# Interacting magnons and critical behaviour of bosons

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## Magnons in frustrated QAFM

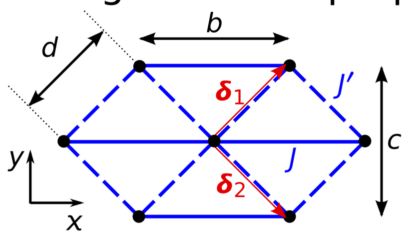
- $Cs_2CuCl_4$ as realization of triangular antiferromagnet
- model Hamiltonian from high field measurements [1]
- ordered phase below $T_N = 0.6$ K (spiral magnet)
- Spin-wave approach for "cone-state" delivers dispersion from linear spin-wave theory [2,3]
- Goal: calculate renormalization of phonons due to magnons

### Spin-Phonon Hamiltonian

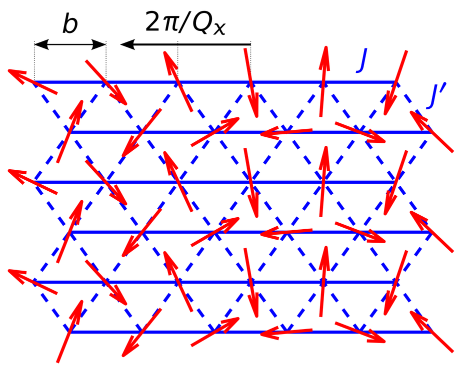
$$\mathcal{H}^{\mathrm{pho}}_{\mathrm{spin}} = \frac{1}{2}\sum_{ij}\left[J_{ij}\boldsymbol{S}_i\cdot\boldsymbol{S}_j + \boldsymbol{D}_{ij}\cdot(\boldsymbol{S}_i\times\boldsymbol{S}_j)\right] \quad (1)$$

$$-\sum_i \boldsymbol{h}\cdot\boldsymbol{S}_i + \sum_{\boldsymbol{k}\lambda}\left[\frac{P_{-\boldsymbol{k}\lambda}P_{\boldsymbol{k}\lambda}}{2M} + \frac{M}{2}\omega^2_{\boldsymbol{k}\lambda}X_{-\boldsymbol{k}\lambda}X_{\boldsymbol{k}\lambda}\right]$$

magnetic field perpendicular to the plane



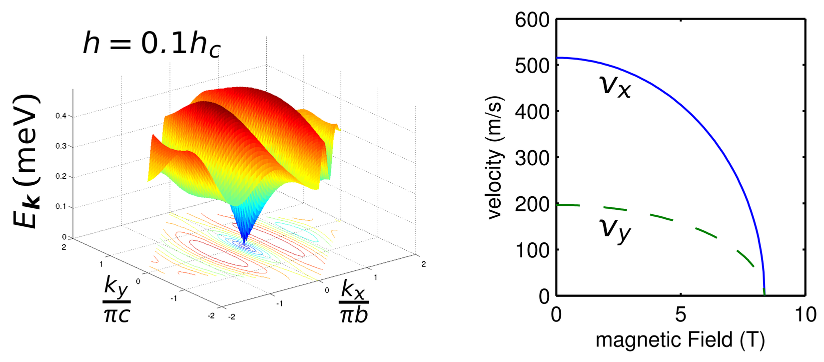

Geometry of a single layer of $Cs_2CuCl_4$: The magnetic ions (black dots) are mainly coupled via the coupling $J$ (solid line) in x-direction and $J'$ along the diagonals (dashed line).




Classical ground state: For $J' < 2J$ and a magnetic field perpendicular to the lattice plane the "cone-state" is realized. Note that the wave-length $2\pi/Q_x$ of the spiral has the indicated length and points along the x-axis in the direction of the arrow.

