Mesh Generation through Delaunay Refinement

3D Meshes

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The 3D meshing problem

Input:

- $C$: a 2D PLC in $\mathbb{R}^3$ (piecewise linear complex)
- $\Omega$: a bounded domain to be meshed. $\Omega$ is bounded by facets in $C$

Output: a mesh of domain $\Omega$

i.e. a 3D triangulation $T$ such that

- vertices of $C$ are vertices of $T$
- edges and facets $C$ are union of faces in $T$
- the tetrahedra of $T$ that are $\subset \Omega$ have controlled size and quality
The 3D meshing problem
Constrained edges and facets

The mesh is a 3D Delaunay triangulation. Edges and facets of the input PLC appear in the mesh as union of Delaunay edges and Delaunay facets, that are called constrained edges and facets.
3D Delaunay refinement

Constrained facets

How input PLC facets are subdivided into constrained facets?
- 3D Delaunay facets are 2D Delaunay in the facet hyperplane.
- Constrained facets are known as soon as PLC edges are refined into Delaunay edges.
- A 2D Delaunay triangulation is maintained for each PLC facet
3D Delaunay refinement

Use a 3D Delaunay triangulation

Constraints
constrained edges are refined into Gabriel edges with empty smallest circumball
encroached edges = edges which are not Gabriel edges

constrained facets are refined into Gabriel facets with empty smallest circumball
encroached facets = facets which are not Gabriel facets

Tetrahedra
Bad tetrahedra are refined by circumcenter insertion.

Bad tetrahedra: radius-edge ratio
\[ \rho = \frac{\text{circumradius}}{l_{\text{min}}} \geq B \]
3D Delaunay refinement algorithm

- **Initialization** Delaunay triangulation of PLC vertices
- **Refinement**

Apply one the following rules, until no one applies. Rule $i$ has priority over rule $j$ if $i < j$.

1. if there is an encroached constrained edge $e$, refine-edge($e$)

2. if there is an encroached constrained facet $f$,
   refine-facet-or-edge($f$) i.e.:
   
   $c = \text{circumcenter}(f)$
   
   if $c$ encroaches a constrained edge $e$, refine-edge($e$).
   
   else insert($c$)

3. if there is a bad tetrahedra $t$,
   refine-tet-or-facet-or-edge($t$) i.e.:
   
   $c = \text{circumcenter}(t)$
   
   if $c$ encroaches a constrained edge $e$, refine-edge($e$).
   
   else if $c$ encroaches a constrained facet $f$,
   
   refine-facet-or-edge($f$).
   
   else insert($c$)
Refinement of constrained facets

Lemma (Projection lemma)

When a point $p$ encroaches a constrained subfacet $f$ of PLC facet $F$ without constrained edges encroachment:

- The projection $p_F$ of $p$ on the supporting hyperplan $H_F$ of $F$, belongs to $F$.
- $p$ encroaches the constrained subfacet $g \subset F$ that contains $p_F$.

Proof.

The algorithm always refine a constrained facet including the projection of the encroaching point.
3D Delaunay refinement algorithm (bis)

- Initialization Delaunay triangulation of PLC vertices
- Refinement

Apply one of the following rules, until no one applies.
Rule $i$ has priority over rule $j$ if $i < j$.

1. If there is an encroached constrained edge $e$, $\text{refine-edge}(e)$
2. If there is a constrained facet $f$ encroached by a vertex $v$, find the constrained facet $g$ that contains the proj. of $v$ on $H_f$
   \[ \text{refine-facet-or-edge}(g) \]
3. If there is a bad tetrahedra $t$,
   \[ \text{refine-tet-or-facet-or-edge}(t) \]
   i.e.:
   \[ c = \text{circumcenter}(t) \]
   if $c$ encroaches a constrained edge $e$, $\text{refine-edge}(e)$.
   else if $c$ encroaches a constrained facet $f$,
   find the constrained facet $g$ that contains the proj. of $v$
   $\text{refine-facet-or-edge}(g)$.
   else insert($c$)
Theorem (3D Delaunay refinement)

The 3D Delaunay refinement algorithm ends provided that:

- the upper bound on radius-edge ratio of tetrahedra is \( B > 2 \)

- all input PLC angles are > 90°
  - dihedral angles: two facets of the PLC sharing an edge
  - edge-facet angles: a facet and an edge sharing a vertex
  - edge angles: two edges of the PLC sharing a vertex

Proof.

