

# Frustration without competition: the SU(N) model of quantum permutations on a lattice

F. Mila

Ecole Polytechnique Fédérale de Lausanne  
Switzerland

## Collaborators

P. Corboz (Zürich), A. Läuchli (Innsbruck), K. Penc (Budapest)  
T. Toth (Toronto), M. Troyer (Zürich), B. Bauer (Santa-Barbara)  
L. Messio (Lausanne), HongYu Yang (Lausanne)

# Scope

- Introduction: **SU(N) models** in condensed matter and cold atoms
- SU(3) on triangular and square lattice
  - **3-sublattice color order**
- SU(4) on square lattice
  - **dimerization and Néel order**
- Conclusions

# Quantum permutations

- Objects with N flavours on a lattice
- Hilbert space =  $\{|\sigma_1 \sigma_2 \dots \sigma_L\rangle\}$   
 $\sigma_i = 1, 2, \dots, N$  or  $\sigma_i = A, B, C, \dots$  or 

$$\mathcal{H} = \sum_{\langle i,j \rangle} P_{ij}$$

$$P_{ij} |\sigma_1 \dots \sigma_i \dots \sigma_j \dots \sigma_N\rangle = |\sigma_1 \dots \sigma_j \dots \sigma_i \dots \sigma_N\rangle$$

# SU(N) formulation

$$H = \sum_{\langle i,j \rangle} S_m^n(i) S_n^m(j)$$

$$S_m^n | \mu \rangle = \delta_{n,\mu} | m \rangle$$

$$[S_m^n, S_k^l] = \delta_{n,k} S_m^l - \delta_{m,l} S_k^n$$

$$S_m^n$$

→ generators of SU(N)

At each site: fundamental N-dimensional representation

# Physical realizations I

## Magnetic insulators

- N=2 → spin-1/2 Heisenberg  $P_{ij} = 2\vec{S}_i \cdot \vec{S}_j + 1/2$
- N=3 → S=1 biquadratic  $P_{ij} = \vec{S}_i \cdot \vec{S}_j + (\vec{S}_i \cdot \vec{S}_j)^2 - 1$
- N=4 → symmetric Kugel-Khomskii model

$$H = \sum_{ij} J_{ij} \left( 2\vec{s}_i \cdot \vec{s}_j + \frac{1}{2} \right) \left( 2\vec{\tau}_i \cdot \vec{\tau}_j + \frac{1}{2} \right)$$

# Physical realizations II

N-flavour fermions in optical lattice  
 $(^{40}\text{K}, ^{87}\text{Sr}, \dots)$

N-flavour Hubbard model

$$\mathcal{H} = -t \sum_{\langle i,j \rangle} \sum_{\alpha=1}^N (c_{i,\alpha}^\dagger c_{j,\alpha} + h.c.) + U \sum_i \sum_{(\alpha,\beta)} n_{i,\alpha} n_{i,\beta}$$

1/N filling



(1 fermion per site)

$$\mathcal{H} = \frac{2t^2}{U} \sum_{\langle i,j \rangle} P_{ij}$$

# General properties

- Soluble in 1D with **Bethe Ansatz**  
→ algebraic correlations with **periodicity  $2\pi/N$**   
**Sutherland, 1974**
- Equivalent of **SU(2) dimer singlet: N sites**

$$|S\rangle = (1/\sqrt{N!}) \sum_P (-1)^P | \sigma_{P(1)} \sigma_{P(2)} \dots \sigma_{P(N)} \rangle$$

with  $\{\sigma_1 \sigma_2 \dots \sigma_N\} = \{1 2 \dots N\}$

**Li, Ma, Shi, Zhang, PRL'98**

# Hartree approximation

$$|\psi\rangle = \prod_i |\varphi_i\rangle$$

$$\langle \varphi_1 \varphi_2 | P_{12} | \varphi_1 \varphi_2 \rangle = \langle \varphi_1 \varphi_2 | \varphi_2 \varphi_1 \rangle = |\langle \varphi_1 | \varphi_2 \rangle|^2$$

→ on 2 sites, energy minimal if  $\langle \varphi_1 | \varphi_2 \rangle = 0$

→ on a lattice, Hartree energy minimal as soon as  
colors on nearest neighbors are different

