Loop statistics of the Coulomb phase

From Spin Ice to Kagome Planes, IIP Natal – 16 December 2011

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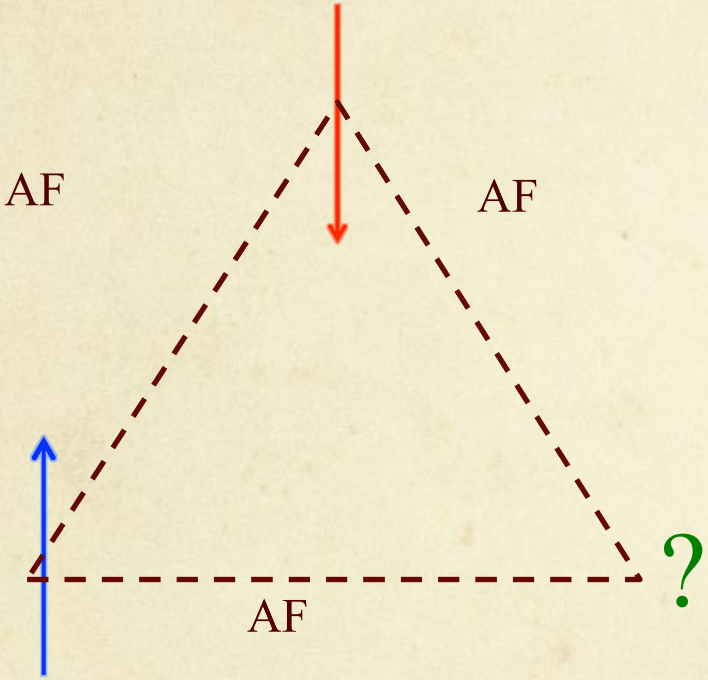
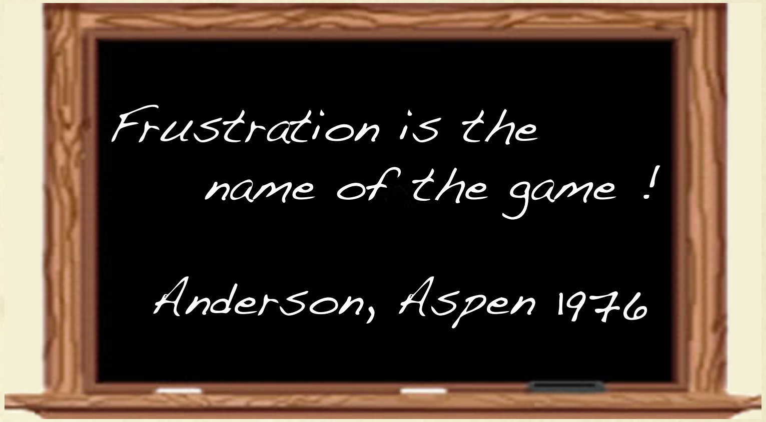
MAX-PLANCK-GESELLSCHAFT

What is frustration ?





Frustration is the
name of the game !

Anderson, Aspen 1976

Source:
GGG, Rev. Modern Phys. 2010

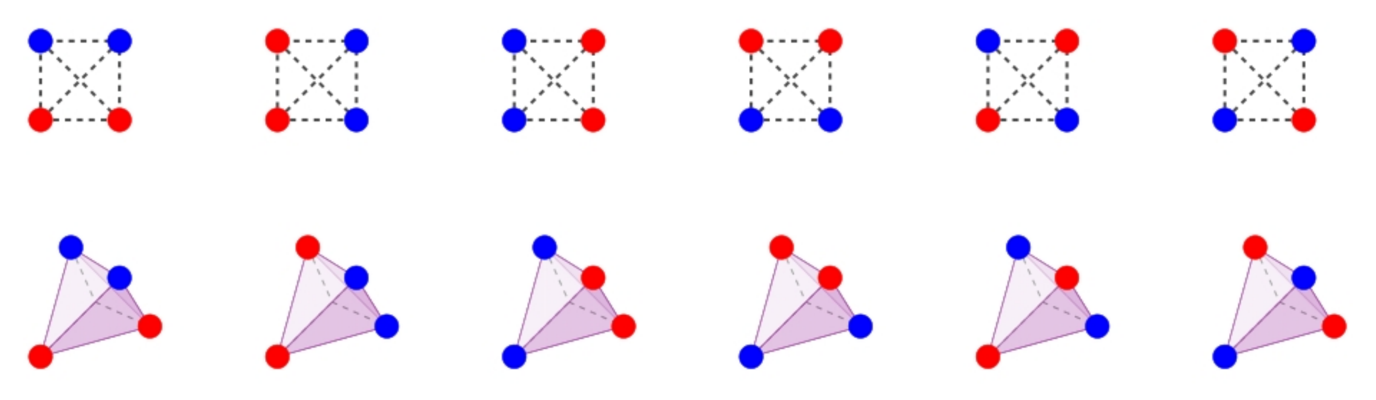
Outline

1. **What is the Coulomb phase ? ✔**

2. **Statistics of the Coulomb phase**
   - Emergence of loops
   - Differences between 2 and 3 dimensions ?

3. **Itinerant electrons on the Coulomb phase**
   - doping dependence, conduction or not ?
   - magnetic order ?

Coulomb phase (thermal spin liquid)



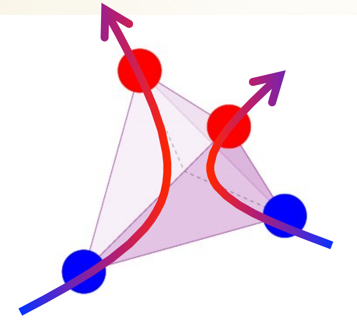
Entering flux **B**

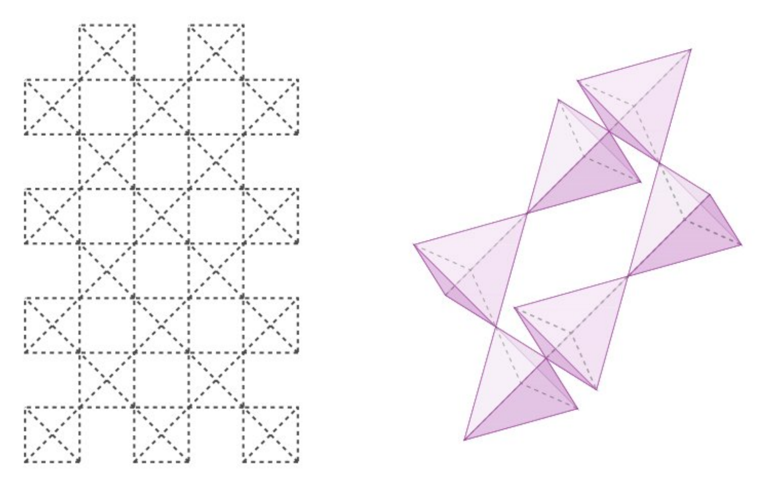
Exiting flux **B**

Flux conservation

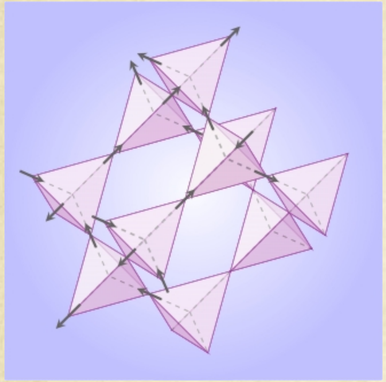
$$\boxed{\nabla \cdot \mathbf{B}(\mathbf{r}) = 0}$$

Talk of Powell

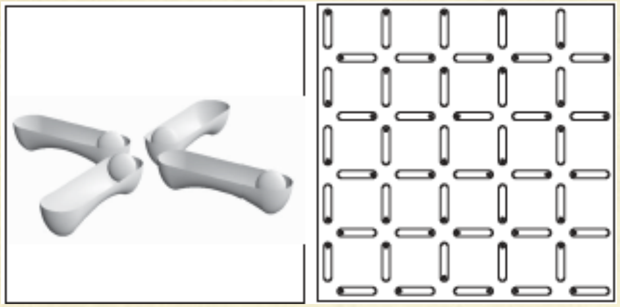




Coulomb phase: realisations



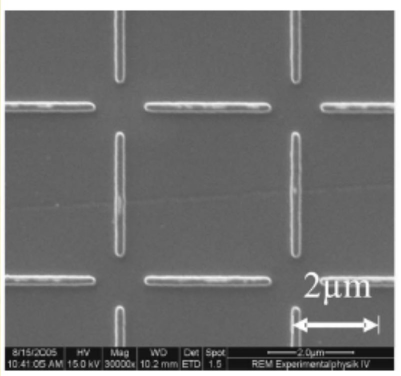
Harris *et al.*
PRL 1997
Spin Ice
$Dy_2Ti_2O_7$



Libal *et al.*
PRL 2006
Colloids in
optical traps

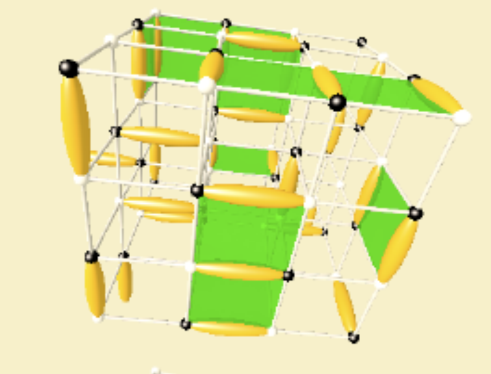
Henley Ann. Rev. Cond. Mat. Phys. 2010

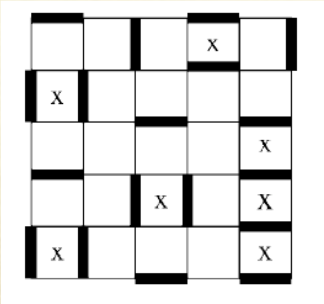
- Ising spins: spin ice, artificial spin ice
- Local half filling:
  atomic disorder, electrons/charges…
- Dimer models, kagome-type lattices…

