\;\alpha} + \dot{x}^\mu K_{\nu\;\alpha}\mathcal{D}_\mu\,\dot{x}^\nu$$
$$= \dot{x}^\mu\dot{x}^\nu\,\underbrace{\partial_{(\mu}K_{\nu)\;\alpha}}_{\text{Killing field}} + K_{\nu\;\alpha}\,\underbrace{(\dot{x}^\mu\,\mathcal{D}_\mu\,\dot{x}^\nu)}_{\text{geodesic equation}} = 0 \tag{138}$$

(32) is the equivalent in the sigma model.

.1.6 Various calculus

From now on the metric is the flat Minskowskian metric, unless specified otherwise.

Time part of a differential form Time has a somehow special role and this requires to seperate time and spatial component of a differential form. Intuitively, the time part of a q-form B is the sum of its dt =dx^0 linear components

$$B_t = B_{0k_2\dots k_q}\,\mathrm{d}x^0\wedge\mathrm{d}x^{k_2}\wedge\cdots\wedge\mathrm{d}x^{k_q} + B_{k_1 0\dots k_q}\,\mathrm{d}x^{k_1}\wedge\mathrm{d}x^0\wedge\cdots\wedge\mathrm{d}x^{k_q} + \cdots$$
$$+ B_{k_1\dots k_{q-1}0}\,\mathrm{d}x^{k_1}\wedge\cdots\wedge\mathrm{d}x^{k_{q-1}}\wedge\mathrm{d}x^0 \tag{139}$$

where the k indices are summed up, but not the 0.

In the sequel it will be useful to define an antisymmetric in 0 part

$$\check{B}_{0k_2\dots k_q} = \frac{1}{q}\sum_{\substack{j=1,\\ \text{position of the 0}}}^{q}(-1)^{j+1}\,B_{k_1\dots k_{j-1}0k_{j+1}\dots k_q} \tag{140}$$

such that

$$B_t = q\,\check{B}_{0k_2\dots k_q}\,\mathrm{d}x^0\wedge\mathrm{d}x^{k_2}\wedge\cdots\wedge\mathrm{d}x^{k_q}$$

The other components give the spatial part. The exterior derivative can also be split into a time d_t and a spatial part d_S and

$$\begin{aligned}\mathrm{d}\,\omega|_S &= \mathrm{d}_S\,\omega_S\\ \mathrm{d}\,\omega|_t &= \mathrm{d}_t\,\omega_S + \mathrm{d}_S\,\omega_t\end{aligned} \tag{141}$$

"Spatial" hodge Let ω be a p-form, the time (respectively spatial) part of its hodge ${}^*\omega$ is related to the spatial hodge of its spatial (respectively time) part. Let's recall the hodge operation:

$${}^*\omega = \frac{1}{(D-p)!\,p!}\,\epsilon^{\lambda_1\dots\lambda_p}{}_{\mu_{p+1}\dots\mu_D}\,\omega_{\lambda_1\dots\lambda_p}\,\mathrm{d}x^{\mu_{p+1}}\wedge\cdots\wedge\mathrm{d}x^{\mu_D} \tag{142}$$

Its time part

$${}^*\omega|_t \equiv \frac{1}{(D-p)!\,p!}\,\epsilon^{j_1\dots j_p}{}_{\mu_{p+1}\dots\mu_D}\,\omega_{j_1\dots j_p}\,\mathrm{d}x^{\mu_{p+1}}\wedge\cdots\wedge\mathrm{d}x^{\mu_D}$$
$$= \frac{1}{(D-p)!\,p!}\sum_{\substack{l=p+1,\\ \text{position of the 0}}}^{D}\epsilon^{j_1\dots j_p}{}_{k_{p+1}\dots k_{l-1}0\,k_{l+1}\dots k_D}\,\omega_{j_1\dots j_p}\,\mathrm{d}x^{k_{p+1}}\wedge\cdots\wedge\mathrm{d}x^{k_{l-1}}\wedge\mathrm{d}x^0\wedge\mathrm{d}x^{k_{l+1}}\wedge\cdots\wedge\mathrm{d}x^{k_D}$$
$$= (-1)^p\,\mathrm{d}x^0\wedge\underbrace{\frac{D-p}{(D-p)!\,p!}\,\epsilon^{j_1\dots j_p}{}_{k_{p+1}\dots k_{D-1}}\,\omega_{j_1\dots j_p}\,\mathrm{d}x^{k_{p+1}}\wedge\cdots\wedge\mathrm{d}x^{k_{D-1}}}_{\equiv\,{}^{*s}\omega_S} \tag{143}$$

