Mesh Generation through Delaunay Refinement

3D Meshes

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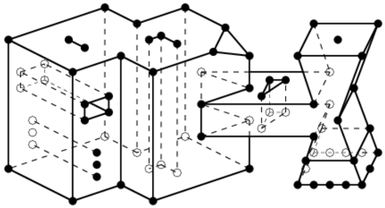
The 3D meshing problem

Input :

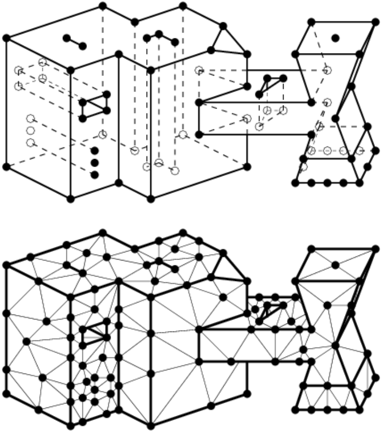


- C : a 2D PLC in $\mathbb{R}^3$
  (piecewise linear complex)
- Ω : a bounded domain
  to be meshed.
  Ω is bounded by facets in C

Output : a mesh of domain Ω
i. e. a 3D triangulation T such that

- vertices of C are vertices of T
- edges and facets C are union of faces in T
- the tetrahedra of T that are $\subset \Omega$
  have controlled size and quality

The mesh is a 3D Delaunay triangulation.
Edges and facets of the input PLC appear in the mesh as union of Delaunay edges and Delaunay facets, that are called constrained edges and facets.

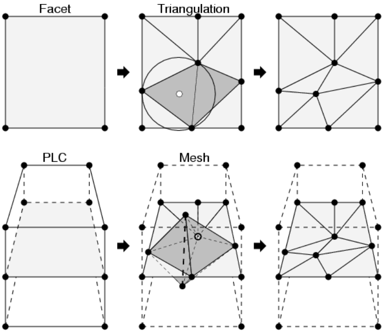


3D Delaunay refinement

Constrained facets

How input PLC facets are subdivided into constrained facets ?

- 3D Delaunay facets are 2D Delaunay in the facet hyperplane.
- Constrained facets are known as soon as PLC edges are refined into Delaunay edges.
- A 2D Delaunay triangulation is maintained for each PLC facet



3D Delaunay refinement

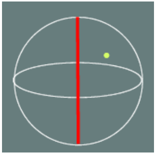
Use a 3D Delaunay triangulation

Constraints
constrained edges are refined into Gabriel edges
with empty smallest circumball
encroached edges = edges which are not
Gabriel edges

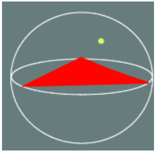
constrained facets are refined into Gabriel facets
with empty smallest circumball
encroached facets = facets which are not
Gabriel facets

Tetrahedra
Bad tetrahedra are refined
by circumcenter insertion.

Bad tetrahedra : radius-edge ratio
$\rho = \frac{\text{circumradius}}{l_{min}} \geq B$



encroached edge



encroached facet

3D Delaunay refinement algorithm

- Initialization Delaunay triangulation of PLC vertices
- Refinement

Apply one the following rules, until no one applies.
Rule i has priority over rule j if $i < j$.

1. if there is an encroached constrained edge e, refine-edge(e)
2. if there is an encroached constrained facet f,
   refine-facet-or-edge(f) i.e.:
   c = circumcenter(f)
   if c encroaches a constrained edge e, refine-edge(e).
   else insert(c)
3. if there is a bad tetrahedra t,
   refine-tet-or-facet-or-edge(t) i.e.:
   c = circumcenter(t)
   if c encroaches a constrained edge e, refine-edge(e).
   else if c encroaches a constrained facet f,
   refine-facet-or-edge(f).
   else insert(c)