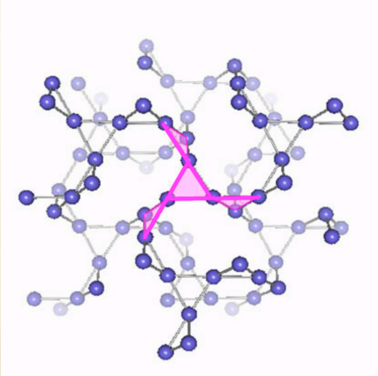


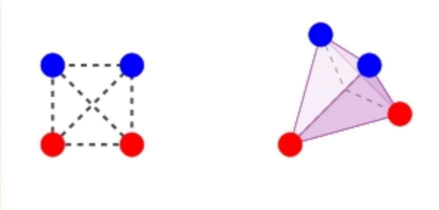
Artificial Spin Ice
Magnetic nano-arrays

Talks Tuesday,
Wednesday



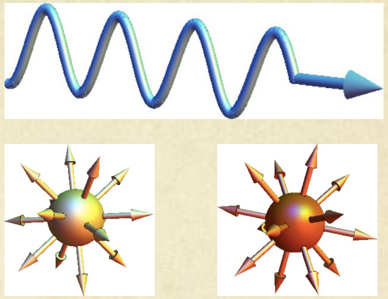






Banks & Bramwell
EPL 2011
Atomic disorder
($CsNiCrF_6$)

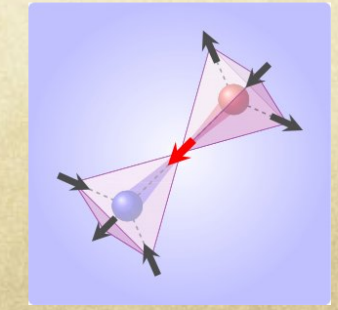
What to do with Coulomb phase ?



- Add quantum fluctuations:
  - U(1) spin liquid with emergent quantum electrodynamics magnetic and electric charges, gapless photon
    Hermele, Fisher, Balents PRB 2004
    talks by Savary and Balents (Coulomb ferromagnet)

  - materials $Yb_2Ti_2O_7$, $Tb_2Ti_2O_7$, $Tb_2Sn_2O_7$, $Pr_2Zr_2O_7$
    Ross, Savary et al. PRX 2011, Onoda & Tanaka PRL 2010
    Bonville, Mirebeau et al. PRB 2011,
    McClarty, Gingras et al. JPCM 2010
    talks on Monday and Tuesday

- Add thermal fluctuations:
  magnetic monopoles Claudio et al. Nature 2008



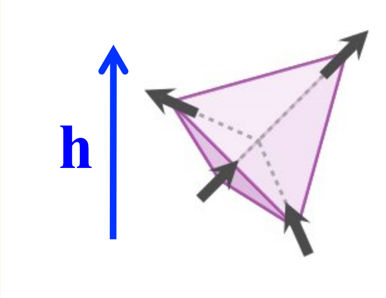
Thank you to Lucille Savary & Kate Ross for the pictures !

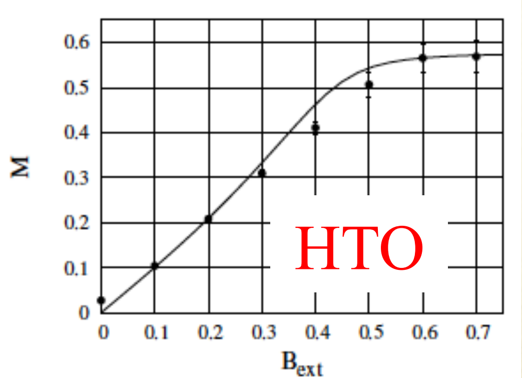
What to do with Coulomb phase ?

Higgs transition (Powell PRB 2011)

Magnetic field: Kasteleyn transition

Moessner, Sondhi PRB 2003
Jaubert et al. PRL 2008
Powell & Chalker PRB 2008
Morris et al. Science 2009
Harmann-Clark et al. in preparation

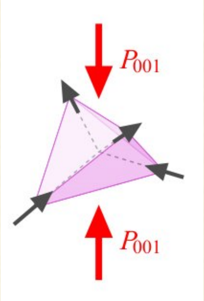


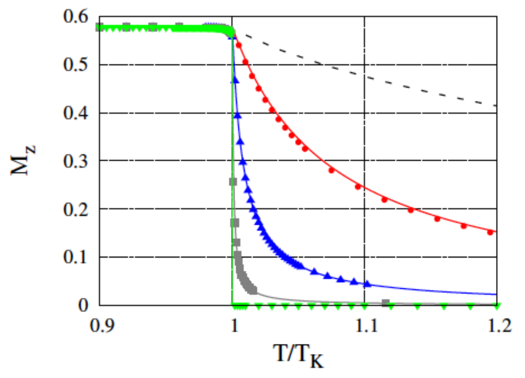


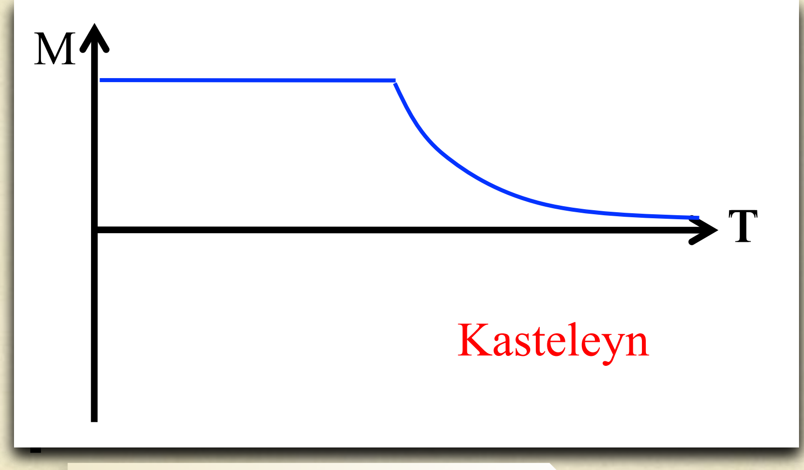
Fennell et al. PRB 2005

Uniaxial Pressure:
KDP transition

Jaubert et al. PRL 2010

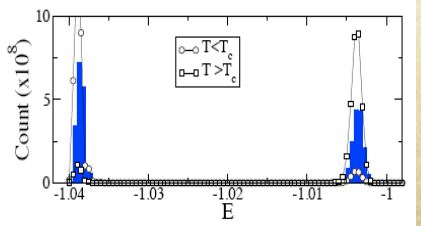


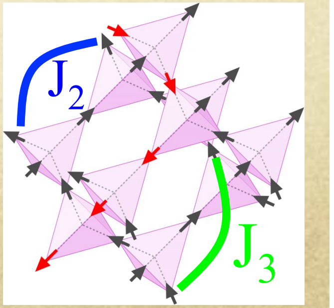




2nd nearest neighbors coupling:
1st order transition

Chou & Kao PRB 2010





Outline

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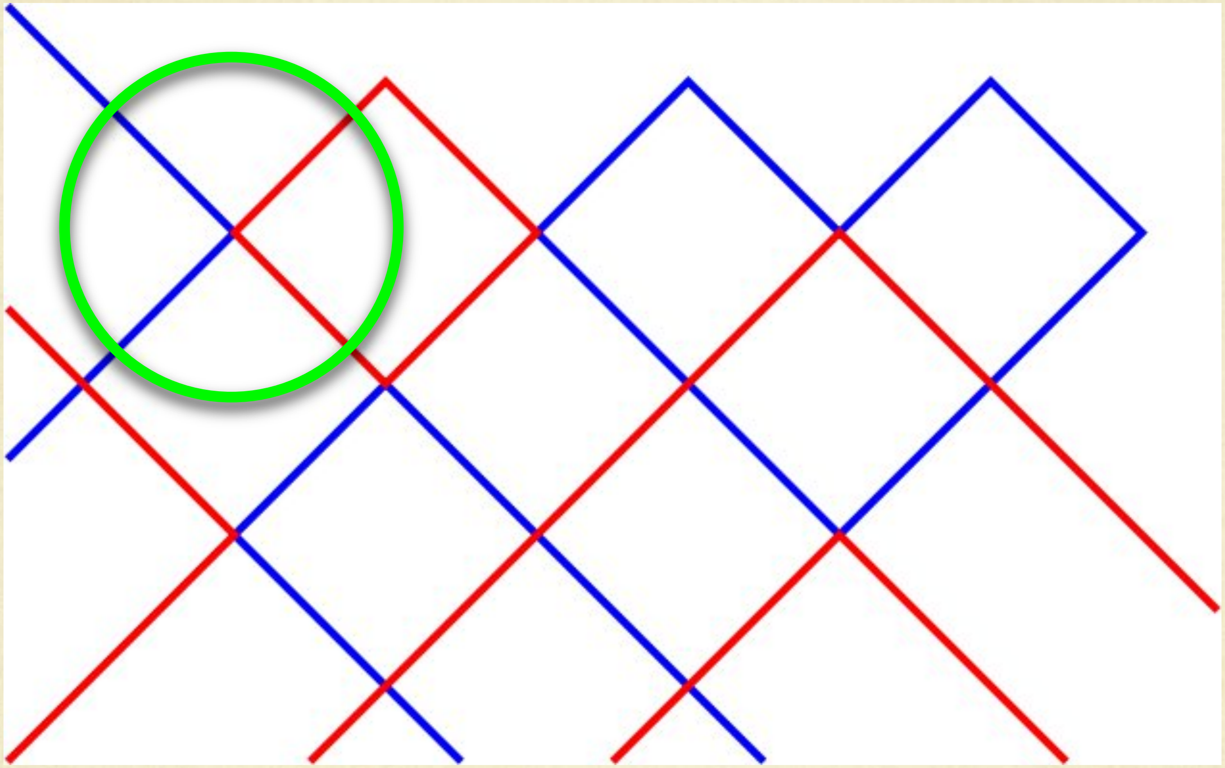
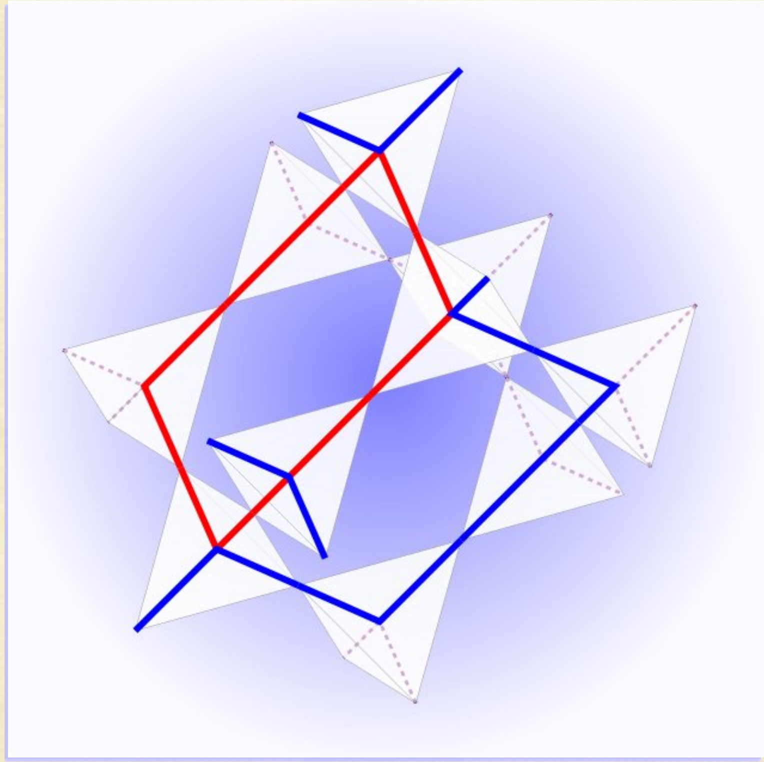
Emergence of loops (2D)