$\omega_{\boldsymbol{k}\lambda} = c_\lambda(\hat{\boldsymbol{k}})|\boldsymbol{k}|$ (acoustic phonons)
dispersion from linear spin-wave theory

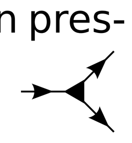
$$E_{-\boldsymbol{k}} \neq E_{\boldsymbol{k}} = \sqrt{(A^+_{\boldsymbol{k}})^2 - B^2_{\boldsymbol{k}}} + A^-_{\boldsymbol{k}} \approx v(\hat{\boldsymbol{k}})|\boldsymbol{k}| \quad (2)$$




(Left) Gapless Magnon dispersion $E_{\boldsymbol{k}}$ of the anisotropic triangular lattice antiferromagnet $Cs_2CuCl_4$ with $J'/J = 0.34$, Dzyaloshinskii-Moriya anisotropy $D/J = 0.054$, and for a magnetic field $h = 0.1h_c$ [2]. (Right) The magnon velocities in the two principal directions as a function of the magnetic field are in the adiabatic limit.

## Velocity and damping of phonons

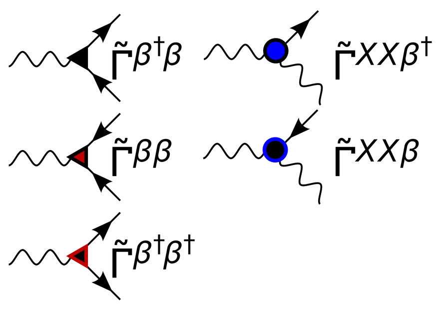
**magnon-magnon interactions** in presence of the magnetic field



$$\Gamma_3^{b^\dagger b^\dagger b}(\boldsymbol{k}_1,\boldsymbol{k}_2;\boldsymbol{k}_3) \approx i\cos\theta\sin\theta\sqrt{\frac{2}{S}}h_c \quad (3)$$

**Problem:** singular in the Bogoliubov basis

**Solution:** Use Hermitian parametrization to sort the relevant degrees of freedom [3]




Interaction processes for the spin-phonon Hamiltonian: One-phonon two-magnon interactions (renormalized by magnon shift, left) and two-phonon one-magnon interactions (right).

### magnon-phonon hybridization

$$\mathcal{H}^{1\mathrm{pho}}_{1\mathrm{mag}} = \frac{1}{2}\sum_{\boldsymbol{k}}\left\{\boldsymbol{\Gamma}^{X\Phi}_{\boldsymbol{k}}\cdot(\boldsymbol{X}_{-\boldsymbol{k}}\Phi_{\boldsymbol{k}} + \boldsymbol{X}_{\boldsymbol{k}}\Phi_{-\boldsymbol{k}}) + \boldsymbol{\Gamma}^{X\Pi}_{\boldsymbol{k}}\cdot(\boldsymbol{X}_{-\boldsymbol{k}}\Pi_{\boldsymbol{k}} - \boldsymbol{X}_{\boldsymbol{k}}\Pi_{-\boldsymbol{k}})\right\} \quad (4)$$

renormalization of the one-phonon two-magnon vertex due to magnon-magnon interactions and the magnon-phonon hybridization

$$\tilde{\boldsymbol{\Gamma}}^{b^\dagger b^\dagger}_{\boldsymbol{k},\boldsymbol{k}'} = \boldsymbol{\Gamma}^{b^\dagger b^\dagger}_{\boldsymbol{k},\boldsymbol{k}'} + \frac{i\Gamma_3^{b^\dagger b^\dagger b}(-\boldsymbol{k},-\boldsymbol{k}';\boldsymbol{k}+\boldsymbol{k}')\boldsymbol{\Gamma}^{X\Pi}_{\boldsymbol{k}}}{\sqrt{2(A^+_{\boldsymbol{k}} - B_{\boldsymbol{k}})}} \quad (5)$$

$G^{\mathrm{pho}}(K,\lambda)$, $G_{\mathrm{mag}}(K)$: Graphical representation of the phonon propagator (wiggly line) and the magnon Green functions (directed lines).