NB: For SU(2), Hartree  $\Leftrightarrow$  classical

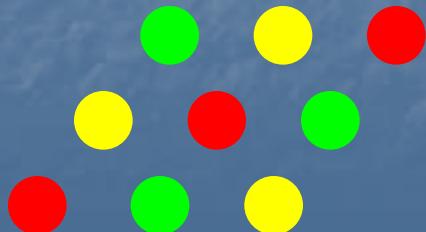
→ fundamental representation: S=1/2  
→ S=1/2: all states are magnetic

# SU(3) on triangular lattice

- Unique 'classical' (Hartree) state
  - 3-sublattice covering of triangular lattice
  - The equivalent of Néel on square lattice
- Schwinger bosons → Flavour wave theory

$$\mathcal{P}_{ij} = \sum_{\mu, \nu \in \{A, B, C\}} a_{\mu, i}^\dagger a_{\nu, j}^\dagger a_{\nu, i} a_{\mu, j}$$

$$\tilde{a}_{A,i}^\dagger, \tilde{a}_{A,i} \rightarrow \sqrt{M - \tilde{a}_{B,i}^\dagger \tilde{a}_{B,i} - \tilde{a}_{C,i}^\dagger \tilde{a}_{C,i}}.$$



3-sublattice order stable

Tsunetsugu, Arikawa, JPSJ 2006  
A. Läuchli, FM, K. Penc, PRL 2006

# SU(3) on square lattice

- Infinite number of 'Hartree' ground states

A B A B

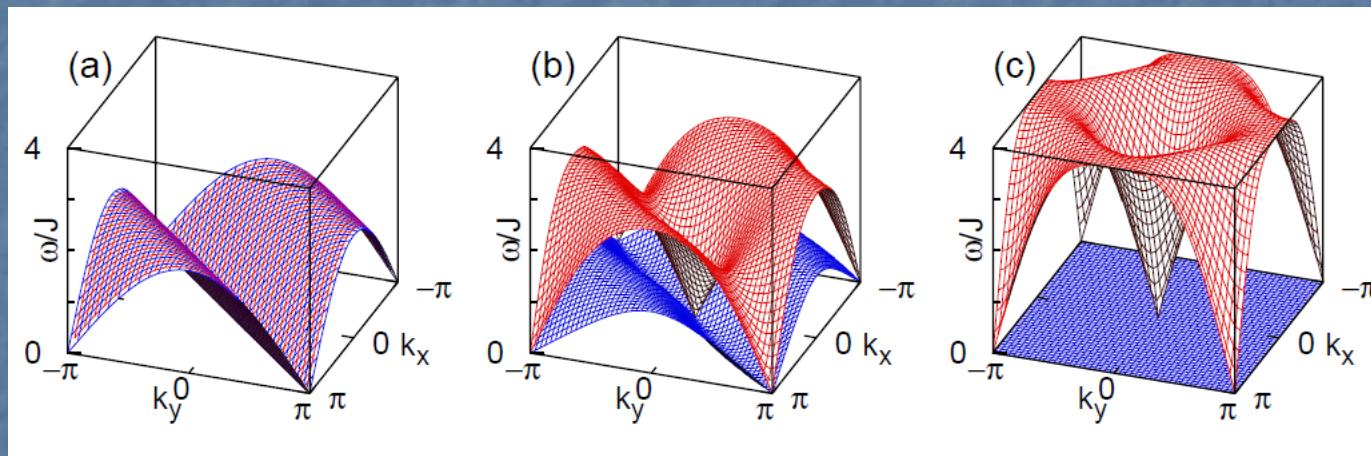
B A B A

A or B → C at any site

A B A B

- Quantum fluctuations: order by disorder?
  - Flavour-wave theory
  - Zero-point energy

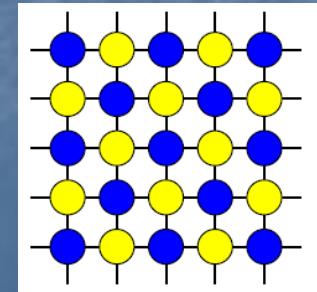
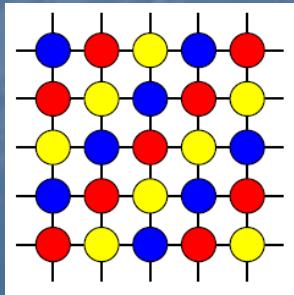
# Flavour wave spectrum