Its spatial part

$${}^*\omega|_S \equiv \frac{1}{(D-p)!\,p!}\,\epsilon^{\lambda_1\dots\lambda_p}{}_{k_{p+1}\dots k_D}\,\omega_{\lambda_1\dots\lambda_p}\,\mathrm{d}x^{k_{p+1}}\wedge\cdots\wedge\mathrm{d}x^{k_D}$$
$$= \frac{1}{(D-p)!\,p!}\sum_{\substack{l=1,\\ \text{position of the 0}}}^{p}\epsilon^{j_1\dots j_{l-1}0\,j_{l+1}\dots j_p}{}_{k_{p+1}\dots k_D}\,\omega_{j_1\dots j_{l-1}0\,j_{l+1}\dots j_p}\,\mathrm{d}x^{k_{p+1}}\wedge\cdots\wedge\mathrm{d}x^{k_D}$$
$$= \underbrace{\frac{p}{(D-p)!\,p!}\,\epsilon^{j_1\dots j_{p-1}}{}_{k_{p+1}\dots k_D}\,\omega_{0\,j_1\dots j_{p-1}}\,\mathrm{d}x^{k_{p+1}}\wedge\cdots\wedge\mathrm{d}x^{k_D}}_{\equiv\,{}^{*s}\check{\omega}_t} \tag{144}$$

where the time part has been redefined as a $(p-1)$-form

$$\breve{\omega}_t \equiv \omega_{0\, j_1 \cdots j_{p-1}}\, \mathrm{d}x^{j_1} \wedge \cdots \wedge \mathrm{d}x^{j_{p-1}}$$

Adjoint [38] In the simple case of a flat space, the adjoint of the exterior derivative of a p-form ω in D dimension, can be explicitly written

$$\begin{aligned}
\mathrm{d}^\dagger\omega &\equiv (-1)^{D(p+1)+\epsilon}\, {}^*\mathrm{d}^*\omega \\
&= (-1)^{D(p+1)+\epsilon}\, {}^*\!\left[\partial_\lambda \left(\frac{1}{(D-p)!\,p!}\, \epsilon^{\mu_1\cdots\mu_p}{}_{\nu_{p+1}\cdots\nu_D}\, \omega_{\mu_1\cdots\mu_p}\right) \mathrm{d}x^\lambda \wedge \mathrm{d}x^{\nu_{p+1}} \wedge \cdots \wedge \mathrm{d}x^{\nu_D}\right] \\
&= \frac{(-1)^{D(p+1)+\epsilon}}{(p-1)!\,(D-p)!\,p!}\, \epsilon^{\lambda\,\nu_{p+1}\cdots\nu_D}{}_{\rho_1\cdots\rho_{p-1}} \partial_\lambda \left(\epsilon^{\mu_1\cdots\mu_p}{}_{\nu_{p+1}\cdots\nu_D}\, \omega_{\mu_1\cdots\mu_p}\right) \mathrm{d}x^{\rho_1} \wedge \cdots \wedge \mathrm{d}x^{\rho_{p-1}} \\
&= \frac{(-1)^{(D+D-p)(1+p)+\epsilon}}{(p-1)!\,(D-p)!\,p!}\, \epsilon^{\nu_{p+1}\cdots\nu_D\,\mu_1\cdots\mu_p} \epsilon_{\nu_{p+1}\cdots\nu_D\,\lambda\,\rho_1\cdots\rho_{p-1}} \left(\partial^\lambda \omega_{\mu_1\cdots\mu_p}\right) \mathrm{d}x^{\rho_1} \wedge \cdots \wedge \mathrm{d}x^{\rho_{p-1}} \\
&= \frac{(-1)^\epsilon}{(p-1)!}\, \partial^\lambda \omega_{\lambda\,\rho_1\cdots\rho_{p-1}} \mathrm{d}x^{\rho_1} \wedge \cdots \wedge \mathrm{d}x^{\rho_{p-1}} \qquad (145)
\end{aligned}$$

where $\epsilon = 0$ for a Riemannian metric, and 1 for Lorentzian one.