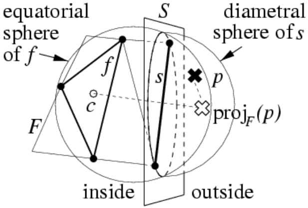
Refinement of constrained facets

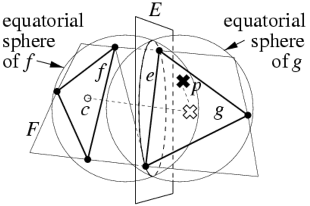
Lemma (Projection lemma)

When a point p encroaches a constrained subfacet f of PLC facet F without constrained edges encroachment :

- *the projection p_F of p on the supporting hyperplan H_F of F, belongs to F*
- *p encroaches the constrained subfacet $g \subset F$ that contains p_F*

Proof.





□

The algorithm always refine a constrained facet including the projection of the encroaching point

3D Delaunay refinement algorithm (bis)

- Initialization Delaunay triangulation of PLC vertices
- Refinement

Apply one the following rules, until no one applies.
Rule i has priority over rule j if $i < j$.

1. if there is an encroached constrained edge e, refine-edge(e)
2. if there is a constrained facet f encroached by a vertex v,
   find the constrained facet g that contains the proj. of v on H_f
   refine-facet-or-edge(g)
3. if there is a bad tetrahedra t,
   refine-tet-or-facet-or-edge(t) i.e.:
   c = circumcenter(t)
   if c encroaches a constrained edge e, refine-edge(e).
   else if c encroaches a constrained facet f,
   find the constrained facet g that contains the proj. of v
   refine-facet-or-edge(g).
   else insert(c)

3D Delaunay refinement theorem

Theorem (3D Delaunay refinement)

The 3D Delaunay refinement algorithm ends provided that ;

- *the upper bound on radius-edge ratio of tetrahedra is*

$$B > 2$$

- *all input PLC angles are* $> 90^o$
  dihedral angles : two facets of the PLC sharing an edge
  edge-facet angles : a facet and an edge sharing a vertex
  edge angles : two edges of the PLC sharing a vertex

Proof.

As in 2D, use a volume argument
to bound the number of Steiner vertices □

Proof of 3D Delaunay refinement theorem

Lemma (Lemma 1)

Any added (Steiner) vertex is inside or on the boundary of the domain Ω to be meshed

Proof.

– trivial for vertices inserted in constrained edges

– as in 2D for facet circumcenters

when facet circumcenters are inserted

there is no encroached edge

– analog proof for tetrahedra circumcenters

when tetrahedra circumcenters are added

there is no encroached edge nor encroached facet. □

Proof of 3D Delaunay refinement theorem

Local feature size lfs(p)
radius of the smallest disk centered in p and intersecting two disjoint faces of C.

Insertion radius r_v
length of the smallest edge incident to v, right after insertion of v, if v is inserted.

Parent vertex p of vertex v

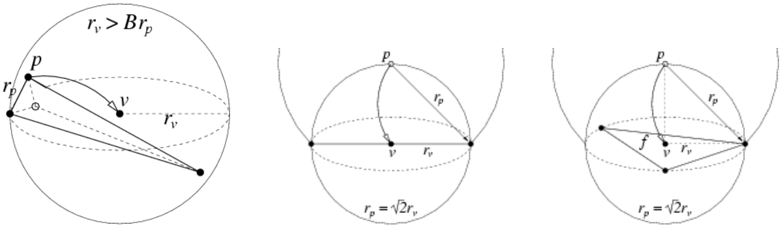
- if v is the circumcenter of a tet t
  p is the last inserted vertex of the smallest edge of t
- if v is inserted on a constrained facet or edge
  p is the encroaching vertex closest to v
  (p may be a mesh vertex or a rejected vertex)

Proof of 3D Delaunay refinement theorem

Lemma (Insertion radius lemma)

Let v be vertex of the mesh or a rejected vertex, with parent p, then $r_v \geq \text{lfs}(v)$ or $r_v \geq C r_p$, with :