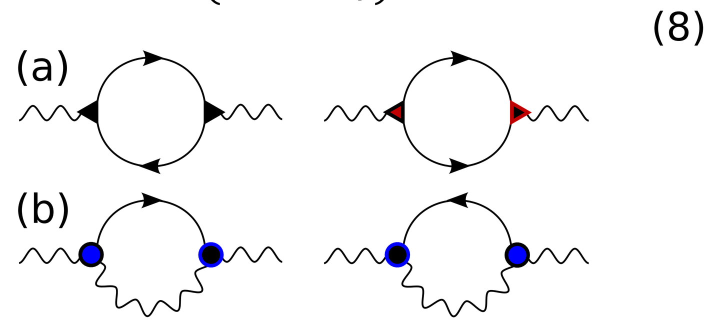
**Shift of elastic constants** in leading order $1/S$: Classical spin background and renormalization due to hybridization

$$\frac{\Delta c_\lambda}{c_\lambda} = \lim_{|\boldsymbol{k}|\to 0}\left(\sqrt{1 - \frac{\Sigma_0^{\mathrm{pho}}(\boldsymbol{k},\lambda)}{\omega^2_{\boldsymbol{k}\lambda}} + \frac{\left|\boldsymbol{\Gamma}^{X\beta}_{\boldsymbol{k}}\cdot\boldsymbol{e}_{\boldsymbol{k}\lambda}\right|^2}{2M\omega^3_{\boldsymbol{k}\lambda}}} - 1\right) \quad (6)$$

### Ultrasonic attenuation rate

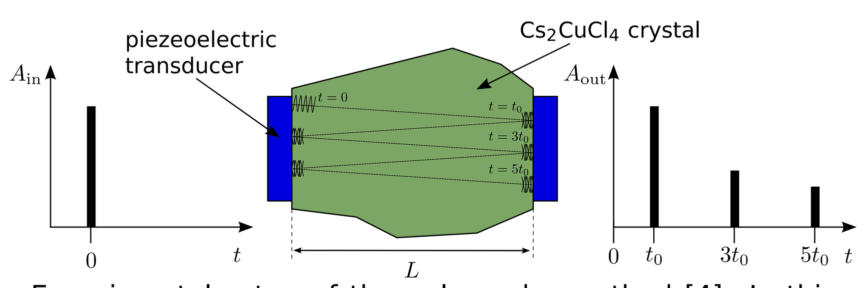
$$\gamma_{\boldsymbol{k}\lambda} = -\frac{\mathrm{Im}\Sigma_2^{\mathrm{pho}}(\omega_{\boldsymbol{k}\lambda}+i0,\boldsymbol{k},\lambda)}{2\omega_{\boldsymbol{k}\lambda}} \quad (7)$$

$$\gamma_{\boldsymbol{k}\lambda} \approx \frac{\pi^2}{64}\left(\frac{\boldsymbol{k}^2}{2M}\right)\left(\frac{S^2c^2_\lambda\boldsymbol{k}^2}{V_{BZ}v_xv_y}\right)\frac{\left[\boldsymbol{f}_1^{X\beta}(\hat{\boldsymbol{k}})\cdot\boldsymbol{e}_{\boldsymbol{k}\lambda}\right]^2}{(1-h/h_c)^2} \quad (8)$$