3-sublattice

helical

2-sublattice

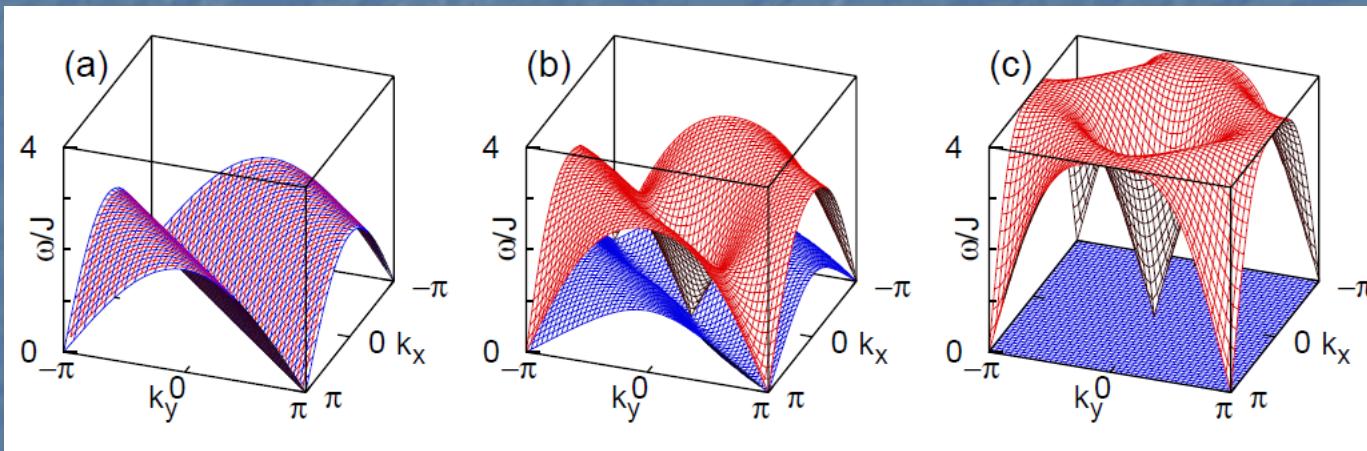


# Order by disorder

3-sublattice

helical

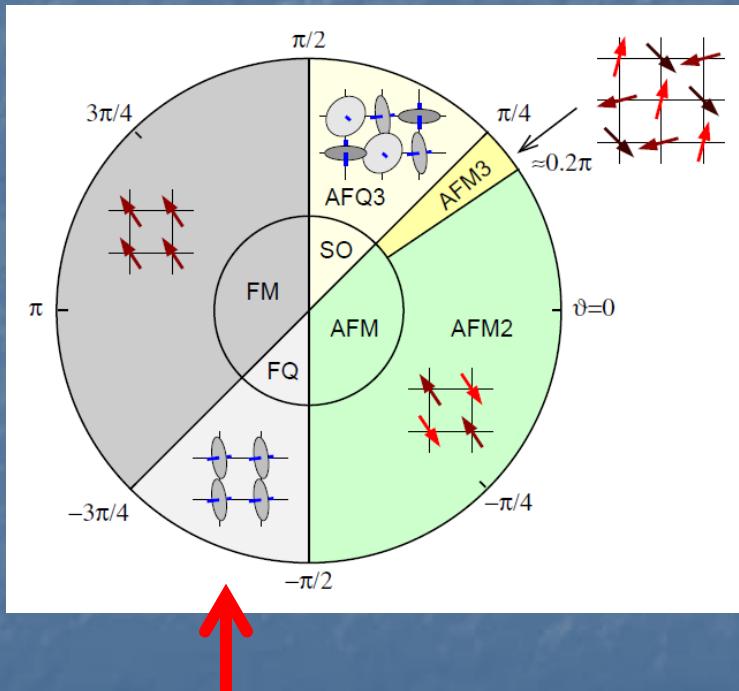
2-sublattice



- Quantum fluctuations:  
minimize  $\sum \omega_q \rightarrow$  3-sublattice order
- Thermal fluctuations:  
maximize # zero modes  $\rightarrow$  2-sublattice order

# Square lattice BBQ model

$$\mathcal{H} = \sum_{\langle i,j \rangle} \cos \theta \vec{S}_i \cdot \vec{S}_j + \sin \theta (\vec{S}_i \cdot \vec{S}_j)^2$$



