Loop Quantum Gravity

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Introduction

Since the beginning of the 20th century with the emergence of Quantum Mechanics and General Relativity, physicists tried to re-conciliate those two completely different theories.
Loop Quantum Gravities (LQG) is one of the current formulation which try to do it, and the following paper deals with what are the mathematical structures used in it, how the theory is built and what is the answer given by it to the topic previously mentioned.
We start by the different formulations for gravity, ending with the used one in LQG. Then we introduce the notion of sympleptic structure, applying it on gravity to finally quantize it.

1 GravitIES

Since Einstein article of 1915, General Relativity is a theory where gravitation is a consequence of space-time geometry. Such geometry is encoded in an object so-called metric g which gives us the infinitesimal space-time measure in a certain coordinate system :

$$ds^2 = g_{\mu_\nu} dx^\mu dx^\nu$$

On the other side the stress-energy tensor T contains the information of the source. Link between matter and geometry is the question that Einstein equations deal with, and show us how matter stretches space-time, i.e. how gravity appears to us. Mathematically, g is a Lorentzian scalar product define above a manifold $\mathcal{M}$ (space-time). On the other hand, a connection ∇ is an object connecting two fibers of $T\mathcal{M}$, the tangential fiber bundle of $T\mathcal{M}$, used to define the covariant derivation ∇_μ, Any connection can thus define a curvature. It is important to notice that metric and connection are independent in general. General Relativity considers a particular connection, called Levi-Civita connection (Γ) which realizes two conditions : it is metric and torsion-free. Connection is so completely determined by the metric g. Curvature associate to the Levi-Civita connection is the Riemann curvature.

1.1 Einstein-Hilbert action

Historically, gravity was firstly introduced in modern mechanics formulation (Lagrangian) by the Einstein-Hilbert action :

$$S_{EH}[g] = \int d^4x \sqrt{-g} R \tag{1}$$

where g is the determinant of the metric components matrix, and R the Scalar curvature. The action depends only on the metric, since we consider the Levi-Civita connection, as said previously.

Equations of motion derived from S_{EH} w.r.t the metric are the well-known Einstein equations :

$$G^{\mu\nu} = 0 \tag{2}$$

where $G^{\mu\nu} \doteq R^{\mu\nu} - \frac{1}{2} R g^{\mu\nu}$ is the Einstein tensor.

It is important to notice that a solution of motion equations is not given by a metric but by an equivalence class of metric for diffeomorphism equivalence relation : two metrics related by a diffeomorphism represent the same solution. The consequence is it exists relations that have to be verified, called diffeomormhism constraints. The notion of constraints will be detailed in a dedicated section, see 2.4.

1.2 1st order formulation

The Einstein-Hilbert theory is said to be a 2nd order formulation, a.k.a. a formulation where 2nd order derivative of the metric appeared in the Lagrangian. An idea to rewrite the action for gravity, historically introduced by Palatini, is to use as dynamical variable a local frame transformation instead of the metric. Indeed, the equivalence Principle tells us we always can find a local frame transformation that we can express the general metric g in an orthonormal base :

$$g = g_{\mu\nu} dx^\mu \otimes dx^\nu = \eta_{IJ} e^I \otimes e^J = \eta_{IJ} = \eta_{IJ} e^I_\mu e^J_\nu dx^\mu \otimes dx^\nu \tag{3}$$

where e^I is the so-called tetrad , a moving frame linking holonom and orthonormal (resp. space-time and Lorentz) base. We have to take care that two tetrads related by a Lorentz transformation represent the same metric. This implies the verification of constraints so-called Gauss constraints. The connection form (so called spin-connection), associated to the connection, is expressed in the tetrad base trough the relation :

$$(\nabla_\alpha e)^I_\mu \doteq \partial_\alpha e^I_\mu + \omega_\alpha{}^I{}_J e^J_\mu - \Gamma^{\mu\nu}{}_\alpha e^I_\nu \doteq 0 \tag{4}$$