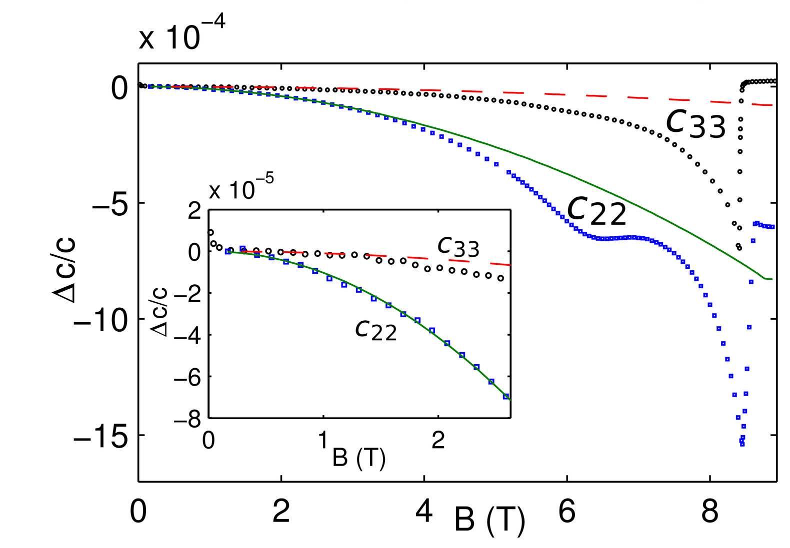



Feynman diagrams for the phonon self-energy: One-phonon two-magnon processes (a), two-phonon one-magnon processes (b).

## Comparison to experiments



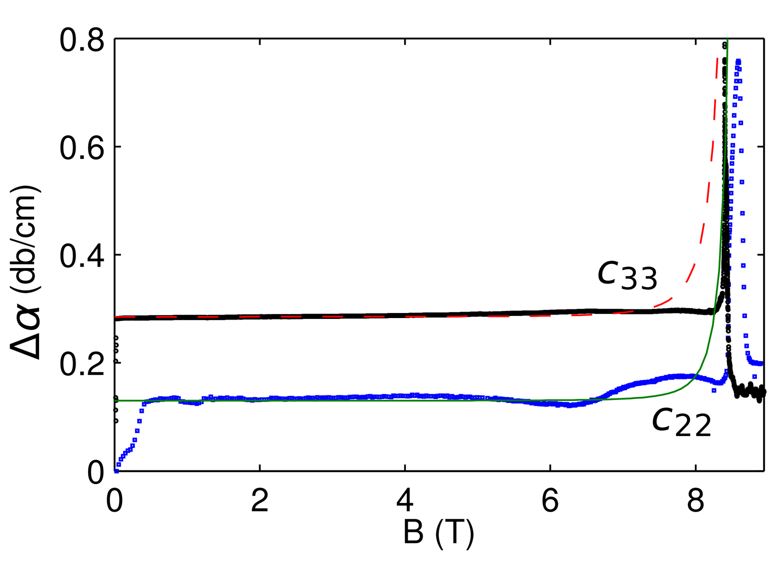

Experimental setup of the pulse echo method [4]: In this phase-sensitive detection technique two ultrasound transducers (blue) are attached to parallel planes of the crystal with distance $L$. On tuning the frequency $f$ such that the echo signal remains unchanged one can measure the change of the sound velocity $dc_\lambda/c_\lambda = df/f$ and the relative attenuation rate $\Delta\alpha \propto 1/L\ln(A_1/A_0)$ simultaneously.




Measured velocity shifts of the longitudinal $c_{22}$-phonon mode (blue) and the $c_{33}$-mode (black) at $T \approx 50$ mK. Solid line: Fit of the data for the $c_{22}$-mode to obtain the parameters $(|\kappa|,|\kappa'|) \approx (15,51)$. Dashed line: Prediction for the $c_{33}$-mode with the same parameters.

spatial dependence of couplings

$$J(x) = J(b)e^{-\kappa(x-b)/b} \quad (9a)$$
$$J'(r) = J'(d)e^{-\kappa'(r-d)/d} \quad (9b)$$




Experimental results for the relative ultrasonic attenuation $\Delta\alpha$ in $Cs_2CuCl_4$ of the longitudinal $c_{22}$-phonon mode (blue) and the $c_{33}$-mode (black) and our corresponding theoretical predictions using the values $(|\kappa|,|\kappa'|) = (15,51)$.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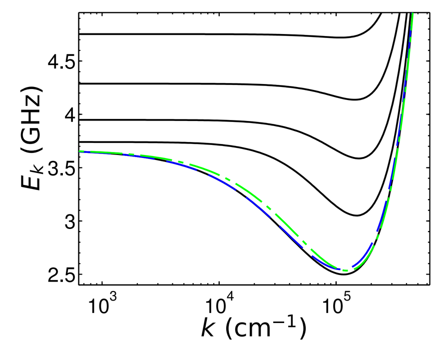
## BEC at finite momentum

### Introduction

- Bose-Einstein condensation (BEC) at finite momenta is of a different symmetry class, the so-called Brazovskii universality class [1].
- Experimental observation of coherence phenomenon of magnons in thin stripes made of the magnetic insulator yttrium-iron garnet (YIG) where the energy dispersion $\epsilon_{\boldsymbol{k}}$ exhibits two degenerate minima at finite wave-vectors $\pm\boldsymbol{q}$ [2].