Fully packed loop model

Self-Avoiding Loops

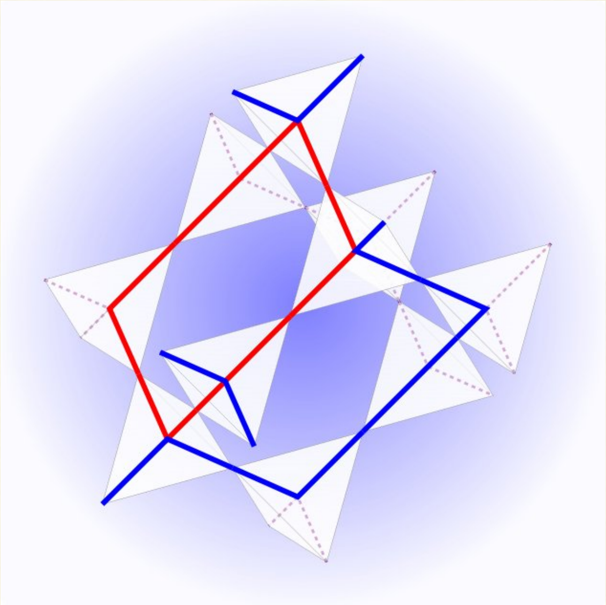
Emergence of loops (3D)



Fully packed loop model

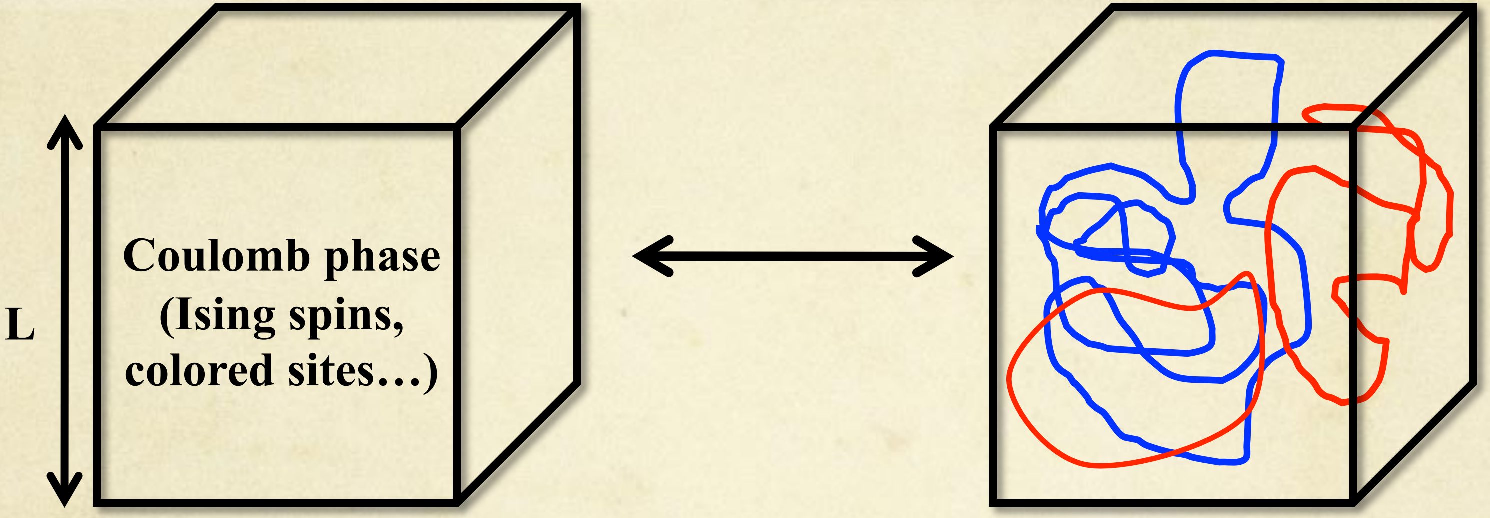
Self-Avoiding Loops

Motivations



- Another point of view on the Coulomb phase via non-local objects (loops)
- Contact with random walks, percolation, polymers (θ-point), cosmic strings, SLE…
- Application to itinerant electrons

Simulations

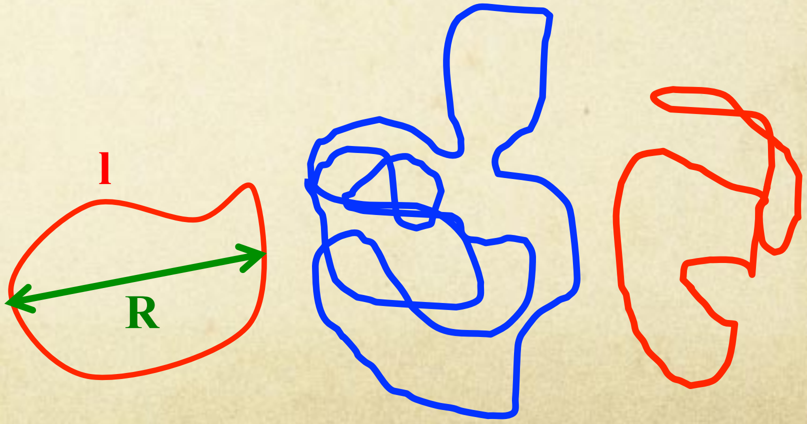


Coulomb phase
(Ising spins,
colored sites…)

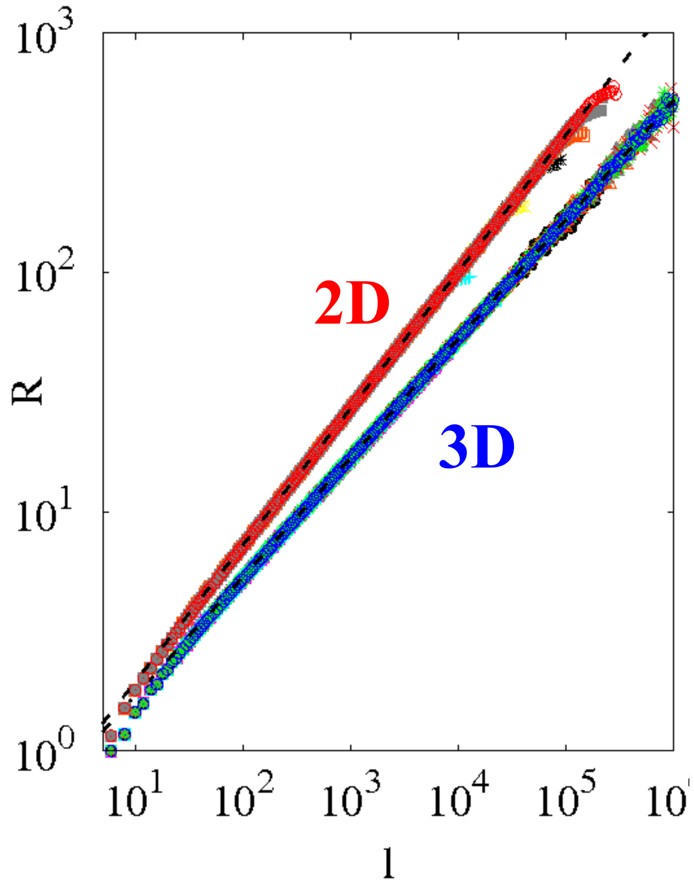
Box full
of loops

l = loop length
R = radius of gyration
L = linear system size

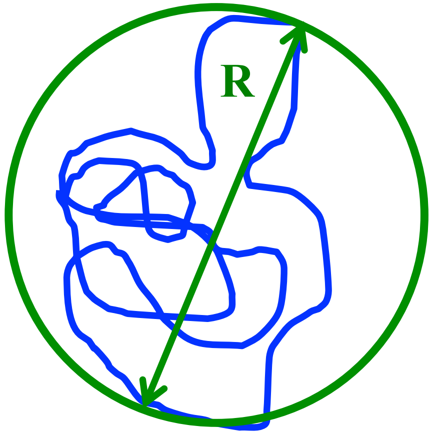
- Monte Carlo simulations (worm algorithm) on checkerboard and pyrochlore lattice
- For different system sizes : 10^3 to 10^7 sites
- Up to $10^6 - 10^7$ independent samples