As in 2D, use a volume argument to bound the number of Steiner vertices
Proof of 3D Delaunay refinement theorem

Lemma (Lemma 1)

Any added (Steiner) vertex is inside or on the boundary of the domain $\Omega$ to be meshed

Proof.

– trivial for vertices inserted in constrained edges
– as in 2D for facet circumcenters
  when facet circumcenters are inserted
  there is no encroached edge
– analog proof for tetrahedra circumcenters
  when tetrahedra circumcenters are added
  there is no encroached edge nor encroached facet.
Proof of 3D Delaunay refinement theorem

Local feature size $lfs(p)$
radius of the smallest disk centered in $p$ and intersecting two
disjoint faces of $C$.

Insertion radius $r_v$
length of the smallest edge incident to $v$, right after insertion of $v$, if $v$ is inserted.

Parent vertex $p$ of vertex $v$
- if $v$ is the circumcenter of a tet $t$
  $p$ is the last inserted vertex of the smallest edge of $t$
- if $v$ is inserted on a constrained facet or edge
  $p$ is the encroaching vertex closest to $v$
  ($p$ may be a mesh vertex or a rejected vertex)
Proof of 3D Delaunay refinement theorem

Insertion radius lemma

Lemma (Insertion radius lemma)

Let $v$ be vertex of the mesh or a rejected vertex, with parent $p$, then $r_v \geq lfs(v)$ or $r_v \geq C r_p$, with:

- $C = B$ if $v$ is a tetrahedra circumcenter
- $C = \frac{1}{\sqrt{2}}$ if $v$ is on a constrained edge or facet and $p$ is rejected
Refinement of constrained facets

$v$ is the circumcenter of constrained facet $f$
$p$ is the vertex encroaching $f$
Because $f$ includes the projection $p_F$ of $p$ on $F$: $r_v \geq \frac{r_p}{\sqrt{2}}$

**Proof.**

$r_v = r$, radius of the smallest circumball of $f$

$r_p \leq p_a$ if $p_a = \min\{p_a, p_b, p_c\}$

$\|p_a\|^2 = \|pp_F\|^2 + \|p_F a\|^2 \leq 2r^2$
Proof of 3D Delaunay refinement theorem

Flow diagram of vertices insertion
Proof of 3D Delaunay refinement theorem

weighted density $d(v) = \frac{lfs(v)}{r_v}$

Lemma (Weighted density lemma 1)
For any vertex $v$ with parent $p$, if $r_v \geq C r_p$, $d(v) \leq 1 + \frac{d(p)}{C}$

Lemma (Weighted density lemma 2)
There are constants $D_e \geq D_f \geq D_t \geq 1$ such that:
for any tet circumcenter $v$, inserted or rejected, $d(v) \leq D_t$
for any facet circumcenter $v$, inserted or rejected, $d(v) \leq D_f$.
for any vertex $v$ inserted in a PLSG edge, $d(v) \leq D_e$.

Thus, for any vertex of the mesh $r_v \geq \frac{lfs(v)}{D_e}$
3D Delaunay refinement theorem
Proof of weighted density lemma

Proof of weighted density lemma
Assume the lemma is true up to the insertion of vertex \( v \), \( p \) parent of \( v \)

- **\( v \) is a tet circumcenter**
  \[ r_v \geq B r_p \implies d(v) \leq 1 + \frac{d_p}{B} \quad \text{assume } 1 + \frac{D_e}{B} \leq D_t \quad (1) \]

- **\( v \) is on a PLC facet \( F \)**
  - \( p \) is a PLC vertex or \( p \in \text{PLC face } F' \) st \( F \cap F' = \emptyset \)
    \[ r_v = \text{lfs}(v) \implies d(v) \leq 1 \]
  - \( p \) is a rejected tet circumcenter
    \[ r_v \geq \frac{r_p}{\sqrt{2}} \implies d(v) \leq 1 + \sqrt{2} d_p \quad \text{assume } 1 + \sqrt{2} D_t \leq D_f \quad (2) \]