The lowest modes of the spin-wave spectrum of a thin YIG film exhibit local minima at finite wave vectors. Especially interesting is the lowest mode where large magnon densities can be created by microwave pumping.

### Model

Interacting boson model on a lattice with Hamiltonian given by

$$\mathcal{H} = \mathcal{H}_2 + \mathcal{H}_3 + \mathcal{H}_4\,, \quad (1)$$

$$\mathcal{H}_2 = \sum_{\boldsymbol{k}}\left[\epsilon_{\boldsymbol{k}}a^\dagger_{\boldsymbol{k}}a_{\boldsymbol{k}} + \frac{\gamma_{\boldsymbol{k}}}{2}a^\dagger_{\boldsymbol{k}}a^\dagger_{-\boldsymbol{k}} + \frac{\gamma_{\boldsymbol{k}}}{2}a_{-\boldsymbol{k}}a_{\boldsymbol{k}}\right]. \quad (2)$$

The energy dispersion $\epsilon_{\boldsymbol{k}}$ is assumed to exhibit two degenerate minima at finite wave-vectors $\pm\boldsymbol{q}$, and the terms proportional to the complex parameter $\gamma_{\boldsymbol{k}}$ explicitly break the $U(1)$ symmetry associated with particle number conservation.
In the presence of a Bose condensate some of the expectation values $\phi_{\boldsymbol{k}} = \langle a_{\boldsymbol{k}}\rangle$ are finite. It is then useful to do a Bogoliubov shift,

$$a_{\boldsymbol{k}} = \phi_{\boldsymbol{k}} + \delta a_{\boldsymbol{k}}. \quad (3)$$

## Condensate at finite momentum

The spatial dependence of the Bose condensate is determined by the Gross-Pitaevskii equation which can be obtained from the extremum condition of the corresponding Euclidean action.
If the dispersion $\epsilon_{\boldsymbol{k}}$ has two degenerate minima at finite wave-vectors $\pm\boldsymbol{q}$ we have to consider the ansatz

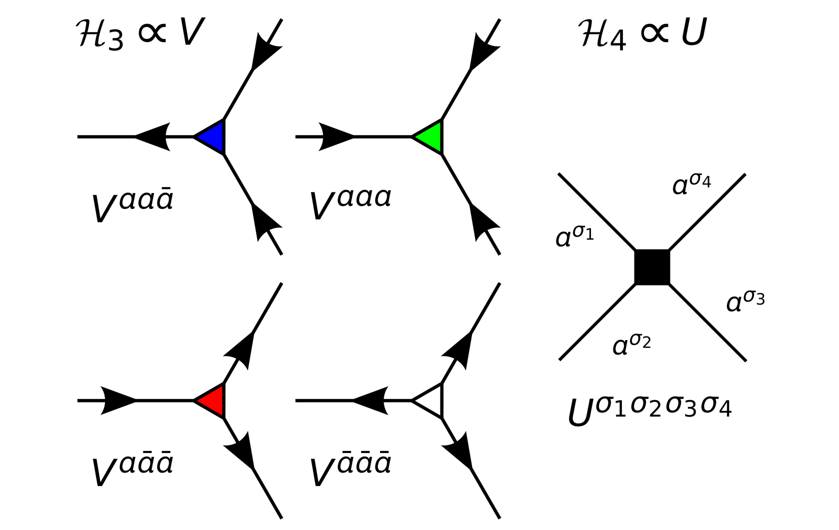
$$\phi^\sigma_{\boldsymbol{k}} = \sqrt{N}\sum_{n=-\infty}^{\infty}\delta_{\boldsymbol{k},n\boldsymbol{q}}\psi^\sigma_n \quad (4)$$