Hamiltonian The Lagrangian can be written

$$\begin{aligned}
&-\frac{1}{2\,(p+1)!}\, F_{\lambda_1\cdots\lambda_{p+1}} F^{\lambda_1\cdots\lambda_{p+1}} \\
&= -\frac{1}{2\,(p+1)!} \left(\sum_{\text{with a time indice}} F_{\lambda_1\cdots\lambda_{p+1}} F^{\lambda_1\cdots\lambda_{p+1}} + \sum_{\text{spatial indices}} F_{\lambda_1\cdots\lambda_{p+1}} F^{\lambda_1\cdots\lambda_{p+1}}\right) \\
&= -\frac{1}{2\,(p+1)!} \left((p+1)\, F_{0\,k_2\cdots k_{p+1}} F^{0\,k_2\cdots k_{p+1}} + F_{k_1\cdots k_{p+1}} F^{k_1\cdots k_{p+1}}\right) \\
&= -\frac{1}{2\,p!}\, F^{0\,k_2\cdots k_{p+1}} F_{0\,k_2\cdots k_{p+1}} - \frac{1}{2\,(D-p-2)!}\, \mathcal{B}_{l_1\cdots l_{D-p-2}} \mathcal{B}^{l_1\cdots l_{D-p-2}} \\
&= +\frac{1}{2}\, \pi^{k_2\cdots k_{p+1}} \left(-\,p!\, g_{00}\, \pi_{k_2\cdots k_{p+1}}\right) - \frac{1}{2\,(D-p-2)!}\, \mathcal{B}_{l_1\cdots l_{D-p-2}} \mathcal{B}^{l_1\cdots l_{D-p-2}} \qquad (146)
\end{aligned}$$

Indeed, there is a minus sign depending on the metric convention. In the following, $(-,+,+,+)$ is choosen as in [1]. In the fourth line, we appealed to the following identity

$$\begin{aligned}
&\frac{1}{(D-p)!} \left({}^*\omega_{\lambda_1\cdots\lambda_{D-p}}\right) \left({}^*\omega^{l_1\cdots l_{D-p-2}}\right) \\
&= \frac{1}{(D-p)!} \left(\frac{1}{p!}\, \epsilon_{\mu_1\,\ldots\,\mu_p\,\lambda_1\,\ldots\,\lambda_{D-p}}\, \omega^{\mu_1\,\ldots\,\mu_p}\, \frac{1}{p!}\, \epsilon^{\nu_1\,\ldots\,\nu_p\,\lambda_1\,\ldots\,\lambda_{D-p}}\, \omega_{\nu_1\,\ldots\,\nu_p}\right) \\
&= \frac{1}{(p!)^2\,(D-p)!}\, \epsilon_{\mu_1\,\ldots\,\mu_p\,\lambda_1\,\ldots\,\lambda_{D-p}} \epsilon^{\nu_1\,\ldots\,\nu_p\,\lambda_1\,\ldots\,\lambda_{D-p}}\, \omega^{\mu_1\,\ldots\,\mu_p}\, \omega_{\nu_1\,\ldots\,\nu_p} \\
&= \frac{1}{p!}\, \delta^{\mu_1\,\ldots\,\mu_p}_{[\nu_1\,\ldots\,\nu_p]}\, \omega^{\mu_1\,\ldots\,\mu_p}\, \omega_{\nu_1\,\ldots\,\nu_p} \\
&= \frac{1}{p!}\, \omega_{\mu_1\,\ldots\,\mu_p}\, \omega^{\mu_1\,\ldots\,\mu_p}
\end{aligned}$$