- **\( v \) is on a PLC edge \( E \)**
  - \( p \) is a PLC vertex or \( p \in \text{PLC face } F' \) st \( E \cap F' = \emptyset \)
    \[ r_v = \text{lfs}(v) \implies d(v) \leq 1 \]
  - \( p \) is a rejected tet or a facet circumcenter
    \[ r_v \geq \frac{r_p}{\sqrt{2}} \implies d(v) \leq 1 + \sqrt{2} d_p \quad \text{assume } 1 + \sqrt{2} D_f \leq D_e \quad (3) \]
3D Delaunay refinement theorem

Proof of weighted density lemma (end)

There are $D_e \geq D_f \geq D_t \geq 1$ such that:

1. $1 + \frac{D_e}{B} \leq D_t$ \hspace{1cm} (1)
2. $1 + \sqrt{2}D_t \leq D_f$ \hspace{1cm} (2)
3. $1 + \sqrt{2}D_f \leq D_e$ \hspace{1cm} (3)

\[
D_e = \left(3 + \sqrt{2}\right) \frac{B}{B - 2}
\]
\[
D_f = \frac{(1 + \sqrt{2}B) + \sqrt{2}}{B - 2}
\]
\[
D_t = \frac{B + 1 + \sqrt{2}}{B - 2}
\]
Proof of 3D Delaunay refinement theorem (end)

Theorem (Relative bound on edge length)

Any edge of the mesh, incident to vertex \( v \), has length \( l \) st:

\[
l \geq \frac{lfs(v)}{D_e + 1}
\]

Proof.
as in 2D

End of 3D Delaunay refinement theorem proof.

Using the above result on edge lengths, prove an upper bound on the number of mesh vertices as in 2D.
Delaunay refinement
meshing domain with small angles

Algorithm Terminator 3D : Delaunay refinement + additionnal rules

1. Clusters of edges : refine edges in clusters along concentric spheres

2. when a facet $f$ in PLC facet $F$ is encroached by $p$
   and circumcenter($f$) encroaches no constrained edge
   refine $f$ iff
   - $p$ is a PLC vertex or belongs to a PLC face $s'$ st $f \cap s' = \emptyset$
   - $r_v > r_g$, where $g$ is the most recently ancestor of $v$.

3. when a constrained edge $e$ is encroached by $p$
   $e$ is refined iff
   - $p$ is a mesh vertex
   - $\min_{w \in W} r_w > r_g$ where
     $g$ is the most recently inserted ancestor of $v$
     $W$ is the set of vertices that will be inserted if $v$ is inserted.
Remarks Notice that some constrained facets remain encroached

• using a constrained Delaunay triangulation is required to respect constrained facets. Fortunately, this constrained Delaunay triangulation exists because constrained edges are Gabriel edges.

• the final mesh may be different from the Delaunay triangulation of its vertices
Nearly degenerated triangles

\[
\rho = \frac{\text{circumradius}}{\text{shortest edge length}}
\]

In both cases the radius-edge ratio is large
Nearly degenerated tetrahedra

Thin tetrahedra

Flat tetrahedra

Slivers: the only case in which radius-edge ratio $\rho$ is not large
**Slivers**

**Definition (Slivers)**
A tetrahedron is a sliver iff
the volume is too small \( \sigma = \frac{V}{l^3} \leq \sigma_0 \)
yet, the radius-edge ratio is not too big \( \rho = \frac{r}{l} \leq \rho_0 \)

\[ r = \text{circumradius}, \; l = \text{shortest edge length}, \; V = \text{volume} \]

**Remark**
Radius-edge ratio is not a fair measure for tetrahedra.
Radius-radius ratio is a fair measure \( \frac{r_{circ}}{r_{insc}} \), and :

\[ \frac{r_{circ}}{r_{insc}} \leq \frac{\sqrt{3} \rho^3}{\sigma} \]