to obtain the discrete Gross-Pitaevskii equation

$$-(\epsilon_{n\boldsymbol{q}} - \mu)\psi^{\bar\sigma}_n - \gamma_n\psi^\sigma_n = \frac{1}{2}\sum_{n_1n_2}\sum_{\sigma_1\sigma_2}\delta_{n,n_1+n_2}V^{\sigma\sigma_1\sigma_2}_{nn_1n_2}\psi^{\sigma_1}_{n_1}\psi^{\sigma_2}_{n_2} + \frac{1}{3!}\sum_{n_1+n_2+n_3=n}\sum_{\sigma_1\sigma_2\sigma_3}U^{\sigma\sigma_1\sigma_2\sigma_3}_{nn_1n_2n_3}\psi^{\sigma_1}_{n_1}\psi^{\sigma_2}_{n_2}\psi^{\sigma_3}_{n_3}\,. \quad (5)$$

Due to the lack of $U(1)$ symmetry all scattering processes are present. Note that we have all combinations of $\sigma_i = a, \bar{a}$ for the four bosons interactions.

If we assume on the right-hand side of Eq. (5) that only the coefficients $\psi^{\sigma_i}_{n_i}$ with $n_i = \pm 1$ are finite, then we find that on the left-hand side all field components $\psi^\sigma_n$ with $n = 0, \pm1, \pm2, \pm3$ must also be finite. For general interactions therefore all integer multiples of $\boldsymbol{q}$ have to be finite. The condensate density can then be strongly localized at the sites of a one-dimensional lattice with spacing $2\pi/|\boldsymbol{q}|$.



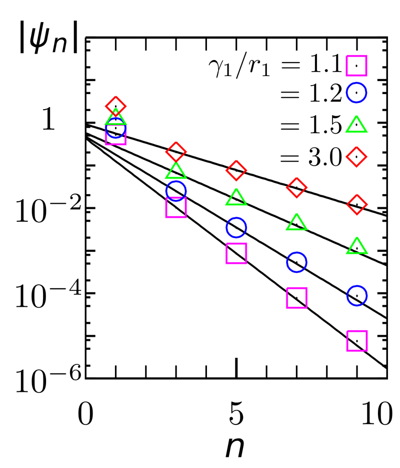

Graphical representation of the interaction processes with three (left) and four (right) bosons of our model. For simplicity the momentum labels are suppressed.

## BEC of magnons in YIG

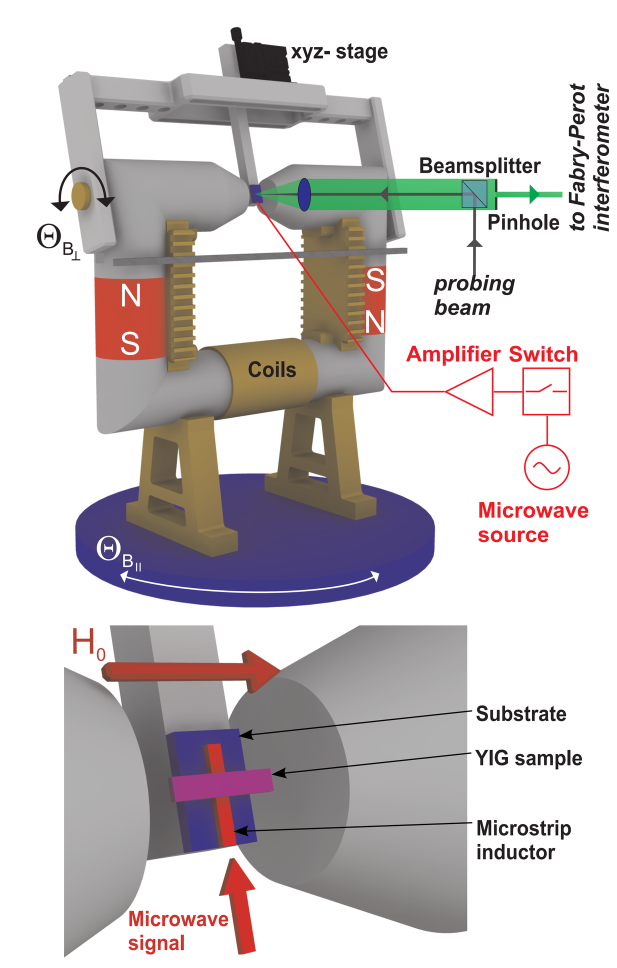
Energy dispersion of thin-film ferromagnets $\epsilon_{\boldsymbol{k}}$ has two degenerate minima at wave-vectors $\pm\boldsymbol{q} = \pm q\boldsymbol{e}_z$ [2-4]. For YIG only the four-point vertices $U^{\sigma\sigma_1\sigma_2\sigma_3}_{nn_1n_2n_3}$ contribute. We truncate the coupled equations (5) at some finite order.