[38] [3] **7.9.4** second edition

The Legendre transformation term actually gives the constraint term:

$$\begin{aligned}
\pi^{\lambda_1\cdots\lambda_p}\dot{A}_{\lambda_1\cdots\lambda_p} &= \pi^{k_1\cdots k_p}\left(\partial_0 A_{k_1\cdots k_p}\right) \\
&= \pi^{k_1\cdots k_p}\left[\frac{F_{0\,k_1\cdots k_p}}{(p+1)} - \frac{1}{p}\left(\sum_{\substack{j=1\\ \text{position of}\\ \text{the 0 indice}}}^{p} (-1)^j\,\partial_{k_1}A_{k_2\cdots 0\cdots k_p}\right)\right] \\
&= ? - p!\,g_{00}\,\pi^{k_1\cdots k_p}\,\pi_{k_1\cdots k_p} - \frac{1}{p}\,\pi^{k_1\cdots k_p}\left(\sum_{\substack{j=1\\ \text{position of}\\ \text{the 0 indice}}}^{p} (-1)^j\,\partial_{k_1}A_{k_2\cdots 0\cdots k_p}\right) \\
&= ?p!\,\pi^{k_1\cdots k_p}\,\pi_{k_1\cdots k_p} + \partial_k\pi^{k\,k_2\cdots k_p}\underbrace{\left(\frac{1}{p}\sum_{\substack{j=1\\ \text{position of}\\ \text{the 0 indice}}}^{p} (-1)^j\,A_{k_2\cdots 0\cdots k_p}\right)}_{\equiv A_{0\,k_2\cdots k_p}}
\end{aligned}$$

where we have defined the antisymmetric part of the time compoent of A. Its cofactor is the constraint.

Thereafter the hamiltonian,

$$\begin{aligned}
&\pi^{\lambda_1\cdots\lambda_p}\dot{A}_{\lambda_1\cdots\lambda_p} - \left(-\frac{1}{2\,(p+1)!}\,F_{\lambda_1\cdots\lambda_{p+1}}F^{\lambda_1\cdots\lambda_{p+1}}\right) \\
&= \frac{1}{2}\left(p!\,\pi^{k_1\cdots k_p}\,\pi_{k_1\cdots k_p} + \frac{1}{(D-p-2)!}\,\mathcal{B}_{l_1\cdots l_{D-p-2}}\mathcal{B}^{l_1\cdots l_{D-p-2}}\right) + A_{0\,k_2\cdots k_p}\,\partial_k\pi^{k\,k_2\cdots k_p} \\
&= \underbrace{\frac{1}{2}\left(\frac{1}{p!}\,\mathcal{E}^{k_1\cdots k_p}\,\mathcal{E}_{k_1\cdots k_p} + \frac{1}{(D-p-2)!}\,\mathcal{B}_{l_1\cdots l_{D-p-2}}\mathcal{B}^{l_1\cdots l_{D-p-2}}\right)}_{\mathcal{H}} + A_{0\,k_2\cdots k_p}\,\mathcal{G}^{k_2\cdots k_p}
\end{aligned} \tag{147}$$

Currents variations

$$\delta V = \Sigma(x)\cdot V \iff \Sigma = \delta V\cdot V^{-1} \tag{148}$$

$$\begin{aligned}
\delta\mathbf{J}_\mu &\equiv \delta\left(\partial_\mu V\cdot V^{-1}\right) = [\partial_\mu \underbrace{\delta V}_{V^{-1}\cdot V}]\cdot V^{-1} - \partial_\mu V\cdot\left[V^{-1}\cdot\delta V\cdot V^{-1}\right] \\
&= \partial_\mu(\delta V\cdot V^{-1})\cdot V\cdot V^{-1} + (\delta V\cdot V^{-1})\cdot(\partial_\mu V\cdot V^{-1}) - (\partial_\mu V\cdot V^{-1})\cdot(\delta V\cdot V^{-1}) \\
&= \partial_\mu(\delta V\cdot V^{-1}) + [(\delta V\cdot V^{-1}),\mathbf{J}_\mu] \\
&= \partial_\mu\Sigma(x) + [\Sigma(x),\mathbf{J}_\mu]
\end{aligned} \tag{149}$$