The tetrad metric compatibility expresses the fact that tetrad is simply the expression of the identity but in two different bases. The curvature form, associated to the curvature, is expressed in the tetrad base by :

$$F = d_\omega \omega = d\omega + \omega \wedge \omega \tag{5}$$

where we use the exterior algebra formulation. It reads in components :

$$F^{IJ}_{\mu\nu} = 2(\partial_{[\mu} \omega^{IJ}_{\nu]} - \omega^{IK}_{[\mu} \omega^J_{\nu]K}) \tag{6}$$

Curvature form components and Riemann tensor are linked through :

$$F^{IJ}_{\mu\nu} = e^I_\rho e^J_\sigma R^{\rho\sigma}{}_{\mu\nu} \tag{7}$$

We notice that when (7) is a tensorial relation, (4) is not, because curvature is a tensor while connection isn't (explaining the so-called Christoffel symbols and not "Christoffel tensor"). We can so now write the action for gravity with tetrad formulation :

$$S_{Pal}[e] = \frac{1}{2} \int d^4x \; \epsilon_{IJKL} \; \epsilon^{\mu\nu\rho\sigma} \; e^I_\mu e^J_\nu \; F^{KL}_{\rho\sigma} = \int e \wedge e \wedge \star F \tag{8}$$

where $\star$ denotes the Lorentzian (i.e. on Lorentz indices) Hodge star in the last. One can note connection is still Levi-Civita connection, i.e. torsion-free and metric. The last is given by :

$$\begin{aligned} \nabla g = \nabla \eta &\doteq 0 \\ d\eta_{IJ} - \omega_I^K \eta_{KJ} - \omega_J^K \eta_{IK} &= 0 \\ \omega_{(IJ)} &= 0 \end{aligned} \tag{9}$$

It's nice to see here that metricity of the connection form (so that $\omega_{IJ} = \omega_{[IJ]}$) shows that connection form takes its value in the Lie algebra of the Lie group (here Lie algebra of the Lorentz group, $\mathfrak{so}(3,1)$).

1.3 Cartan Theory

An interesting idea, introduced by Cartan [1], is to consider that the action could depend on both tetrad and connection as independent dynamical variables, removing the torsion-free assumption but keeping connection metricity (9). Introducing the action we'll call Palatini-Cartan action, let's do the variation of the action w.r.t. the connection :

$$\frac{\delta S_{PC}}{\delta \omega}[e,\omega] = \frac{1}{2}\int \frac{\delta}{\delta \omega}\left(e \wedge e \wedge d_\omega \omega\right)$$

using the Palatini equation $\delta d_\omega \omega = d_\omega \delta \omega$, a by-part integration and assuming the tetrad is invertible, we got without source the torsion-free condition !

$$d_\omega e \doteq T = 0 \tag{10}$$

Equations (10) and (9) completely determines the connection : $\omega = \omega[e]$, which is Levi-Civita Connection, and gives us back the original tetrad action for gravity. The strength of that formulation is both the possibility to generalize the theory to non-Levi-Civita connection and let the torsion-free condition status from assumption to a solution of a motion equation. If torsion source is present (typically fermions), connection could be written like :

$$\omega = \Gamma + C \tag{11}$$

[1] Historically, Cartan introduces connection dependency before development of tetrad formulation; but for more technical simplicity, we chose to present theories in this order.