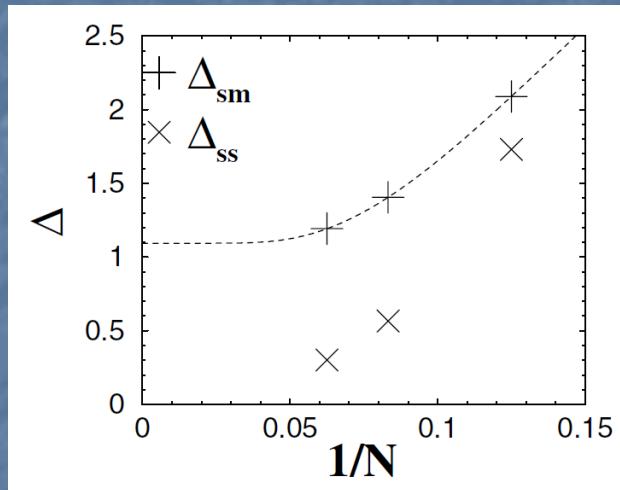
Ferroquadrupolar

3-sublattice order

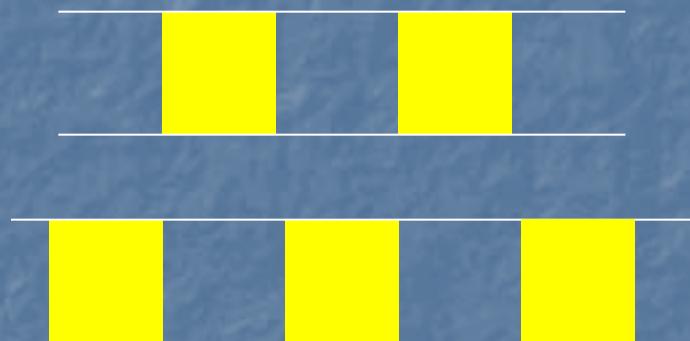
T. Toth, A. Läuchli, K. Penc,  
FM, unpublished

Harada, Kawashima, PRB 2002

# SU(4) ladder



2-fold degenerate GS



Spontaneous SU(4) plaquette singlet formation

Confirmed by field theory in weak and strong rung limits

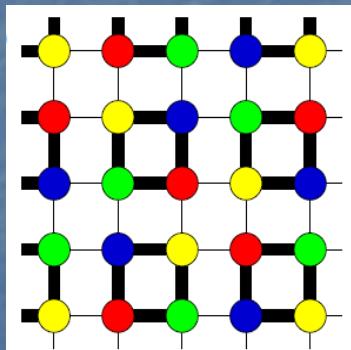
M. Van Den Bossche, P. Azaria, P. Lecheminant, FM, PRL 2001

# SU(4) on square lattice: early results

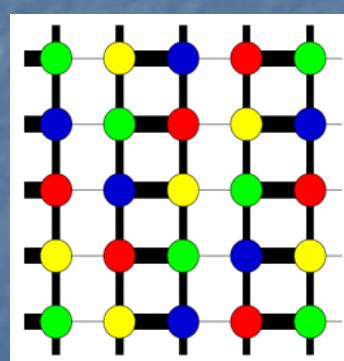
- Low-lying SU(4) singlets: plaquette coverings?  
M. Van den Bossche, F.-C. Zhang, FM, EPJB 2001
- Plaquette long-range order  
H.-H. Hung, Y. Wang, and C. Wu, Modern Phys Lett 2006
- Liquid with emergent nodal fermions  
Fang, Vishwanath, PRB 2009
- Possible chiral spin liquid ground state with topological order for  $N>4$   
Hermele et al, PRL 2009
- Stripe color order?

# SU(4) on square lattice

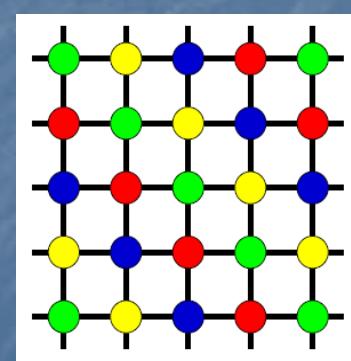
- Hartree: infinite number of coverings
- Flavor-wave theory
  - small clusters favored (2 and 4 sites)
  - stripe order not stabilized