$$\tilde{\delta}V(x) = V\cdot\Sigma(x) \iff \Sigma = V^{-1}\cdot\delta V \tag{150}$$

$$\begin{aligned}
\tilde{\delta}\tilde{\mathbf{J}}_\mu &\equiv \tilde{\delta}\left(V^{-1}\cdot\partial_\mu V\right) = \partial_\mu(\tilde{\delta}V\cdot V^{-1}) + [\tilde{\mathbf{J}}_\mu,(\tilde{\delta}V\cdot V^{-1})] \\
&= \partial_\mu\Sigma(x) + [\tilde{\mathbf{J}}_\mu,\Sigma(x)]
\end{aligned} \tag{151}$$

For a field $X(x)\in\mathfrak{g}$

$$\begin{aligned}
\partial_\mu\left(V^{-1}\cdot\mathbf{X}(x)\cdot V\right) &= -[V^{-1}\cdot(\partial_\mu V)\cdot V^{-1}]\cdot\mathbf{X}(x)\cdot V + V^{-1}\cdot\mathbf{X}(x)\cdot(\partial_\mu V) + V^{-1}\cdot(\partial_\mu\mathbf{X}(x))\cdot V \\
&= \left[V^{-1}\cdot\mathbf{X}(x)\cdot V\,,\,V^{-1}\cdot(\partial_\mu V)\right] + V^{-1}\cdot(\partial_\mu\mathbf{X}(x))\cdot V \\
&= V^{-1}\cdot\left(\left[\mathbf{X}(x),(\partial_\mu V)\cdot V^{-1}\right] + \partial_\mu\mathbf{X}(x)\right)\cdot V
\end{aligned} \tag{152}$$

$$\partial_\mu\left(V\cdot\mathbf{X}(x)\cdot V^{-1}\right) = V\cdot\left(\left[V^{-1}\cdot(\partial_\mu V),\mathbf{X}(x)\right] + \partial_\mu\mathbf{X}(x)\right)\cdot V^{-1} \tag{153}$$

Gauge invariance The action of $h(x) = \lambda^\alpha(x)\, t_\alpha \in \mathfrak{h}$ leaves the system (92) unchanged: (right action for right current...)

$$\begin{aligned}
\delta\mathbf{J}_\mu &= -V^{-1}\cdot (V\cdot h)\cdot V^{-1}\mathcal{D}_\mu V + V^{-1}\cdot(\mathcal{D}_\mu V)\cdot h \\
&= -\left[h(x), J_\mu\right] && (154)\\
\delta\mathbf{H}_{k_1\dots k_{D-1}} &= \delta\left(\mathcal{D}_{[k_1}\mathbf{B}_{k_2\dots k_{D-1}]}\right) \;=\; \left(\mathcal{D}_{[k_1}\mathbf{B}_{k_2\dots k_{D-1}]}\right)\cdot h(x) \\
&= -\left[h, \mathbf{H}_{k_1\dots k_{D-1}}\right] && (155)\\
\Longrightarrow \qquad \delta\mathrm{Tr}\left(\mathbf{J}_0\cdot\mathbf{H}_{j_1\dots j_{D-1}}\right) &= \mathrm{Tr}\left(-\left[h,\mathbf{J}_0\right]\cdot\mathbf{H}_{j_1\dots j_{D-1}} - \mathbf{J}_0\cdot\left[h,\mathbf{H}_{j_1\dots j_{D-1}}\right]\right)\\
&= 0 && (156)\\
\& \qquad \delta\mathrm{Tr}\left(\mathbf{J}^i\cdot\mathbf{J}_i\right) &= \delta\mathrm{Tr}\left(\mathbf{H}^{j_1\dots j_{D-1}}\cdot\mathbf{H}_{j_1\dots j_{D-1}}\right) \;=\; 0 && (157)
\end{aligned}$$

In (155) and (154) we have used the fact that a field and its covariant derivative transform in the same way under the action of the gauge group.