- *$C = B$ if v is a tetrahedra circumcenter*
- *$C = 1/\sqrt{2}$ if v is on a constrained edge or facet and p is rejected*

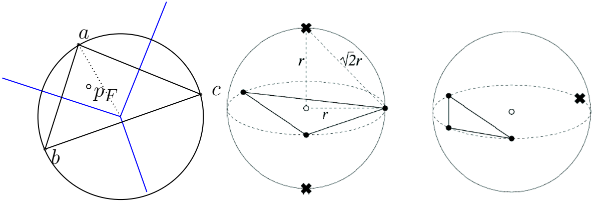


Refinement of constrained facets

v is the circumcenter of constrained facet f
p is the vertex encroaching f
Because f includes the projection p_F of p on F: $r_v \geq \frac{r_p}{\sqrt{2}}$

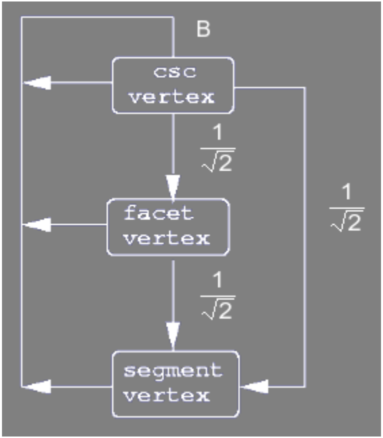
Proof.

$r_v = r$, radius of the smallest circumball of f
$r_p \leq pa$ if $pa = \min\{pa, pb, pc\}$
$\|pa\|^2 = \|pp_F\|^2 + \|p_F a\|^2 \leq 2r^2$ □



Proof of 3D Delaunay refinement theorem

Flow diagram of vertices insertion



Proof of 3D Delaunay refinement theorem

weighted density

weighted density $d(v) = \frac{\text{lfs}(v)}{r_v}$

Lemma (Weighted density lemma 1)

For any vertex v with parent p, if $r_v \geq C r_p$, $d(v) \leq 1 + \frac{d(p)}{C}$

Lemma (Weighted density lemma 2)

There are constants $D_e \geq D_f \geq D_t \geq 1$ such that :
for any tet circumcenter v, inserted or rejected, $d(v) \leq D_t$
for any facet circumcenter v, inserted or rejected, $d(v) \leq D_f$.
for any vertex v inserted in a PLSG edge, $d(v) \leq D_e$.
Thus, for any vertex of the mesh $r_v \geq \frac{lfs(v)}{D_e}$

Proof of weighted density lemma

Assume the lemma is true up to the insertion of vertex v,
p parent of v

- v is a tet circumcenter
  $r_v \geq Br_p \Longrightarrow d(v) \leq 1 + \frac{d_p}{B}$ assume $1 + \frac{D_e}{B} \leq D_t$ (1)
- v is on a PLC facet F
  - p is a PLC vertex or $p \in$ PLC face F' st $F \cap F' = \emptyset$
    $r_v = \text{lfs}(v) \Longrightarrow d(v) \leq 1$
  - p is a rejected tet circumcenter
    $r_v \geq \frac{r_p}{\sqrt{2}} \Longrightarrow d(v) \leq 1 + \sqrt{2}d_p$ assume $1 + \sqrt{2}D_t \leq D_f$ (2)
- v is on a PLC edge E
  - p is a PLC vertex or $p \in$ PLC face F' st $E \cap F' = \emptyset$
    $r_v = \text{lfs}(v) \Longrightarrow d(v) \leq 1$
  - p is a rejected tet or a facet circumcenter
    $r_v \geq \frac{r_p}{\sqrt{2}} \Longrightarrow d(v) \leq 1 + \sqrt{2}d_p$ assume $1 + \sqrt{2}D_f \leq D_e$ (3)

3D Delaunay refinement theorem

Proof of weighted density lemma (end)