Radius of Gyration



$$\boxed{R \sim \ell^{\nu}}$$



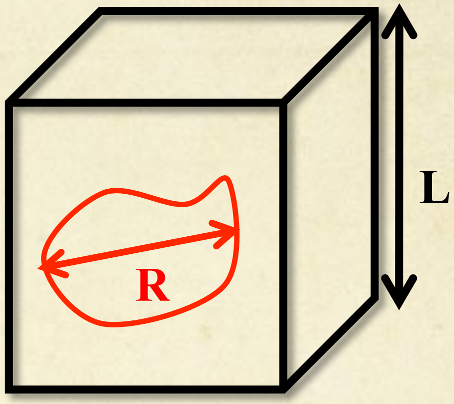
Fractal dimension $D_f = 1/\nu$

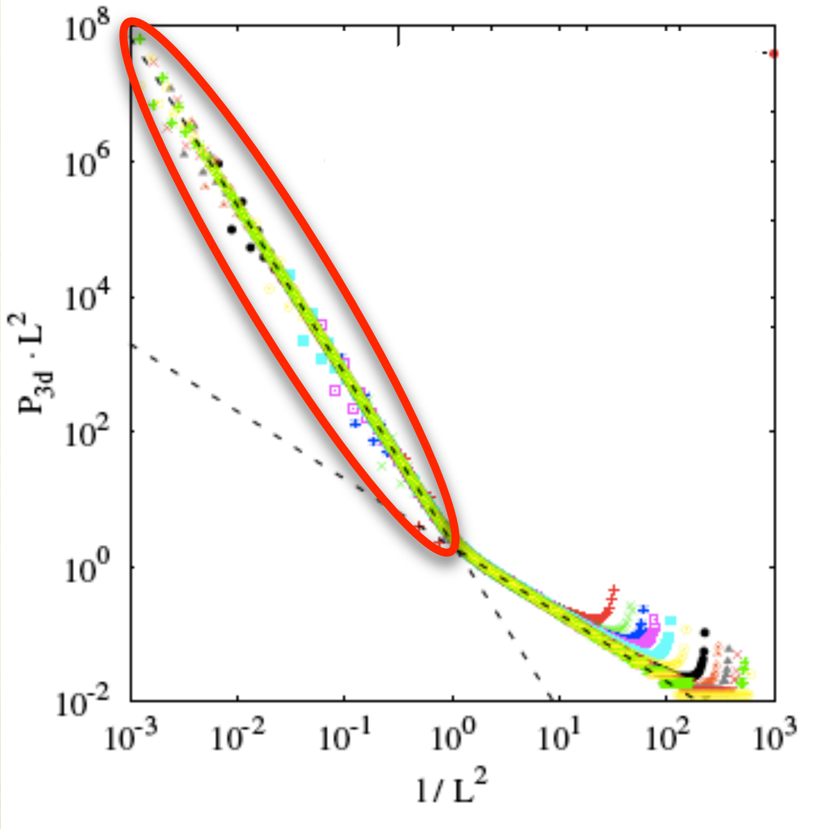
$$\nu_{3D} = 1/2 \Rightarrow \text{RandomWalk}$$

$$\underset{\substack{\uparrow \\ \text{Random Walk}}}{1/2} < \nu_{2D} = 4/7 < \underset{\substack{\uparrow \\ \text{Self-Avoiding Walk}}}{3/4}$$

Same exponent as polymers at θ-point

Number of loops per sample



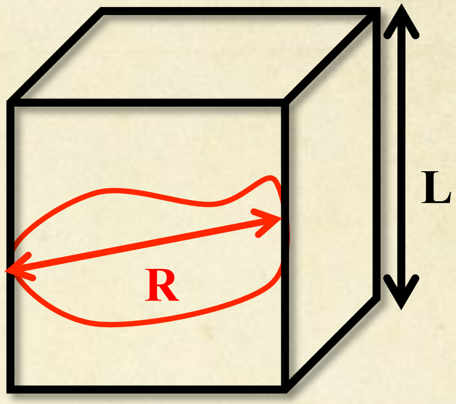


Two regimes:

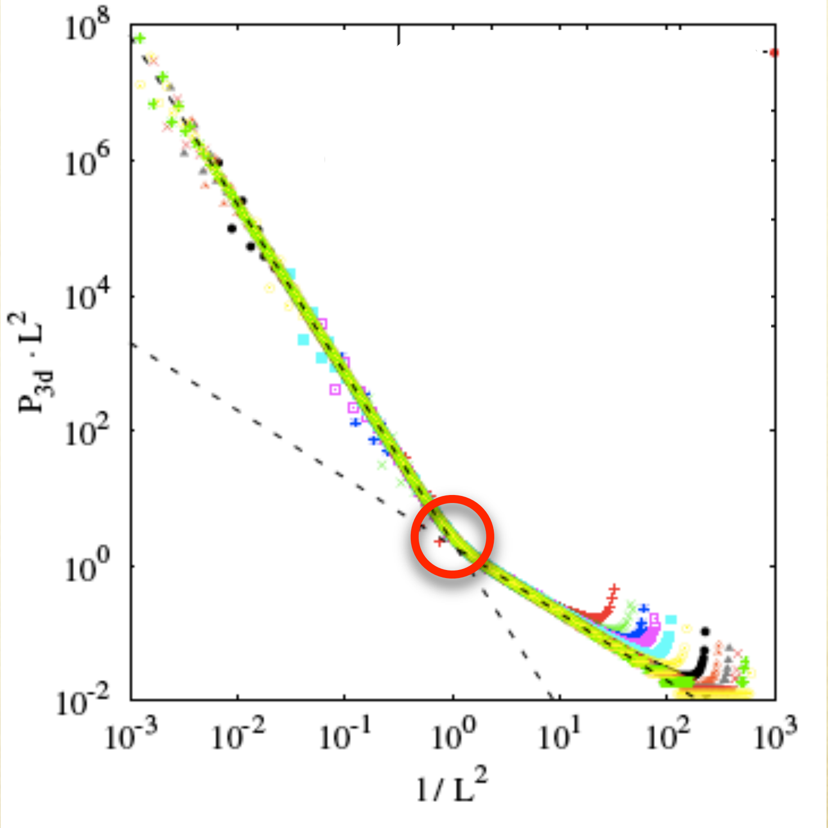
$$P_{3d}(\ell) \propto \begin{cases} \dfrac{L^3}{\ell^{5/2}} & (\ell < L^2) \text{ non-winding} \\[2em] \dfrac{1}{\ell} & (\ell > L^2) \text{ winding} \end{cases}$$

$$P_{3d}(\ell) \approx \frac{1}{L^2}\, f\left(\frac{\ell}{L^2}\right)$$

Crossover



$$R \propto \sqrt{\ell}$$



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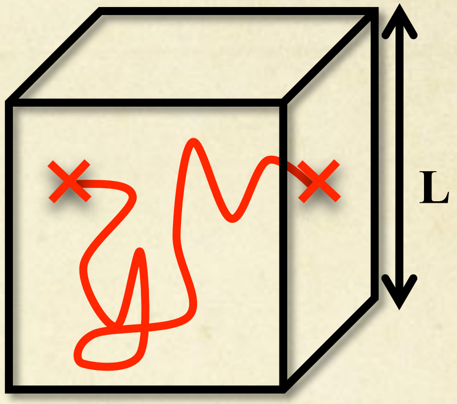
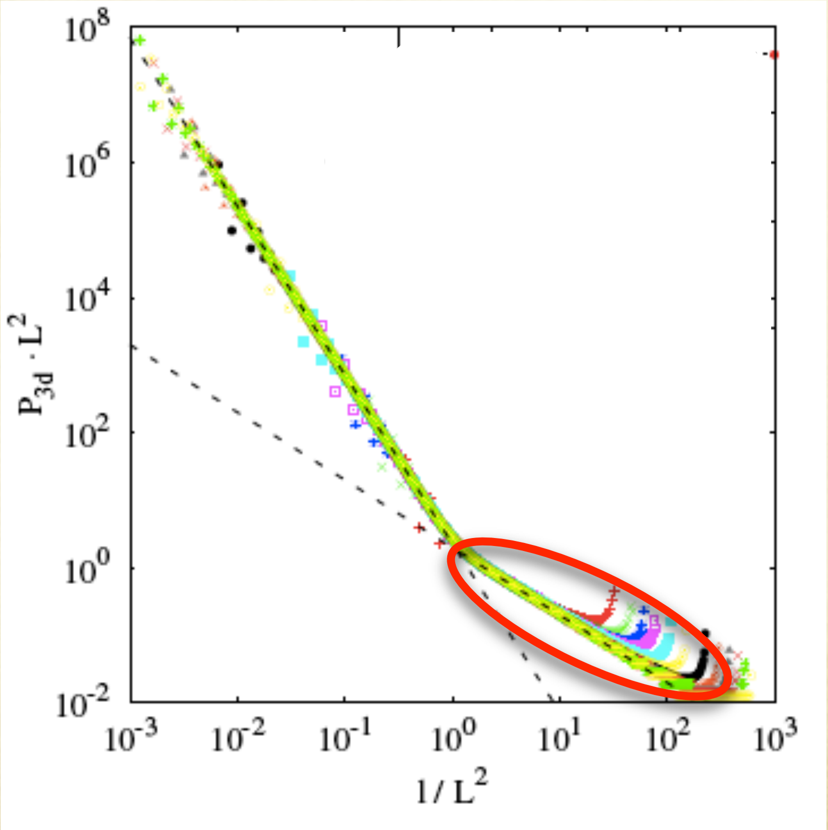
Macroscopic winding loops