Equation of motion for (92) Integration by part for covariant derivatives comes from leibniz rule:

$$\mathcal{D}_\mu\left(J_{0\,\alpha}B_{\nu_2\dots\nu_{D-1}}{}^\alpha\right) = \left(\mathcal{D}_\mu J_{0\,\alpha}\right)B_{\nu_2\dots\nu_{D-1}}{}^\alpha + J_{0\,\alpha}\left(\mathcal{D}_\mu B_{\nu_2\dots\nu_{D-1}}{}^\alpha\right) \tag{158}$$

Algebraic relations: left and right current A right action is equivalent to a left action of the "inverse" group. The complete Noether theorem does precise that ...symmetry equivalence classes...

A group $(G, *)$ can always act $(\cdot)$ on itself and on its algebra by conjugation:

$$\forall S, V \in G \text{ and } X \in \mathfrak{g}\,, \quad \begin{cases} V\cdot S \equiv V * S * V^{-1} \\ V\cdot X \equiv V * X * V^{-1}\end{cases} \tag{159}$$

-More generaly derivation of a group representation yields an algebra representation, and exponentiation does the opposite. -Does $Aut(G)$ have a particular structure, is it much bigger than G... -Algebra acting on the left on a group/a group acting on the right on its algebra? ¡-¿ bigger algebra

Conjugation matrix $S(V)$

$$(V\cdot X)^\alpha \equiv S^\alpha{}_\beta\, X^\beta \tag{160}$$

then

$$\begin{aligned}
L^\alpha{}_a &= S^\alpha{}_\beta\,\tilde{L}^\beta{}_a && (161)\\
\mathbf{J}_{\mu\,\alpha} &= \eta_{\alpha\beta}\,S^\beta{}_\gamma\eta^{\gamma\delta}\,\tilde{\mathbf{J}}_{\mu\,\delta} \;=\; (S^{-1})_\alpha{}^\delta\,\tilde{\mathbf{J}}_{\mu\,\delta} && (162)
\end{aligned}$$

2nd order equations for B Using the first order equations again

$$\begin{aligned}
\tilde{\mathbf{J}}_0 &= \mathbf{f} - {}^{*s}\mathbf{H}_S && (163)\\
\tilde{\mathbf{J}}^i &= (-1)^D\,({}^{*s}\tilde{\mathbf{H}}_t)^i - \left[\frac{\epsilon^{i\,j_2\dots j_{D-1}}}{(D-2)!}\,\mathbf{B}_{j_2\dots j_{D-1}}\,,\;\mathbf{f}\right] && (164)
\end{aligned}$$

we obtain for μ, ν spatial indices

$$\begin{aligned}
&\partial_{[i}\left((-1)^D({}^{*s}\tilde{\mathbf{H}}_t)_{j]} - \left[\frac{\epsilon_{j]\,k_2\dots k_{D-1}}}{(D-2)!}\,\mathbf{B}^{k_2\dots k_{D-1}}\,,\;\mathbf{f}\right]\right) && (165)\\
&= -\frac{1}{2}\left[\left((-1)^D({}^{*s}\tilde{\mathbf{H}}_t)_i - \left[\frac{\epsilon_{i\,k_2\dots k_{D-1}}}{(D-2)!}\,\mathbf{B}^{k_2\dots k_{D-1}}\,,\;\mathbf{f}\right]\right),\left((-1)^D({}^{*s}\tilde{\mathbf{H}}_t)_j - \left[\frac{\epsilon_{j\,l_2\dots l_{D-1}}}{(D-2)!}\,\mathbf{B}^{l_2\dots l_{D-1}}\,,\;\mathbf{f}\right]\right)\right]
\end{aligned}$$