There are $D_e \geq D_f \geq D_t \geq 1$ such that :
$1 + \frac{D_e}{B} \leq D_t \quad (1)$
$1 + \sqrt{2} D_t \leq D_f \quad (2)$
$1 + \sqrt{2} D_f \leq D_e \quad (3)$

$$
\begin{aligned}
D_e &= \left(3 + \sqrt{2}\right) \frac{B}{B-2} \\
D_f &= \frac{(1 + \sqrt{2} B) + \sqrt{2}}{B-2} \\
D_t &= \frac{B + 1 + \sqrt{2}}{B-2}
\end{aligned}
$$

Proof of 3D Delaunay refinement theorem (end)

Theorem (Relative bound on edge length)

Any edge of the mesh, incident to vertex v, has length l st :

$$l \geq \frac{lfs(v)}{D_e + 1}$$

Proof.

as in 2D □

End of 3D Delaunay refinement theorem proof.

Using the above result on edge lengths,
prove an upper bound on the number of mesh vertices
as in 2D. □

Delaunay refinement

meshing domain with small angles

Algorithm Terminator 3D : Delaunay refinement + additionnal rules

1. Clusters of edges : refine edges in clusters along concentric spheres
2. when a facet f in PLC facet F is encroached by p
   and circumcenter(f) encroaches no constrained edge
   refine f iff
   - p is a PLC vertex or belongs to a PLC face s' st $f \cap s' = \emptyset$
   - $r_v > r_g$, where g is the most recently ancestor of v.
3. when a constrained edge e is encroached by p
   e is refined iff
   - p is a mesh vertex
   - $\min_{w \in W} r_w > r_g$ where
     g is the most recently inserted ancestor of v
     W is the set of vertices that will be inserted if v is inserted.

Remarks Notice that some constrained facets remain encroached

- using a constrained Delaunay triangulation is required to respect constrained facets.
  Fortunately, this constrained Delaunay triangulation exists because constrained edges are Gabriel edges.
- the final mesh may be different from the Delaunay triangulation of its vertices

Nearly degenerated triangles

Radius-edge ratio $\rho = \frac{\text{circumradius}}{\text{shortest edge lentgh}}$

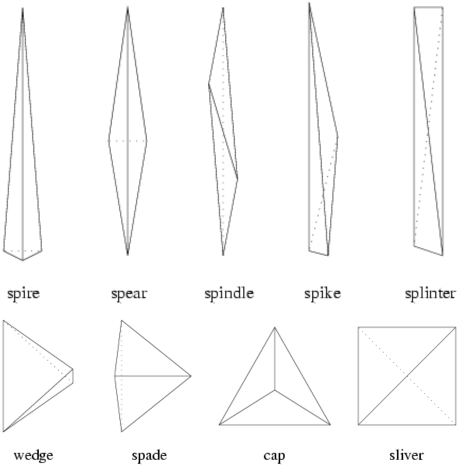
In both cases the radius-edge ratio is large

Nearly degenerated tetrahedra

Thin tetrahedra

Flat tetrahedra

Slivers : the only case in which radius-edge ratio ρ is not large



Definition (Slivers)

A tetrahedron is a sliver iff
the volume is too small $\sigma = \frac{V}{l^3} \leq \sigma_0$
yet, the radius-edge ratio is not too big $\rho = \frac{r}{l} \leq \rho_0$

r = circumradius, l = shortest edge length, V = volume

Remark

Radius-edge ratio is not a fair measure for tetrahedra.
Radius-radius ratio is a fair measure $\frac{r_{circ}}{r_{insc}}$, and :

$$\frac{r_{circ}}{r_{insc}} \leq \frac{\sqrt{3}\rho^3}{\sigma}$$

Proof.

area of facets of t : $S_i \leq \frac{3\sqrt{3}}{4} r_{circ}^2$

$\sqrt{3} r_{circ}^2 r_{insc} \geq \sum_{i=1}^{4} \frac{1}{3} S_i r_{insc} = V = \sigma l^3 = \sigma \left(\frac{r_{circ}}{\rho}\right)^3$ □