where Γ is the Levi-Civita (i.e. metric and torsion-free) connexion and C is the contortion tensor, related to the torsion by :

$$C_{IJK} = \frac{1}{2}\left(T_{KIJ} - T_{JKI} - T_{IJK}\right) \tag{12}$$

1.4 Holst Action

The Holst action is an extension of the the Palatini action. We added to the last a topological term, i.e. a term classically null which does not change the motion equations. This term is :

$$\delta_{IJKL}\; e^I \wedge e^J \wedge F^{KL}$$

where $\delta_{IJKL} \doteq \delta_{I[K}\delta_{L]J}$. Indeed we can see that term is identically null :

$$\begin{aligned} \epsilon^{\mu\nu\rho\sigma}\delta_{IJKL}e^I_\mu e^J_\nu F^{KL}_{\rho\sigma} &= \epsilon^{\mu\nu\rho\sigma}\delta_{IJKL}e^I_\mu e^J_\nu e^K_\alpha e^L_\beta R^{\alpha\beta}{}_{\rho\sigma} \\ &= \epsilon^{\mu\nu\rho\sigma}\delta_{\mu\nu\alpha\beta}R^{\alpha\beta}{}_{\rho\sigma} \\ &= \epsilon^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma} \\ &= 0 \end{aligned}$$

where we used the Bianchi identity in the last equation assuming we got the Levi-Civita connection.

We then have the Holst action :

$$S_\gamma \doteq S_{HP} + \frac{1}{\gamma}\int \delta_{IJKL}e^I \wedge e^J \wedge F^{KL} = \int \left(\frac{1}{2}\epsilon_{IJKL} + \frac{1}{\gamma}\delta_{IJKL}\right) e^I \wedge e^J \wedge F^{KL}$$

where γ is a proportionality term called the Barbero-Immirzi parameter. It will play an important role in the Hamiltonian analysis of the theory, because an elegant choice of γ can drastically simplify the analysis.

2 Sympleptic Structure

In the first part we have described different actions for gravity. We now have to focus on the description of the structure which will give us the quantum aspects of the theory. We then need to understand what is a sympleptic form, and several useful properties of it which will be necessary to understand the construction of the theory.

2.1 Sympleptic form and properties

Given a manifold $\mathcal{M}$, a sympleptic form ω on $\mathcal{M}$ is a closed and non-degenerated 2-form (bilinear, antisymmetric)

$$\omega(u,v) = -\omega(v,u)$$
$$\omega(\alpha u + \beta v, w) = \alpha\omega(u,w) + \beta\omega(v,w)$$
$$d\omega = 0$$

Given a function f on $\mathcal{M}$, we can so define a vectorial field X_f called the sympleptic gradient of f such as :

$$-df = i_\omega X_f = \omega(X_f, .)$$

where i_ω is the inner product. A common notation of this field is $X_f \doteq \nabla_\omega f$ and reads in components :

$$X^\mu = -\omega^{\mu\nu}\partial_\nu f$$

An important theorem we will use in the following is the Darboux theorem : We can always find a coordinate system (p_i, q^i) in which we have :

$$\omega = dp_i \wedge dq^i$$

The sympleptic structure of a manifold allows us to define an other structure called Poisson structure : mathematically, we define the Poisson tensor (or bi-vector) as the inverse of the sympleptic 2-form generating the sympleptic structure. Such tensor defines a product between two functions of the manifold, called Poisson Bracket [2].

[2] A Poisson Bracket and so Poisson structure could be defined more generally, it doesn't need a sympleptic form since Poisson Bracket hasn't to be non-singular

2.2 sympleptic dynamics

In Hamiltonians dynamics, a system is governed by an Hamiltonian $\mathcal{H}(p,q)$ and the physical states are given bye the curves $q(t), p(t)$, solutions of motion equations, the Hamilton equations. We will now see how the sympleptic formalism is powerful : with a correct choice of ω, the sympleptic gradient of the Hamiltonian are solutions of the motion equations by construction. Indeed, we first define the sympleptic gradient of Hamiltonian, ξ :

$$\xi \doteq \nabla_\omega H$$

Then in virtue of the Darboux theorem, we can chose a base in which $\xi = (\xi^{q^i}, \xi^{p_i})$ where the previous equation becomes (in components) :

$$\begin{cases} \xi^{q^i} = \frac{\partial \mathcal{H}}{\partial p_i} \\ \xi^{p_i} = -\frac{\partial \mathcal{H}}{\partial q^i} \end{cases} \tag{13}$$