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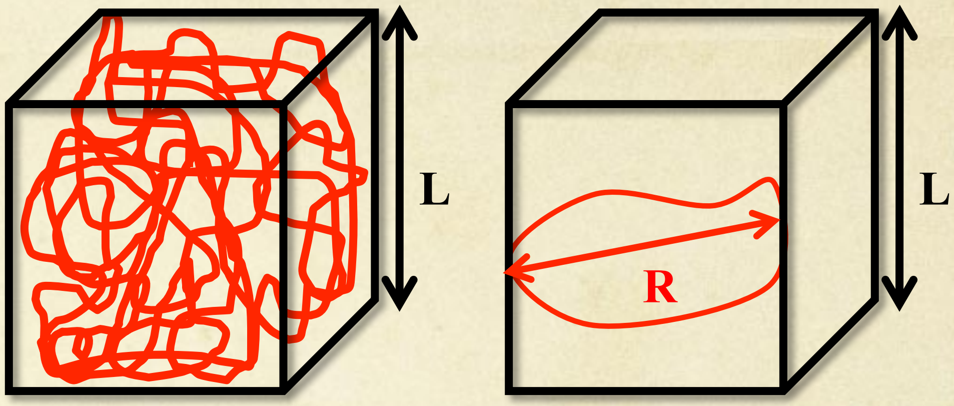
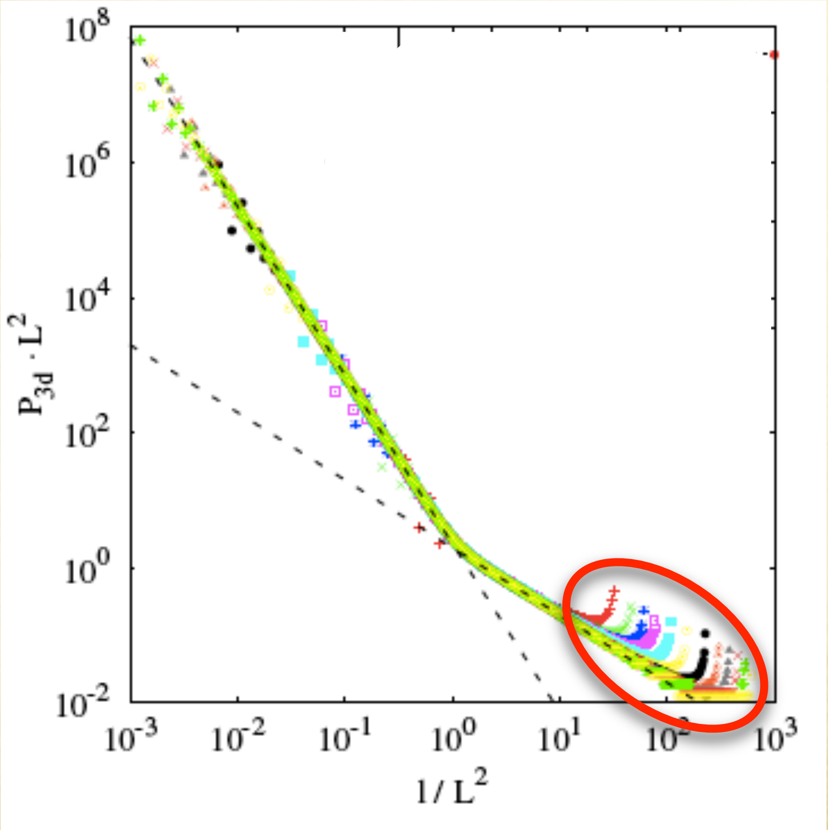
Two length scales





Two regimes:

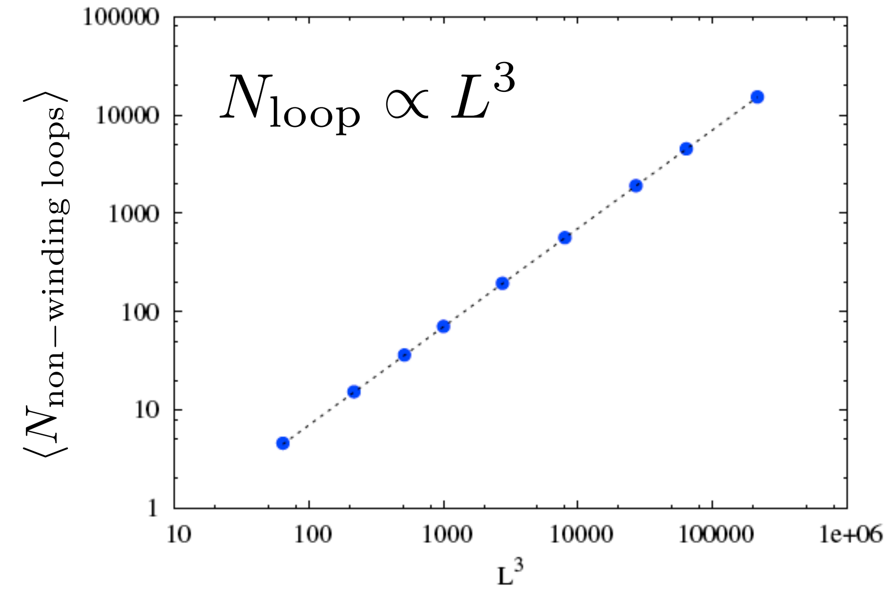
$$P_{3d}(\ell) \propto \begin{cases} \dfrac{L^3}{\ell^{5/2}} & (\ell < L^2) \text{ non-winding} \\ \\ \dfrac{1}{\ell} & (\ell > L^2) \text{ winding} \end{cases}$$

Two length scales:

despite a unique “natural” length scale “L”, two of them are appearing: L^2 and L^3.

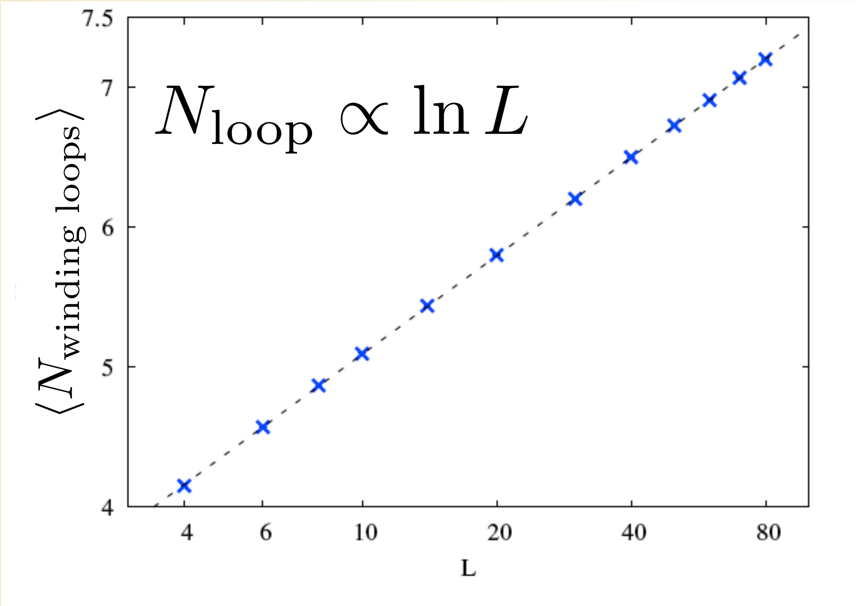
Number of loops per sample

Number of “short” non-winding loops



Number of sites belonging to
- non-winding loops : 6.31 %
- winding loops : 93.69 %