Delaunay meshes with bounded radius-edge ratio

Theorem (Delaunay meshes with bounded radius-edge ratio)

Any Delaunay mesh with bounded radius-edge ratio is such that :

1. *The ratio between the length of the longest edge and the length of shortest edge incident to a vertex v is bounded.*
2. *The number of edges, facets or tetrahedra incident to a given vertex is bounded*

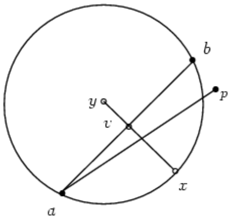
Delaunay meshes with bounded radius-edge ratio

Lemma

In a Delaunay mesh with bounded radius-edge ratio ($\rho \leq \rho_0$), edges ab, ap incident to the same vertex and forming an angle less than $\eta_0 = \arctan\left[2\left(\rho_0 - \sqrt{\rho_0^2 - 1/4}\right)\right]$ *are such that* $\frac{\|ab\|}{2} \leq \|ap\| \leq 2\|ab\|$

Proof.

$\Sigma(y, r_y) =$ Intersection of the hyperplan spanned by (ap, ab) with the circumsphere of a tetrahedron incident to ab



$$\|xv\| = r_y - \sqrt{r_y^2 - \|ab\|^2/4}$$

$$\|xv\| \geq \left(\rho_0 - \sqrt{\rho_0^2 - 1/4}\right)\|ab\|$$

$$(\widehat{ab, ax}) = \arctan\left(\frac{2\|xv\|}{\|ab\|}\right) \geq \eta_0$$

$$(\widehat{ab, ap}) \leq \eta_0 \Longrightarrow \|ap\| \geq \|ax\| \geq \frac{\|ab\|}{2}$$

Delaunay meshes with bounded radius-edge ratio theorem

proof of Part 1

ρ_0 radius-edge ratio bound

$m_0 = \frac{2}{(1-\cos(\eta_0/4))}$

$\eta_0 = \arctan\left[2\left(\rho_0 - \sqrt{\rho_0^2 - 1/4}\right)\right]$

$\nu_0 = 2^{2m_0-1}\rho_0^{m_0-1}$

Two mesh edges ab and ap incident to a are such that :
$\frac{\|ab\|}{\nu_0} \leq \|ap\| \leq \nu_0\|ab\|$

Proof.

$\Sigma(a,1)$ unit sphere around a
Max packing on Σ of spherical caps with angle $\eta_0/4$
There is at most m_0 spherical caps
Doubling the cap's angles form a covering of Σ.
Graph G = traces on $\Sigma(a,1)$ of edges and facets incident to a.
Path in G from ab to ap, ignore detours when revisiting a cap.
At most, the path visits m_0 caps and crosses $m_0 - 1$ boundary. □

The number of edges incident to a given vertex is bounded by $\delta_0 = (2\nu_0^2 + 1)^3$

Proof.

ap : shortest edge incident to a, let $\|ap\| = 1$
ab : longest edge incident to a, $\|ap\| \le \nu_0$
for any vertex c adjacent to a, $1 \le \|ac\| \le \nu_0$
for any vertex d adjacent to c, $\|cd\| \ge \frac{1}{\nu_0}$
Spheres $\Sigma_c(c, \frac{1}{2\nu_0})$ are empty of vertices except c, disjoint and included in $\Sigma(a, \nu_0 + \frac{1}{2\nu_0}))$

$$V_\Sigma = \frac{4}{3}\pi\left(\nu_0 + \frac{1}{2\nu_0}\right)^3 = \left(2\nu_0^2 + 1\right)^3 V_{\Sigma_c}$$