The Lie algebra of the Lie group allows us to define a diffeomorphism using ξ,

$$\varphi_t = \exp t\underline{\xi}$$

We can so make a one-to-one correspondence between the vector field ξ and the equivalence class of curves (i.e. one solution of motion equations) $C_m(t) \doteq \tilde{m}(t) = (q(t)_i, p(t)^i)$ $(m \approx m_0 \Leftrightarrow m = \phi_t m_0\ )$: ξ is the tangential vector field of C_m An important property is then we have :

$$\xi(m) = \frac{d}{dt}(\varphi_t(m) = \dot{m} \tag{14}$$

Then using the Darboux theorem to introduce canonical variables in (14), and injecting the last in (13) gives us the well-known Hamilton equations.

So we have the evidence that the sympleptic gradient of the Hamiltonian are solutions of motion equations.

2.3 Towards quantization

An important point to clear from now on is why it is important to define a sympleptic structure in order to quantize, and so what is canonical quantization. Canonical quantization is a correspondence made between a theory said "classical" to an other called "quantum". The physical quantities measured in the classical theory are now eigenvalues of operators acting on an Hilbert Space (a space where you define an hermitian product). The rigorous correspondence is given between the Poisson algebra defined through the sympleptic structure and the non-commutative algebra of operators on Hilbert Space. That's why defining a sympleptic structure is important in order to quantize the theory.

2.4 Constraints

It happens that dynamical systems have to deal with so-called constraints, i.e. relations that have to be verified by the variables which describe the system. They are classified in two types, said to be first or second class. The first class constraints are a set of relations that form a close algebra. They are quite important because they have a physical sense : they represent the symmetries of the system. The diffeomorphism and Gauss constraints we saw previously are part of it, because they are the algebraic expressions of resp. space-time and Lorentz invariance. The second class constraints are more difficult to deal with : they expressed redundancy in the set of dynamical variables. Dirac developed a protocol to follow in order to deal with them for any case but it is technically hard to use; that's why it is more elegant to find a formulation where no second class constraints appear. This was the historic meaning of introduction of Immirzi parameter by Ashtekar : a correct choice of γ allow us to change second class constraints into first class constraints during Hamiltonian analysis of Holst action. The physical meaning is that the Lorentz group can be split in two subgroups : $SO(3,1) \approx SO(2) \oplus SO(2)$, and changing second to first class constraints tells us we only need to deal with one of the subgroup of the Lorentz instead of the entire group.

3 Sympleptic structure for gravity

3.1 ADM action

In order to perform a canonical analysis of the theory, we make a foliation of the manifold, assuming : $\mathcal{M} \cong \mathbb{R} \times \Sigma$. We decompose the time flow vector $\tau^\mu \doteq (1,0,0,0)$ into its normal and tangential parts to Σ , $\tau^\mu = Nn^\mu + N^\mu$, and with the correct parametrization, we have $N^\mu = (0, N^a)$.

We could define an induced metric q_{ab} on Σ through $q_{\alpha\beta} \doteq g_{\alpha\beta} - n_\alpha n_\beta$, then the object $q^\alpha_\beta \doteq g^{\alpha\gamma} q_{\gamma\beta}$ acts as a projector on Σ. We can so define the covariant derivative, the extrinsic curvature and the curvature of Σ using the induced metric :

$$\begin{cases} {}^3\nabla_\mu X_\nu \doteq q^\rho_\mu q^\sigma_\nu \nabla_\rho X_\sigma \\ K_{\alpha\beta} \doteq {}^3\nabla_\mu n_\nu = q^\gamma_\alpha \nabla_\gamma n_\beta \\ {}^3R^\alpha{}_{\beta\gamma\delta} \doteq q^\alpha_\mu q^\nu_\beta q^\rho_\gamma q^\sigma_\delta R^\mu{}_{\nu\rho\sigma} - 2K_{\beta\{\gamma} K^\alpha_{\delta\}} \end{cases} \tag{15}$$