### Results

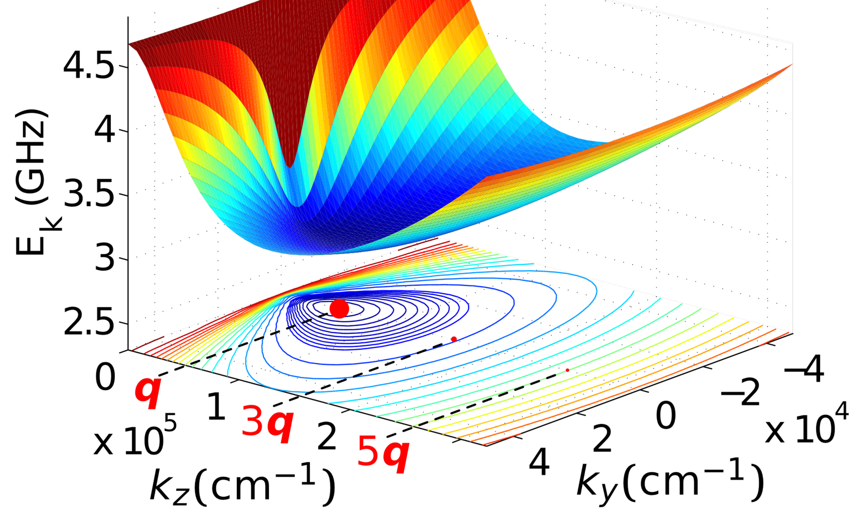
- Non-trivial solutions for $\gamma_1 > \epsilon_{\boldsymbol{q}} - \mu$
- Fourier coefficients $\psi^\sigma_n$ decay rapidly for large $n$




Absolute values $|\psi_n| = |\psi^\sigma_n|$ of the Fourier components of the order parameter for BEC in YIG for different values of the dimensionless ratio $\gamma_1/r_1 = \gamma_{\boldsymbol{q}}/(\epsilon_{\boldsymbol{q}} - \mu)$. The Fourier components $\psi^\sigma_1$ are dominant and the higher order Fourier components $\psi^\sigma_n$ decay approximately exponentially as a function of $n$. The data has been obtained from the numerical solution of the discrete Gross-Pitaevskii equation using interaction parameters for YIG.




Experimental setup to investigate the magnon gas in thin-films of YIG using the Brillouin light scattering spectroscopy technique in combination with parallel pumping [5]. In analizing the scattered light it is possible to measure the occupation number of the magnons with energy and wave-vector resolution. Unfortunately, the wave vectors are restricted to the regime $k_* \lesssim 1.6\cdot10^5(\mathrm{cm})^{-1}$.




Spin-wave dispersion of the lowest mode of a YIG film with $H_e = 1000$ Oe and $d = 5.1\mu$m using the approach [4]. Once a condensate at finite momentum $\boldsymbol{q}$ at minimum Energy $E_{\boldsymbol{q}}$ is present, interactions lead to finite condensate densities at $3\boldsymbol{q}$ and $5\boldsymbol{q}$. The condensate fraction of the higher Fourier components is rather small.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