Number of “long” winding loops

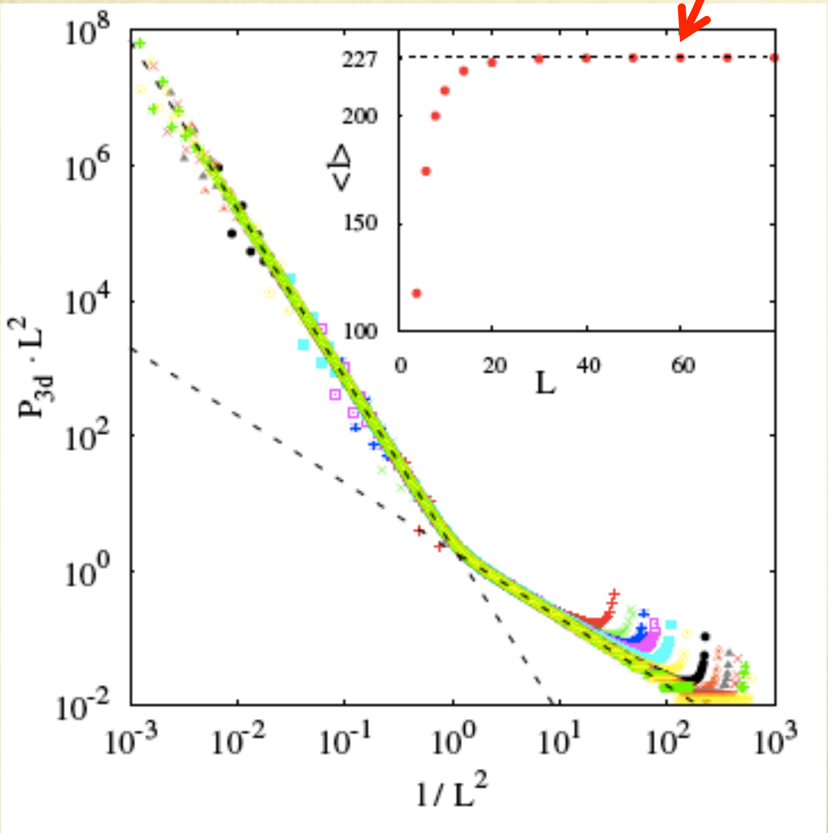


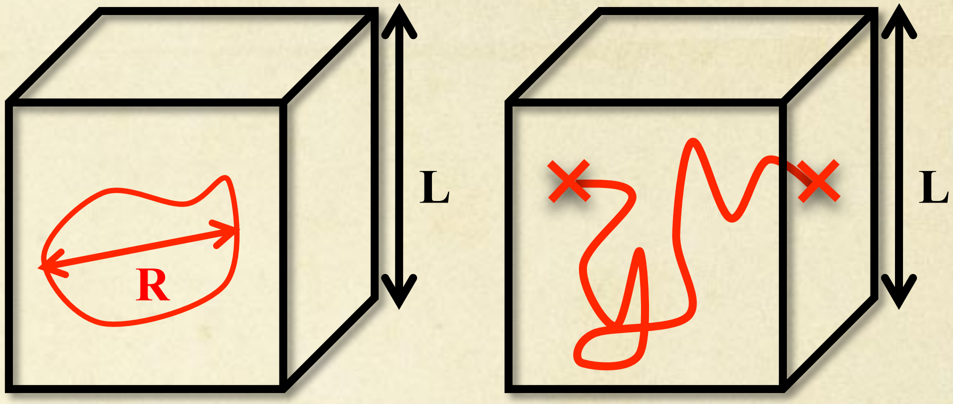
$\langle N_{\text{non-winding loops}} \rangle$

Finite average loop length

Finite average loop length:

$$\langle \ell \rangle = 227.3 \pm 0.1$$





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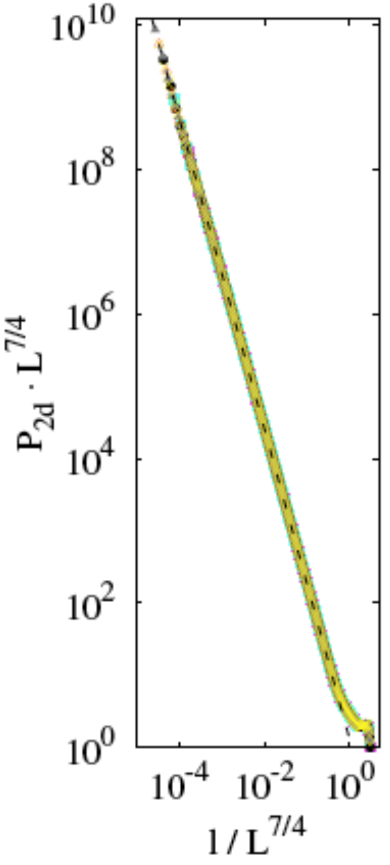
2D & 3D scaling

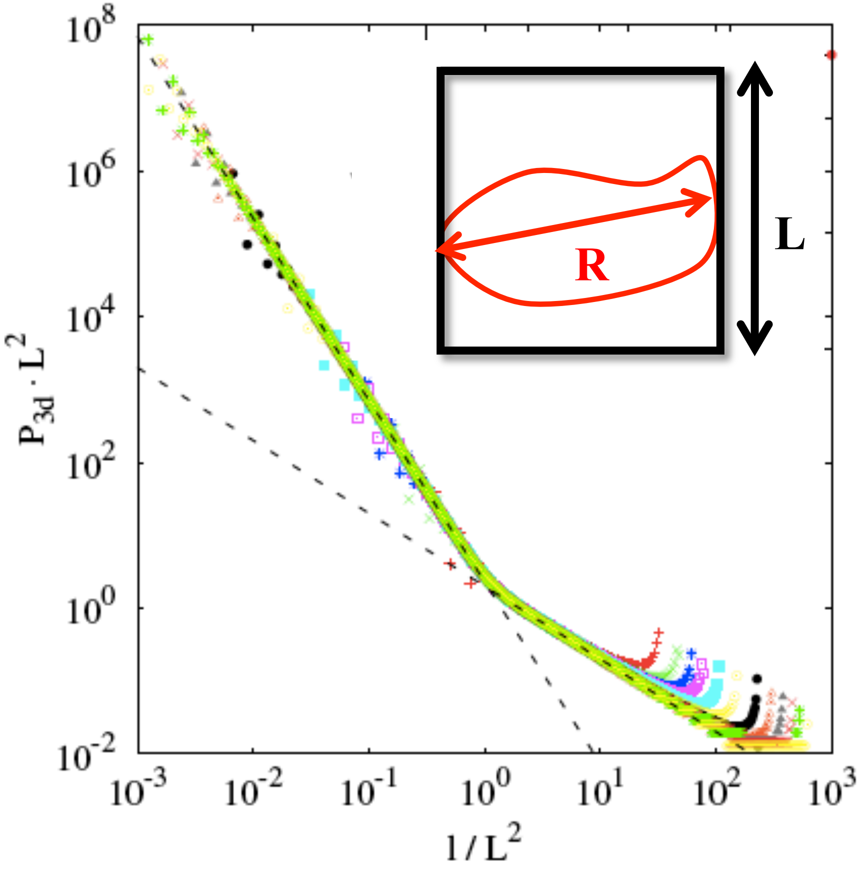
2D - checkerboard

$$\langle \ell \rangle_{2d} = 24.678$$



3D - pyrochlore

$$\langle \ell \rangle_{3d} = 227.3$$



$$P_{2d}(\ell) \propto \frac{L^2}{\ell^{2+1/7}}$$

Jacobsen, Vannimenus
J. Phys. A (1999)

Very good scaling in 2d
= unique length scale

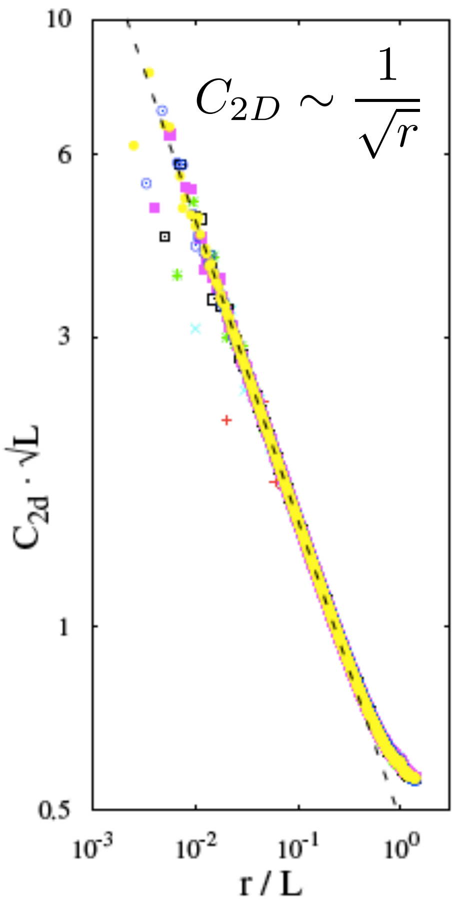
$$P(\ell) \approx \frac{1}{L^\alpha} \, f\left(\frac{\ell}{L^\alpha}\right)$$

$$\begin{cases} \alpha_{2d} = 7/4 \\ \alpha_{3d} = 2 \end{cases}$$

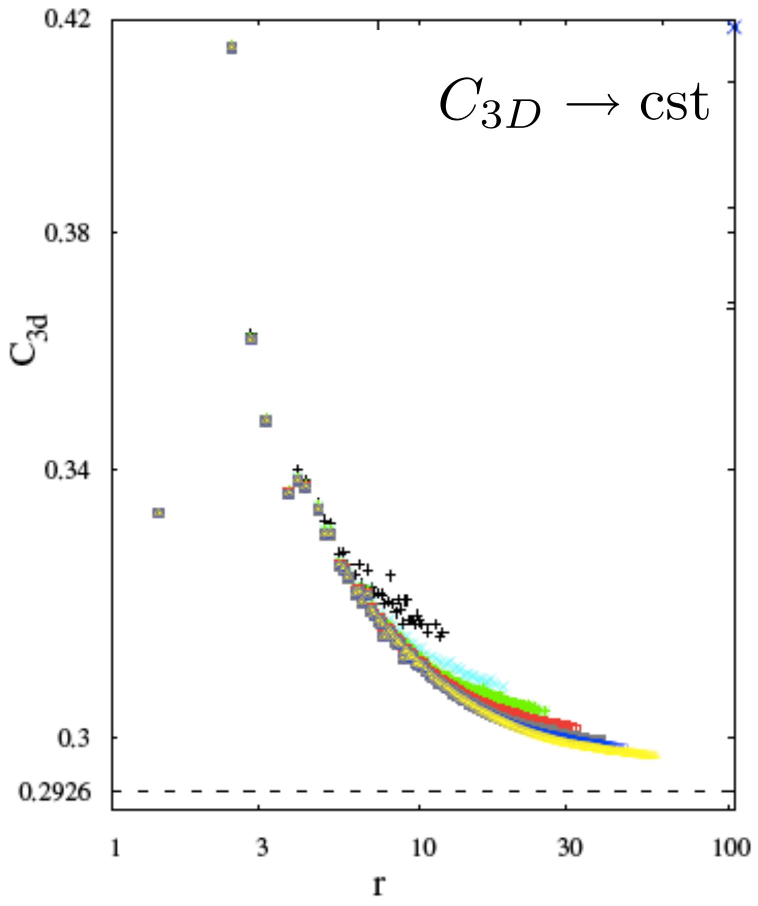
In 2d, a loop “sees” the boundaries at the same time as it fills the system.