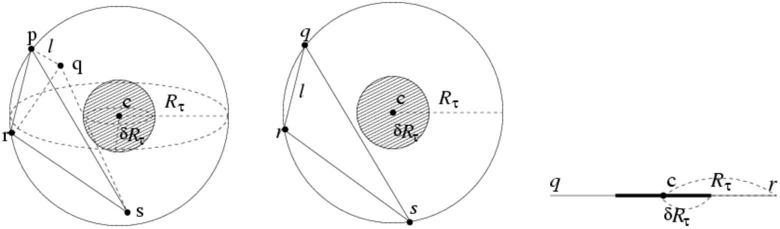
□

Sliver elimination

Method of Li [2000]
Choose each Steiner vertex in a refinement region :

Refinement region

refining a tetrahedra t with circumsphere (c_t, r_t) :	3D ball $(c_t, \delta r_t)$
refining a facet f with circumcircle (c_f, r_f) :	2D ball $(c_f, \delta r_f)$
refining an edge (c_s, r_s)	1D ball $(c_s, \delta r_s)$



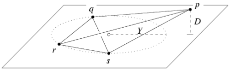
Sliver lemma

Definition (Slivers)

$r =$ circumradius, $l =$ shortest edge length, $V =$ volume
$\rho = \frac{r}{l} \leq \rho_0 \quad \sigma = \frac{V}{l^3} \leq \sigma_0$

Lemma

If pqrs is a tet with $\sigma \leq \sigma_0$, $\frac{d}{r_y} \leq 12\sigma_0$
d *: distance from p to the hyperplan of qrs*
r_y *: circumradius of triangle qrs*



Proof.

$$\sigma l^3 = V = \frac{1}{3} S d \geq \frac{1}{3}\left(\frac{1}{2} l^2 \frac{l}{2r_y}\right) d = \frac{l^3}{12 r_y} d$$

□

Sliver lemma

Lemma (Sliver lemma)

Let $\Sigma(y, r_y)$ be the circumcircle of triangle qrs.
If the tet pqrs is a sliver,
$d' = d(p, \Sigma(y, r_y)) \leq 12\sigma_0 f(\rho_0)\ r_y$
with $f(\rho_0) = 1 + \frac{1}{1 - \sqrt{1 - \frac{1}{4\rho_0^2}}}$.

Proof.

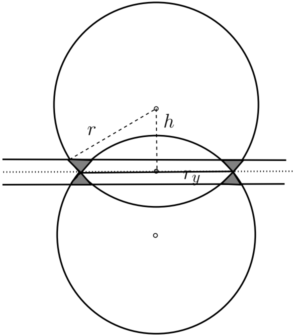
$r \leq 2\rho_0 r_y$, $d \leq 12\sigma_0 r_y$

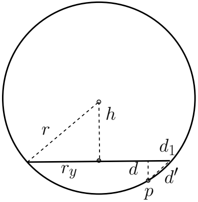
$h = \sqrt{r^2 - r_y^2} \leq r\sqrt{1 - \frac{1}{4\rho_0^2}}$

$\frac{d}{d_1} \geq \frac{r-h}{r_y} \geq \frac{r-h}{r}$

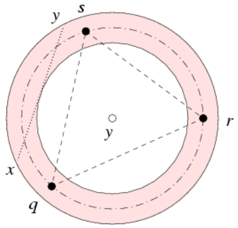
$d' \leq d + d_1 \leq d\left(1 + \frac{r}{r-h}\right)$

□





Sliver elimination



Forbidden volume

For any triangle qrs with circumradius r_y,
p should not be in a torus with volume :
$V(\text{ torus}(qrs)) = 2\pi r_y.\pi d'^2$
$V = c_3 r_y^3$
$c_3 = 488\pi^2\left(\sigma_0 f(\rho_0)\right)^2$

Forbidden area on any plane h

$S(\text{ torus}(qrs) \cap h) \leq \pi(r_y + d')^2 - \pi(r_y - d')^2 = 4\pi d' r_y$
$S(\text{ torus}(qrs) \cap h) \leq c_2 r_y^2$
$c_2 = 48\pi\left(\sigma_0 f(\rho_0)\right)$