$$E/J = -1.5$$



$$E/J = -1.29$$

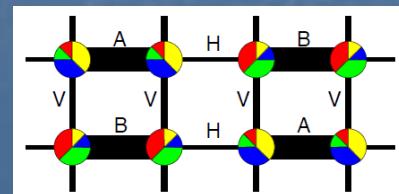
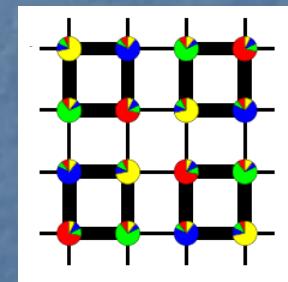
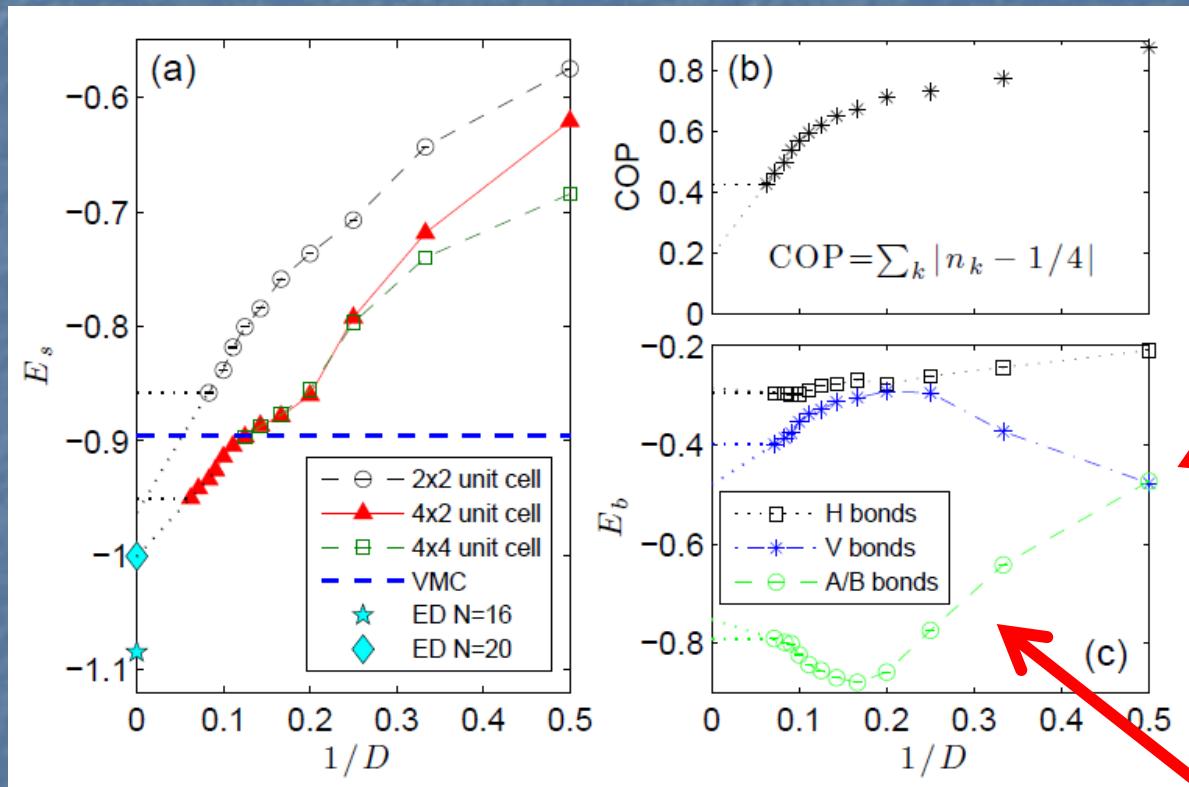