Probability to be on the same loop

2D - checkerboard

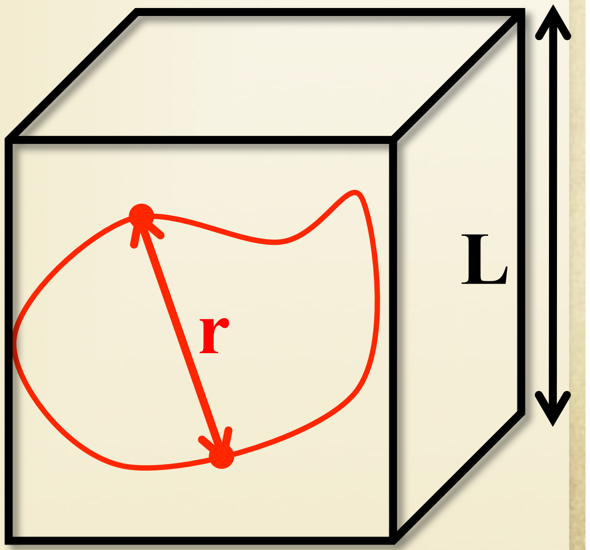


3D - pyrochlore

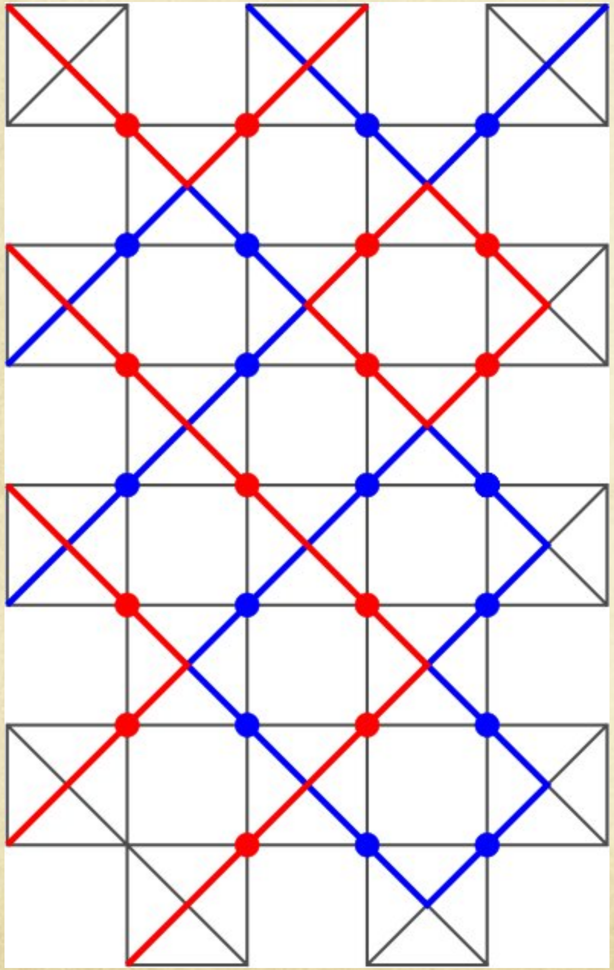


Probability for 2 sites to be on the same loop:

- power law decay in 2d Jacobsen & Kondev Nucl. Phys. (1998)
- decrease to a constant in 3D = winding loops occupy most of the system.



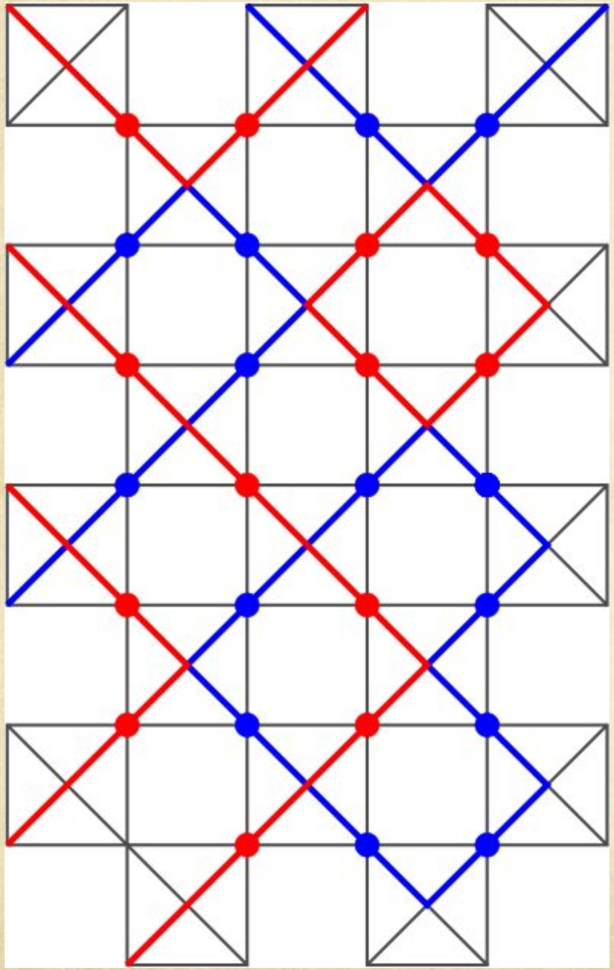
Consequences

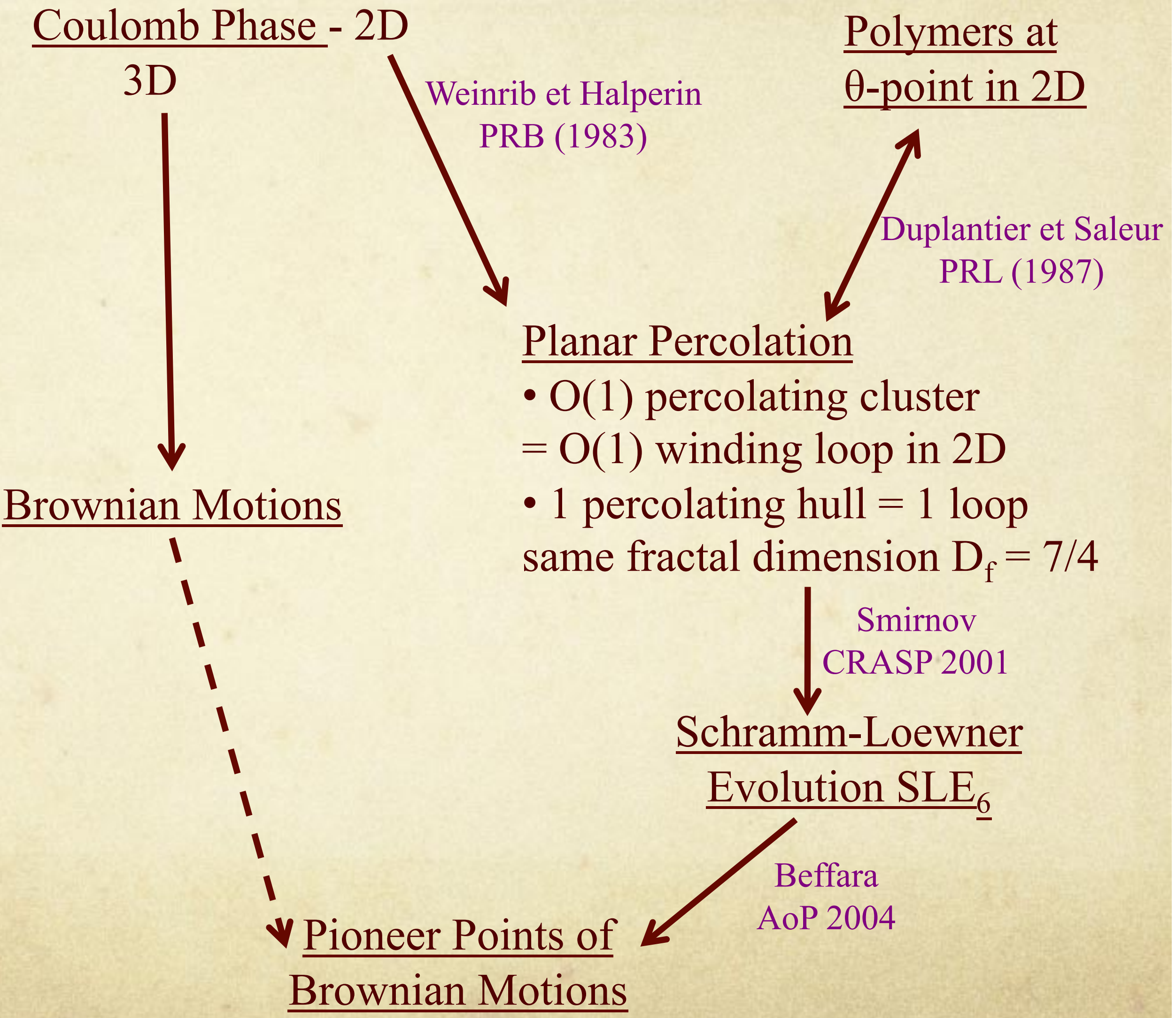


Heisenberg magnets ($CsNiCrF_6$ on pyrochlore lattice)

a) 1 color = 1 kind of magnetic atom (blue = Cr, red = Ni)
b) Antiferromagnetic exchange along a loop, no interaction between loops
c) Correlations along a loop, but perfectly random between loops
d) Finite probability to be on the same loop
   = Long Range Magnetic Order !