Some calculations we can see in appendix of [1] lead us to :

$$S = \int d^4x \left({}^3R + K^2 - Tr(KK) \right) \tag{16}$$

where ${}^3R = {}^3R^{\alpha\beta}{}_{\alpha\beta}$ is defined through the induced curvature on Σ, $K^2 \doteq (K^\mu_\mu)^2$ and $Tr(KK) \doteq K^{\alpha\beta}K_{\beta\alpha}$

We now have to perform a Legendre transform in order to make appear the sympleptic structure of the theory, i.e. the conjugate canonical variables. We first notice that N and N_a are not dynamical variables (no time derivative of them appeared in the action). It's relevant due to the arbitrary foliation of space-time ; they are indeed Lagrange multipliers. The dynamical variables are so the components of the induced metric q_{ab} and their conjugate momentum are :

$$p^{ab} \doteq \frac{\partial \mathcal{L}}{\partial(\partial_0 q_{ab})} = \sqrt{(det(q)} \left(K^{ab} - Kq^{ab} \right)$$

this bring us to the Hamiltonian formulation of the action :

$$S_{EH} = \int d^4x \left(p^{ab}\partial_0 q_{ab} - NH - N^a H_a \right)$$

where the scalar and vectorial constraints are respectively defined through

$$\begin{cases} H_a \doteq -2q^3_{ab} \nabla_c p^{bc} \approx 0 \\ H \doteq -\sqrt{detq}^3 R + \frac{1}{\sqrt{detq}} \mathcal{G}_{abcd} p^{ab} p^{cd} = \doteq -\sqrt{detq}^3 R + \frac{1}{\sqrt{detq}} \left(p^{ab}p_{ab} - \frac{p^2}{2} \right) \approx 0 \end{cases}$$

where we have the so-called DeWitt super-metric :

$$\mathcal{G}_{abcd} \doteq \frac{1}{2}\left(q_{ac}q_{bd} + q_{ad}q_{bc} - q_{ab}q_{cd}\right)$$

The important thing to observe here is that the Hamiltonian is only the sum of constraints : this shows us that the real Hamiltonian is classically null, due to the time (and more generally space-time) covariance. But the arbitrary foliation allow us to study the constraints, which will contain all the dynamics of the system.

3.2 Hamiltonian analysis of Holst-Action

We will do here the same work as previously but using Holst Action. First the same foliation we used previously allows us to separate tetrad components into its purely space parts (so-called triad), the shift and lapse. To simplify the problem we use the time gauge . Now we need to perform the most important variable change, due to Ashtekar, called Ashtekar parameters, introducing the densitized triad and the Ashtekar-Barbero connection :

$$\begin{cases} E^a_i \doteq \frac{1}{2}\epsilon_{ijk}\epsilon_{abc}e^j_b e^k_c \\ A^i_a \doteq \gamma {\omega_a}^{0i} + \frac{1}{2}{\epsilon^i}_{jk}{\omega_a}^{jk} \end{cases} \tag{17}$$

With such variables we can rewrite the action and see how scalar and vectorial constraints are written, and see an additional constraint (so called Gauss constraint), expressing the Lorentz invariance :

$$S_\gamma = \int d^4x \left[\dot{A}^i_a E^i_a - A^i_0 G_i - NH - N^a H_a\right] \tag{18}$$

where

$$\begin{cases} G_i \doteq \partial_a E^a_j + \epsilon_{ijk} A^j_a E^{al} \\ H_a = \gamma^{-1} F^j_{ab} - \gamma^{-1}(\gamma^2+1) K^i_a G_i \\ H = \left[F^j_{ab} - (\gamma^2+1)\epsilon^j_{mn} K^m_a K^n_b\right] \frac{\epsilon^{kl}_j E^a_k E^b_l}{detE} + \gamma^{-1}(\gamma^2+1) G^i \partial_a \frac{E^a_i}{detE} \end{cases} \tag{19}$$