$$E/J = -0.73$$

# iPEPS

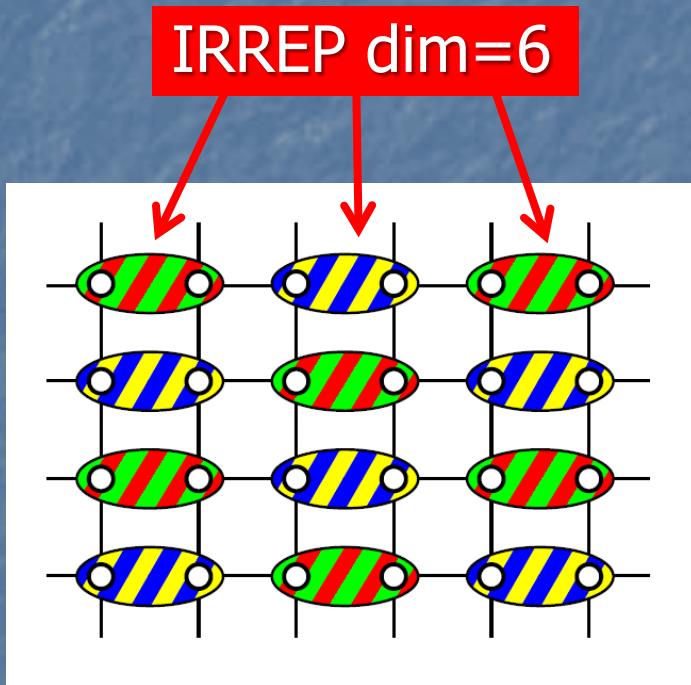
- iPEPS = infinite Projected Entangled Pair States
- Variational method based on a tensor product wave-function
- Becomes exact if the dimension D of the tensors  $\rightarrow$  infinity
- Can be seen as a generalization of DMRG  
Verstraete and Cirac, 2004

# iPEPS: SU(4) on square lattice



# $SU(4)$ on square lattice

IPEPS, ED, Hartree + flavour-wave theory,...



Dimerized ground state  
+ Néel order

P. Corboz, A. Läuchli, K. Penc,  
M. Troyer, F. Mila, PRL 2011

IRREP 6 on square lattice: Algebraic order? Assaad 2005  
Long-range order? Paramekanti and Marston, 2007

# SU(4) spin-orbital model

$$A = |\uparrow, a\rangle, \quad B = |\downarrow, a\rangle, \quad C = |\uparrow, b\rangle, \quad D = |\downarrow, b\rangle$$

IRREP6 {

$(AB - BA)/\sqrt{2}$	$\rightarrow$	$ \text{spin singlet}\rangle \otimes  a, a\rangle$
$(CD - DC)/\sqrt{2}$	$\rightarrow$	$ \text{spin singlet}\rangle \otimes  b, b\rangle$
<hr/>		
$(AC - CA)/\sqrt{2}$	$\rightarrow$	$ \uparrow, \uparrow\rangle \otimes  \text{orbital singlet}\rangle$
$(BD - DB)/\sqrt{2}$	$\rightarrow$	$ \downarrow, \downarrow\rangle \otimes  \text{orbital singlet}\rangle$
<hr/>		
$(AD - DA)/\sqrt{2}$	$\rightarrow$	$ \text{spin singlet}\rangle \otimes  \text{orbital } T_0\rangle$
$(BC - CB)/\sqrt{2}$	$\rightarrow$	$ \text{spin } T_0\rangle \otimes  \text{orbital singlet}\rangle$

# SU(4) spin-orbital model

$$A = |\uparrow, a\rangle, \quad B = |\downarrow, a\rangle, \quad C = |\uparrow, b\rangle, \quad D = |\downarrow, b\rangle$$

IRREP6

$$\begin{aligned} (AB - BA)/\sqrt{2} &\rightarrow |\text{spin singlet}\rangle \otimes |a, a\rangle \\ (CD - DC)/\sqrt{2} &\rightarrow |\text{spin singlet}\rangle \otimes |b, b\rangle \end{aligned}$$

$$\begin{aligned} (AC - CA)/\sqrt{2} &\rightarrow |\uparrow, \uparrow\rangle \otimes |\text{orbital singlet}\rangle \\ (BD - DB)/\sqrt{2} &\rightarrow |\downarrow, \downarrow\rangle \otimes |\text{orbital singlet}\rangle \end{aligned}$$

$$\begin{aligned} (AD - DA)/\sqrt{2} &\rightarrow |\text{spin singlet}\rangle \otimes |\text{orbital } T_0\rangle \\ (BC - CB)/\sqrt{2} &\rightarrow |\text{spin } T_0\rangle \otimes |\text{orbital singlet}\rangle \end{aligned}$$

# Conclusions

- SU(3) on triangular lattice
  - canonical example of **color order**
- SU(3) on square lattice
  - **3-sublattice order** at zero temperature
  - 2-sublattice correlations at large T?
- SU(4) on square lattice
  - **dimerization + Néel order**

# Perspectives

- Experimental realizations with quantum magnets:
  - BLBQ spin-1 model on square lattice
  - spin-orbital model on square lattice
- Quantum liquids for larger N?
- Probing order with cold atoms?
  - real space measurement of correlations
  - multiple occupancy