$$\begin{aligned}
&\overset{\times\epsilon^{i\,j\,h_3\dots h_{D-1}}}{\Longrightarrow}\quad \partial_k\left(\left[\mathbf{B}^{k\,h_3\dots h_{D-1}},\;\mathbf{f}\right] + \mathbf{H}_0{}^{k\,h_3\dots h_{D-1}}\right)\\
&= \frac{1}{2}\left(\left[\left[\mathbf{B}^{k\,h_3\dots h_{D-1}},\;\mathbf{f}\right],\left[\frac{\epsilon_{k\,j_2\dots j_{D-1}}}{(D-2)!}\,\mathbf{B}^{j_2\dots j_{D-1}},\;\mathbf{f}\right]\right] - 2\,(-1)^D\left[\mathbf{B}^{k\,h_3\dots h_{D-1}},\;({}^{*s}\tilde{\mathbf{H}}_t)_k\right]\right.\\
&\qquad\left.\left[({}^{*s}\tilde{\mathbf{H}}_t)_i,\;({}^{*s}\tilde{\mathbf{H}}_t)_j\right]\epsilon^{i\,j\,h_3\dots h_{D-1}}\right) && (166)
\end{aligned}$$

and

$$\partial_0 \left((-1)^D ({}^{*s}\tilde{\mathbf{H}}_t)_k \; - \; \left[\; \frac{\epsilon_{k\, j_2 \ldots\, j_{D-1}}}{(D-2)!} \, \mathbf{B}^{j_2 \ldots\, j_{D-1}} \, , \; \mathbf{f} \; \right] \right) \; - \; \partial_k \left(\mathbf{f} - {}^{*s}\mathbf{H}_S \right) \tag{167}$$

$$= \; - \left[\mathbf{f} - ({}^{*s}\mathbf{H}_S) \, , \, (-1)^D \, ({}^{*s}\tilde{\mathbf{H}}_t)_k \; - \; \left[\; \frac{\epsilon_{k\, l_2 \ldots\, l_{D-1}}}{(D-2)!} \, \mathbf{B}^{l_2 \ldots\, l_{D-1}} \, , \; \mathbf{f} \; \right] \right]$$

$$\overset{\times \epsilon^{k\, h_2 \ldots h_{D-1}}}{\Longrightarrow} \qquad \partial_\mu \, \mathbf{H}^{\mu\, h_2 \ldots h_{D-1}} \; - \; \partial_0 \left[\mathbf{B}^{h_2 \ldots h_{D-1}} , \, \mathbf{f} \right] \; - \; \partial_k \, (\epsilon^{k\, h_2 \ldots h_{D-1}} \, \mathbf{f})$$

$$= \; \left[(-1)^D \, \tilde{\mathbf{H}}^{0\, h_2 \ldots h_{D-1}} \; - \; \left[\; \mathbf{B}^{h_2 \ldots\, h_{D-1}} , \; \mathbf{f} \; \right] \, , \, \mathbf{f} - ({}^{*s}\mathbf{H}_S) \right] \tag{168}$$

Adjoint representation We are looking for matrices $(t_i)_\alpha{}^\beta$ that satisfy the Lie algebra structure $[\, t_i, \, t_j \,] \; = \; f_{ij}{}^l \, t_l$:

$$(t_i)_\alpha{}^\beta \, (t_j)_\beta{}^\gamma - (t_j)_\alpha{}^\beta \, (t_s)_\beta{}^\gamma \quad = \quad f_{ij}{}^k \, (t_k)_\alpha{}^\gamma \tag{169}$$

Identifying (169) with Jacobi's identity for the structure constants:

$$f_{il}{}^p \, f_{jk}{}^l \; + \; f_{jl}{}^p \, f_{ki}{}^l \; + \; f_{kl}{}^p \, f_{ij}{}^l \; = \; 0 \tag{170}$$

suggests to take :

$$(t_i)_j{}^k \equiv f_{ij}{}^k \tag{171}$$

Bibliography

[1] Marc Henneaux Claudio Bunster. The action for twisted self-duality. *arXiv:1103.3621v1 [hep-th]*, 2011.

[2] Christoph Jürgen FUCHS. *Symmetries, Lie algebras and Representations.* Cambridge University Press, 1997.

[3] Mikio NAKAHARA. *Geometry, Topology and Physics.* Institute of physics publishing, Bristol and Philadephia, 1995.

[4] Henning Samtleben. Actions for non-abelian twisted self-duality.

[5] Henning Samtleben. Lectures on gauged supergravity and flux compactifications, 2008.

[6] Christianne SCHOMBLOND. *Théories hamiltonienne à contraintes.*