The study of constraint algebra detailed in [1] shows us that giving to the Immirzi parameter the value $\pm i$ just remove the second class terms appearing in the constraints. The key to understand such a variable change is that the original connection took his values in $\mathfrak{so}(3,1)$, while the new Ashtekar connection did in $\mathfrak{su}(2)$. The problem of that choice is that the action is now complex, and real conditions are so hard to deal with that research conducted currently focused on real value of Immirzi Parameter. The last object we need to introduce are the holonomies and the flux: they are the resp. integrals of the triad and the connection on reps. paths and surface of $\mathcal{M}$. The paths are the central objects defining the Hilbert space we will use in quantization.

4 Quantization

As we said in 2.3, quantization is about giving a correspondence between Poisson algebra of classical theory and operator algebra of an Hilbert space. The following section deals with how we define the Hilbert space for gravity.

4.1 Quantization Scheme

In constrained quantum Hamiltonian dynamics, the physical Hilbert space is the reduction of a first "naive" Hilbert space to the kernels of the constraint operators :

$$\mathcal{H}_0 \overset{\hat{G}^i=0}{\longrightarrow} \mathcal{H}_{kin} \overset{\hat{H}^a=0}{\longrightarrow} \mathcal{H}_{Diff} \overset{\hat{H}=0}{\longrightarrow} \mathcal{H}_{phys} \tag{20}$$

Current research topics deals with resolving the last constraint. The following will explain how we define the naive Hilbert space, and how an elegant choice of elements of that space gives us a gauge invariant subspace, resolving automatically the Gauss constraint.

4.2 The naive Hilbert space

As we said in the end of previous section, the final conjugate variables (holonomy-flux) are defined on graphs, i.e. a collection of oriented paths of Σ. We can define then so-called cylindrical function on those graphs, which are complex evaluated function of holonomy values of graph links (e_i) :

$$\psi_{(\Gamma,f)}[A] = f(h_{e_i}[A]) \tag{21}$$

Space of cylindrical functions of a graph Γ is noted Cyl_Γ. This space could be turned into an Hilbert space by defining the following scalar product, thanks to a particular measure called Haar measure, which is the unique gauge-invariant and normalized measure.

$$\langle \psi_{(\Gamma,f)} | \psi_{(\Gamma,f')} \rangle \doteq \int \prod_e dh_e \overline{f(h_{e_i}[A])} f'(h_{e_i}[A]) \tag{22}$$

The Hilbert space associated to the graph Γ is noted $\mathcal{H}_{Gamma}$. Our naive Hilbert space is then the sum of Hilbert spaces associated to the all graphs of Σ, with a measure found by Ashtekar and Lewandowsky called the Ashtekar-Lewandowsky measure, noted $d\mu_{AL}$

4.3 Gauge Invariant space

Solving the Gauss constraints could be done easily by defining an averaging of cylindrical functions. We can associate to any cylindrical function f its averaged function $\tilde{f}$ following :

$$\tilde{f}(h_i) \doteq \int \prod_n dg_n f(g_{s_i} h_i g_{t_i}^{-1})$$

Those functions automatically solve the gauss constraints, because they are $SU(2)$-invariant by construction.

Conclusion

Loop Quantum Gravity gives us a canonical way to quantize gravity, and kinematics of it are well understood. But LQG still has open topics: dynamics are not completely solved, and research currently focus on finding solution, like the spinfoam formalism: one can define an object called spinfoam using a path integral, which represents the time propagation of a spin-network. An other problem is the status of the Immirzi : when we know that it has to disappear in the classical theory, we don't know if it plays a physical role in the quantum theory or if it is just an elegant mathematical trick.

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