Forbidden length on any line l

$L(\text{ torus}(qrs) \cap h) \leq 2\sqrt{(r_y + d')^2 - (r_y - d')^2} = 4\sqrt{r_y d'}$
$L(\text{ torus}(qrs) \cap l) \leq c_1 r_y$
$c_1 = 8\sqrt{3}\sqrt{\sigma_0 f(\rho_0)}$

Sliver elimination

Main Idea

Start from a Delaunay mesh with bounded edge-radius ratio
Then refine bad tets ($\rho > \rho_0$) and slivers ($\rho \leq \rho_0, \sigma \leq \sigma_0$)
- with refinement point carefully choosen in the refinement regions
- to avoiding forbidden volumes, areas and segments

When refining a mesh element τ (τ may be a tet, a facet or an edge)
it is not always possible to avoid producing new slivers
but it is possible to avoid producing small slivers,
i. e. slivers *pqrs* with circumradius(*pqrs*) $\leq \gamma r_\tau$
where r_τ is the radius of the smallest circumsphere of τ.

Sliver elimination

Lemma

Assume that the shortest edge length in the mesh stays $\geq l_2$*.*
For any refinement region $(c_\tau, \delta r_\tau)$
there is a finite number W *of facets* (qrs)
that are likely to form a tetrahedron $pqrs$ *in the mesh*
if some point $p \in (c_\tau, \delta r_\tau)$ *is inserted.*

Proof.

$\mathcal{Q} =$ set of vertices of facets qrs forming a tet with $p \in (c_\tau, \delta r_\tau)$
r_1 maximum circumradius in the mesh before the sliver elimination phase.
Any point in Ω is at distance less than r_1 from a mesh vertex :
circumradius$(pqrs) \leq r_1$
$\|c_\tau q\| \leq \|c_\tau p\| + \|pq\| \leq \delta r_\tau + 2r_1 \leq (2+\delta)r_1$
Empty circle of radius $l_2/2$ around an vertex in $\mathcal{Q}$
$|\mathcal{Q}| \leq \left(\frac{2[(2+\delta)r_1)+l_2]}{l_2}\right)^3$
$W \leq |\mathcal{Q}|^3$ □

Sliver elimination

Lemma

For any set of constants ρ_0, γ and δ,
if σ_0 is small enough,
there is a point p in refinement region $(c_\tau, \delta r_\tau)$
such that the insertion of p creates no sliver with radius smaller than γr_τ.

Proof.

If circumradius($pqrs$) $\leq \gamma r_\tau$, circumcircle(pqr) $\leq \gamma r_\tau$

$$\begin{aligned} W\ 488\pi^2 \left(\sigma_0 f(\rho_0)\right)^2 \gamma^3 r_\tau^3 &< \frac{4}{3}\pi\delta^3 r_\tau^3 \\ W\ 48\pi \left(\sigma_0 \rho_0\right) \gamma^2 r_\tau^2 &< \pi\delta^2 r_\tau^2 \\ W\ 8\sqrt{3}\sqrt{\sigma_0\rho_0}\ \gamma r_\tau &< 2\delta r_\tau \end{aligned}$$

□

Sliver elimination

- Initial phase

Build a bounded radius-edge ratio mesh
using usual Delaunay refinement

- Sliver elimination phase

Apply one of the following rules, until no one applies
Rule i has priority over rule j if $i < j$.