Connections





Coulomb Phase - 2D
3D

Weinrib et Halperin
PRB (1983)

Polymers at
θ-point in 2D

Duplantier et Saleur
PRL (1987)

Brownian Motions

Planar Percolation
- O(1) percolating cluster = O(1) winding loop in 2D
- 1 percolating hull = 1 loop same fractal dimension $D_f = 7/4$

Smirnov
CRASP 2001

Schramm-Loewner
Evolution SLE_6

Beffara
AoP 2004

Pioneer Points of
Brownian Motions

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Double exchange + frustration

Ising spins on the checkerboard or pyrochlore lattice + itinerant electrons via double exchange

$$\begin{aligned}\mathcal{H} &= J\sum_{\langle i,j\rangle}\mathbf{S}_i\cdot\mathbf{S}_j\\ &\quad-\sum_{\langle i,j\rangle,\alpha} t_{ij}\,(c^\dagger_{i,\alpha}c_{j,\alpha}+c^\dagger_{j,\alpha}c_{i,\alpha})\\ &\quad-J_h\sum_{i,\alpha,\beta}c^\dagger_{i,\alpha}(\sigma_{\alpha,\beta}\cdot\mathbf{S}_i)\,c_{i,\beta}\end{aligned}$$

$$t_{ij}\ll J_h\ll J$$

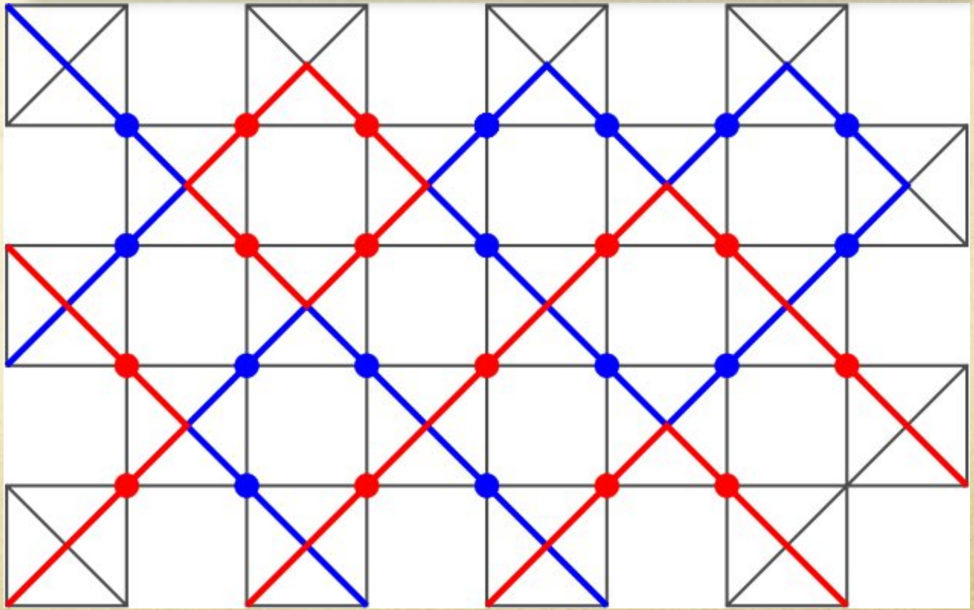
- 1d filaments of conduction predicted in Mn: Viret et al. PRL 2004
- $Nd_2Mo_2O_7$, $Pr_2Ir_2O_7$

Frustration + itinerant electrons:

Kagome: Ohgushi et al. PRB 2000, Taillefumier et al. PRB 2006, Shimomura et al. JPSJ 2006

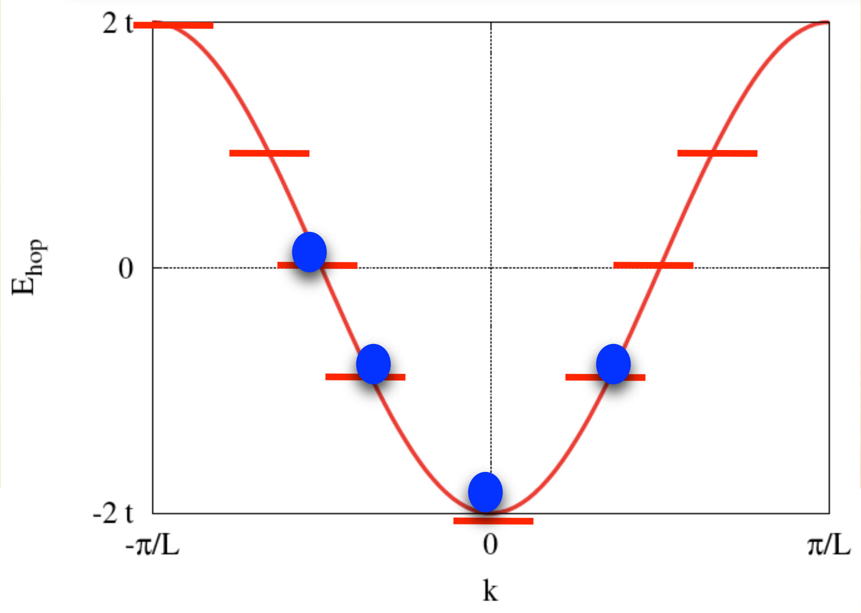
Triangular: Fujimoto PRL 2009

Pyrochlore: Onoda et al. PRL 2003, Motome et al. PRL 2010, Kalitsov et al. JPCS 2009



Hopping on 1d finite chains

$$
\begin{aligned}
\mathcal{H}_{\text{hop}} &= -\,t\sum_{j=1}^{\ell}\left(c_j^\dagger\, c_{j+1} \;+\; c_{j+1}^\dagger\, c_j\right) \\
&= -\,2t\sum_{k}\cos(k)\, c_k^\dagger c_k \;=\; -\,2t\sum_{k}\cos(k)\,\widehat{n}_k
\end{aligned}
$$

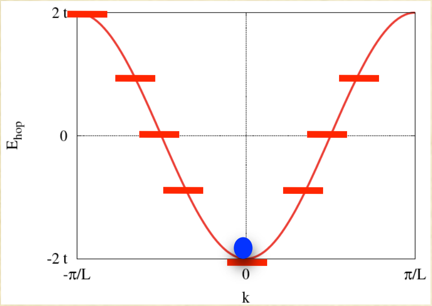


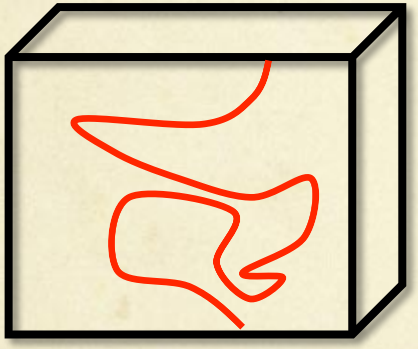
Energy levels on a loop of 8 sites

1 spin configuration
= loop configuration (h_i of length l_i)
= a given energy spectrum
= a given Fermi level according to the density of electrons ρ.

What configuration corresponds to the lowest energy / ground state ?

Phase diagram with doping ρ

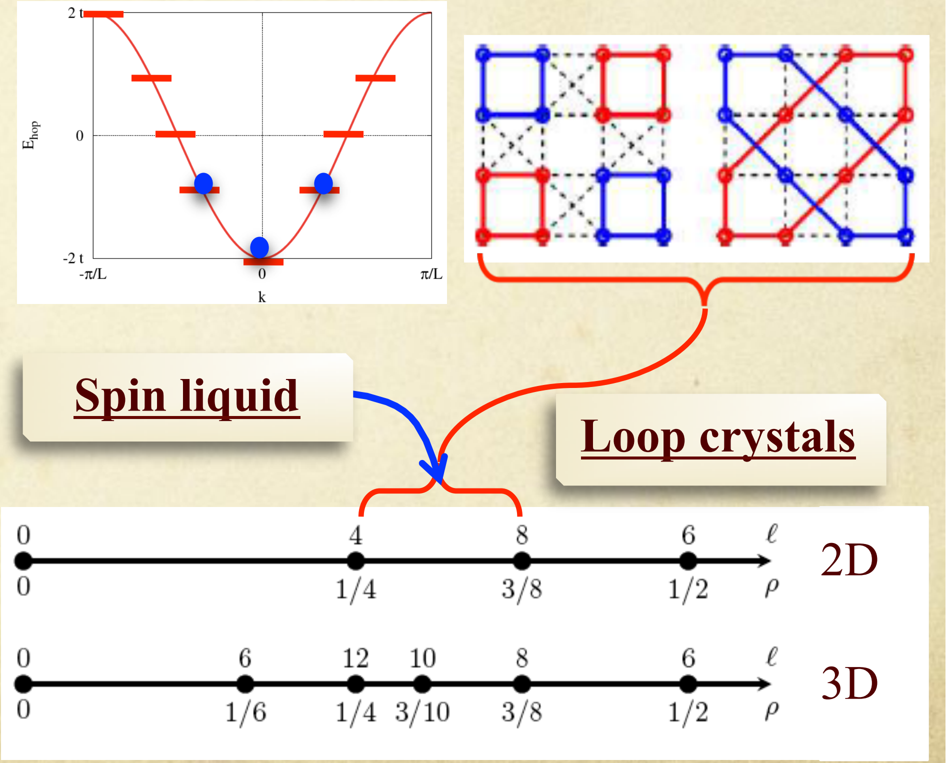




$N_{electrons} < N_{loops}$: 1 electron / loop at most

Doping = cutoff of number of loops

Number of conducting channels ~ cst with ρ



Spin liquid

Loop crystals

Conclusion

Statistical physics of the Coulomb phase: Jaubert, Haque & Moessner, PRL 2011
- mapped onto a fully packed loop model

- Loops in 3D ~ *Random Walk*
- Loops in 2D ~ *Percolation Hull*, polymers at θ-point

- two length scales appear in 3D, *probability to be on the same loop goes to a constant in 3D* because of winding loops occupying ~ 94% of the system

Influence of itinerant electrons: Jaubert, Piatecki, Haque & Moessner, on arxiv soon
- loops = conducting paths for itinerant electrons
- 2 regimes for doping:
  - low doping: itinerant electrons are a perturbation, but most of the "ρ=0" physics remains, *conducting channels* spanning the system still exist
  - intermediate doping: *loop crystal* (when possible), non-trivial spin ordering, extensive degeneracy recovered for some values of doping ρ = classical spin liquid