**Proof.**
area of facets of \( t \) : \( S_i \leq \frac{3\sqrt{3}}{4} r_{circ}^2 \)

\[ \sqrt{3} r_{circ}^2 r_{insc} \geq \sum_{i=1}^{4} \frac{1}{3} S_i r_{insc} = V = \sigma l^3 = \sigma \left( \frac{r_{circ}}{\rho} \right)^3 \]
Delaunay meshes with bounded radius-edge ratio

Theorem (Delaunay meshes with bounded radius-edge ratio)

Any Delaunay mesh with bounded radius-edge ratio is such that:

1. The ratio between the length of the longest edge and the length of shortest edge incident to a vertex $v$ is bounded.

2. The number of edges, facets or tetrahedra incident to a given vertex is bounded.
Delaunay meshes with bounded radius-edge ratio

Lemma
In a Delaunay mesh with bounded radius-edge ratio \( \rho \leq \rho_0 \), edges \( ab, ap \) incident to the same vertex and forming an angle less than \( \eta_0 = \arctan \left[ 2 \left( \rho_0 - \sqrt{\rho_0^2 - 1/4} \right) \right] \) are such that \( \frac{\|ab\|}{2} \leq \|ap\| \leq 2\|ab\| \)

Proof.
\[ \Sigma(y, r_y) = \text{Intersection of the hyperplan spanned by } (ap, ab) \text{ with the circumsphere of a tetrahedron incident to } ab \]

\[ \|xv\| = r_y - \sqrt{r_y^2 - \|ab\|^2/4} \]

\[ \|xv\| \geq \left( \rho_0 - \sqrt{\rho_0^2 - 1/4} \right) \|ab\| \]

\[ (ab, ax) = \arctan \left( \frac{2\|xv\|}{\|ab\|} \right) \geq \eta_0 \]

\[ (ab, ap) \leq \eta_0 \implies \|ap\| \geq \|ax\| \geq \frac{||ab||}{2} \]
Delaunay meshes
with bounded radius-edge ratio theorem
proof of Part 1

\[ \rho_0 \text{ radius-edge ratio bound} \]
\[ \eta_0 = \arctan \left[ 2 \left( \rho_0 - \sqrt{\rho_0^2 - 1/4} \right) \right] \]
\[ \nu_0 = 2^{2m_0-1} \rho_0^{m_0-1} \]

Two mesh edges \( ab \) and \( ap \) incident to \( a \) are such that :
\[ \frac{\|ab\|}{\nu_0} \leq \|ap\| \leq \nu_0 \|ab\| \]

Proof.
\( \Sigma(a, 1) \) unit sphere around \( a \)
Max packing on \( \Sigma \) of spherical caps with angle \( \eta_0/4 \)
There is at most \( m_0 \) spherical caps
Doubling the cap’s angles form a covering of \( \Sigma \).
Graph \( G = \) traces on \( \Sigma(a, 1) \) of edges and facets incident to \( a \).
Path in \( G \) from \( ab \) to \( ap \), ignore detours when revisiting a cap.
At most, the path visits \( m_0 \) caps and crosses \( m_0 - 1 \) boundary.
Delaunay meshes
with bounded radius-edge ratio
proof of Part 2

The number of edges incident to a given vertex is bounded by
$$\delta_0 = (2\nu_0^2 + 1)^3$$

Proof.

- $ap$ : shortest edge incident to $a$, let $\|ap\| = 1$
- $ab$ : longest edge incident to $a$, $\|ap\| \leq \nu_0$

for any vertex $c$ adjacent to $a$, $1 \leq \|ac\| \leq \nu_0$

for any vertex $d$ adjacent to $c$, $\|cd\| \geq \frac{1}{\nu_0}$

Spheres $\Sigma_c(c, \frac{1}{2\nu_0})$ are empty of vertices except $c$, disjoint
and included in $\Sigma(a, \nu_0 + \frac{1}{2\nu_0})$)

$$V_{\Sigma} = \frac{4}{3} \pi \left(\nu_0 + \frac{1}{2\nu_0}\right)^3 = (2\nu_0^2 + 1)^3 V_{\Sigma_c}$$
Sliver elimination

Method of Li [2000]
Choose each Steiner vertex in a refinement region:

Refinement region
refining a tetrahedra $t$ with circumsphere $(c_t, r_t)$: 3D ball $(c_t, \delta r_t)$
refining a facet $f$ with circumcircle $(c_f, r_f)$: 2D ball $(c_f, \delta r_f)$
refining an edge $(c_s, r_s)$: 1D ball $(c_s, \delta r_s)$
Definition (Slivers)

\[ r = \text{circumradius}, \quad l = \text{shortest edge length}, \quad V = \text{volume} \]
\[ \rho = \frac{r}{l} \leq \rho_0 \quad \sigma = \frac{V}{l^3} \leq \sigma_0 \]

Lemma

If pqrs is a tet with \( \sigma \leq \sigma_0 \), \( \frac{d}{r_y} \leq 12\sigma_0 \)

\( d \): distance from \( p \) to the hyperplan of \( qrs \)
\( r_y \): circumradius of triangle qrs

Proof.

\[ \sigma l^3 = V = \frac{1}{3} Sd \geq \frac{1}{3} \left( \frac{1}{2} l^2 \frac{l}{2r_y} \right) d = \frac{l^3}{12r_y} d \]
Lemma (Sliver lemma)

Let $\Sigma(y, r_y)$ be the circumcircle of triangle $qrs$.
If the tet $pqrs$ is a sliver, $d' = d(p, \Sigma(y, r_y)) \leq 12\sigma_0 f(\rho_0) r_y$ with $f(\rho_0) = 1 + \frac{1}{1 - \sqrt{1 - \frac{1}{4\rho_0^2}}}$.

Proof.

$r \leq 2\rho_0 r_y, \quad d \leq 12\sigma_0 r_y$

$h = \sqrt{r^2 - r_y^2} \leq r \sqrt{1 - \frac{1}{4\rho_0^2}}$

$d/d_1 \geq \frac{r-h}{r_y} \geq \frac{r-h}{r}$

$d' \leq d + d_1 \leq d \left(1 + \frac{r}{r-h}\right)$
Forbidden volume
For any triangle $qrs$ with circumradius $r_y$, $p$ should not be in a torus with volume:

\[ V(\text{torus}(qrs)) = 2\pi r_y \cdot \pi d'^2 \]

\[ V = c_3 r_y^3 \]

\[ c_3 = 488\pi^2 (\sigma_0 f(\rho_0))^2 \]

Forbidden area on any plane $h$

\[ S(\text{torus}(qrs) \cap h) \leq \pi (r_y + d')^2 - \pi (r_y - d')^2 = 4\pi d' r_y \]

\[ S(\text{torus}(qrs) \cap h) \leq c_2 r_y^2 \]

\[ c_2 = 48\pi (\sigma_0 f(\rho_0)) \]

Forbidden length on any line $l$

\[ L(\text{torus}(qrs) \cap h) \leq 2\sqrt{(r_y + d')^2 - (r_y - d')^2} = 4\sqrt{r_y d'} \]

\[ L(\text{torus}(qrs) \cap l) \leq c_1 r_y \]

\[ c_1 = 8\sqrt{3} \sqrt{\sigma_0 f(\rho_0)} \]
Sliver elimination

Main Idea
Start from a Delaunay mesh with bounded edge-radius ratio
Then refine bad tets (\( \rho > \rho_0 \)) and slivers (\( \rho \leq \rho_0, \sigma \leq \sigma_0 \))
- with refinement point carefully chosen in the refinement regions
- to avoiding forbidden volumes, areas and segments

When refining a mesh element \( \tau \) (\( \tau \) may be a tet, a facet or an edge)
it is not always possible to avoid producing new slivers
but it is possible to avoid producing small slivers,
i. e. slivers \( pqrs \) with circumradius(\( pqrs \)) \( \leq \gamma r_\tau \)
where \( r_\tau \) is the radius of the smallest circumsphere of \( \tau \).
Lemma
Assume that the shortest edge length in the mesh stays $\geq l_2$.
For any refinement region $(c_\tau, \delta r_\tau)$
there is a finite number $W$ of facets $(qrs)$
that are likely to form a tetrahedron $pqrs$ in the mesh
if some point $p \in (c_\tau, \delta r_\tau)$ is inserted.