1. if there is an encroached constrained edge e,
   sliver-free-refine-edge(e)
2. if there is an encroached constrained facet f,
   sliver-free-refine-facet-or-edge(f)
3. if there is a tet t with $\rho \geq \rho_0$,
   sliver-free-refine-tet-or-facet-or-edge(t)
4. if there is a sliver t ($\rho \leq \rho_0, \sigma \leq \sigma_0$),
   sliver-free-refine-tet-or-facet-or-edge(t)

Sliver elimination

Sliver-free versions of refine functions
sliver-free-refine-edge(e)
sliver-free-refine-facet-or-edge(f)

sliver-free-refine-tet-or-facet-or-edge(t)

- pick q sliver free in refinement region
- if q encroaches a constrained edge e,
  sliver-free-refine-edge(e).
- else if q encroaches a constrained facet f,
  sliver-free-refine-facet-or-edge(f).
- else insert(q)

picking q sliver free in refinement region means :

- pick a random point q in refinement region
- while q form small slivers
  pick another random point q in refinement region

Sliver elimination

Theorem

If the hypothesis of Delaunay refinement theorem are satisfied and if the constants δ, ρ_0 and γ are such that

$$\frac{(1-\delta)^3\rho_0}{2} \geq 1 \text{ and } \frac{(1-\delta)^3\gamma}{4} \geq 1$$

the sliver elimination phase terminates
yielding a sliver free bounded radius-edge ratio mesh
i.e. for any tetrahedron $\rho \leq \rho_0$ and $\sigma \geq \sigma_0$

Proof.

Two lemmas to show that
if l_1 is the shortest edge length before sliver elimination phase
the shortest edge length after sliver elimination phase is $l_2 = \frac{(1-\delta)^3 l_1}{4}$ □

Original mesh = bounded radius-edge ratio mesh obtained in first phase
original sliver = sliver of the original mesh

Lemma

Any point q whose insertion is triggered by an original sliver, has an insertion radius $r_q \geq l_2$ with $l_2 = \frac{(1-\delta)^3 l_1}{4}$

Proof.

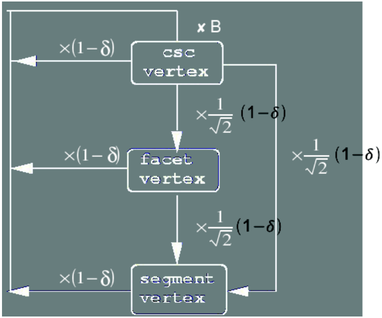
Assume that original sliver t with circumradius r_t
triggers the insertion of point q in the refinement region $(v, \delta r_v)$
of either the original sliver t, or a constrained facet, or a constrained edge.

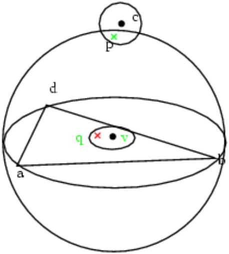
$$r_q \geq (1-\delta) r_v \geq \left| \begin{array}{l} (1-\delta) r_t \\ (1-\delta)^2 \frac{r_t}{\sqrt{2}} \\ (1-\delta)^3 \frac{r_t}{2} \end{array} \right. \qquad r_t \geq \frac{l_1}{2}$$

□

Insertion radius

Flow diagram





$$r_q \geq (1-\delta) r_v$$
$$r_p \geq (1-\delta) r_c$$
$$r_p \leq \min(\|pa\|, \|pb\|, \|pd\|) \leq \sqrt{2} r_v$$
$$r_q \geq \frac{(1-\delta)^2}{\sqrt{2}} r_c$$

Lemma

If $\frac{(1-\delta)^3\rho_0}{2} \geq 1$ *and* $\frac{(1-\delta)^3\gamma}{4} \geq 1$
any vertex inserted during the sliver elimination phase has an insertion radius at least $l_2 = \frac{(1-\delta)^3 l_1}{4}$.

Proof.

By induction, let t be the tetrahedron that triggers the insertion of p
- done if t is an original sliver
- otherwise

$$r_p \geq (1-\delta) r_v \geq \left| \begin{array}{l} (1-\delta) r_t \\ (1-\delta)^2 \frac{r_t}{\sqrt{2}} \\ (1-\delta)^3 \frac{r_t}{2} \end{array} \right. \quad \text{with} \quad \begin{array}{l} r_t \geq \rho_0 l_2 \\ r_t \geq \gamma r'_t \geq \gamma \frac{l_2}{2} \end{array}$$

□