Proof.
$Q =$ set of vertices of facets $qrs$ forming a tet with $p \in (c_\tau, \delta r_\tau)$
$r_1$ maximum circumradius in the mesh before the sliver elimination phase.
Any point in $\Omega$ is at distance less than $r_1$ from a mesh vertex:
circumradius($pqrs$) $\leq r_1$
$\|c_\tau q\| \leq \|c_\tau p\| + \|pq\| \leq \delta r_\tau + 2r_1 \leq (2 + \delta)r_1$
Empty circle of radius $l_2/2$ around an vertex in $Q$
$|Q| \leq \left(\frac{2[(2+\delta)r_1+l_2]}{l_2}\right)^3$
$W \leq |Q|^3$
Lemma
For any set of constants $\rho_0$, $\gamma$ and $\delta$, if $\sigma_0$ is small enough, there is a point $p$ in refinement region $(c_\tau, \delta r_\tau)$ such that the insertion of $p$ creates no sliver with radius smaller than $\gamma r_\tau$.

Proof.
If circumradius($pqrs$) $\leq \gamma r_\tau$, circumcircle($pqr$) $\leq \gamma r_\tau$

\[
W 488\pi^2 (\sigma_0 f(\rho_0))^2 \gamma^3 r_\tau^3 < \frac{4}{3} \pi \delta^3 r_\tau^3 \\
W 48\pi (\sigma_0 \rho_0) \gamma^2 r_\tau^2 < \pi \delta^2 r_\tau^2 \\
W 8\sqrt{3}\sqrt{\sigma_0 \rho_0} \gamma r_\tau < 2\delta r_\tau
\]
Sliver elimination

- Initial phase
  Build a bounded radius-edge ratio mesh
    using usual Delaunay refinement
- Sliver elimination phase
  Apply one of the following rules, until no one applies
  Rule $i$ has priority over rule $j$ if $i < j$.

1. if there is an encroached constrained edge $e$,
   sliver-free-refine-edge($e$)

2. if there is an encroached constrained facet $f$,
   sliver-free-refine-facet-or-edge($f$)

3. if there is a tet $t$ with $\rho \geq \rho_0$,
   sliver-free-refine-tet-or-facet-or-edge($t$)

4. if there is a sliver $t$ ($\rho \leq \rho_0$, $\sigma \leq \sigma_0$),
   sliver-free-refine-tet-or-facet-or-edge($t$)
Sliver elimination

Sliver-free versions of refine functions

sliver-free-refine-edge(e)
sliver-free-refine-facet-or-edge(f)
sliver-free-refine-tet-or-facet-or-edge(t)

• pick \( q \) sliver free in refinement region
• if \( q \) encroaches a constrained edge \( e \),
  sliver-free-refine-edge(e).
• else if \( q \) encroaches a constrained facet \( f \),
  sliver-free-refine-facet-or-edge(f).
• else insert(\( q \))

picking \( q \) sliver free in refinement region means :

• pick a random point \( q \) in refinement region
• while \( q \) form small slivers
  pick another random point \( q \) in refinement region
Sliver elimination

**Theorem**

*If the hypothesis of Delaunay refinement theorem are satisfied and if the constants \( \delta \), \( \rho_0 \) and \( \gamma \) are such that*

\[
\frac{(1 - \delta)^3 \rho_0}{2} \geq 1 \quad \text{and} \quad \frac{(1 - \delta)^3 \gamma}{4} \geq 1
\]

*the sliver elimination phase terminates yielding a sliver free bounded radius-edge ratio mesh i.e. for any tetrahedron \( \rho \leq \rho_0 \) and \( \sigma \geq \sigma_0 \)*

**Proof.**

Two lemmas to show that

if \( l_1 \) is the shortest edge length before sliver elimination phase

the shortest edge length after sliver elimination phase is \( l_2 = \frac{(1 - \delta)^3 l_1}{4} \)
Original mesh = bounded radius-edge ratio mesh obtained in first phase
original sliver = sliver of the original mesh

Lemma
Any point q whose insertion is triggered by an original sliver,
has an insertion radius \( r_q \geq l_2 \) with \( l_2 = \frac{(1-\delta)^3 l_1}{4} \)

Proof.
Assume that original sliver \( t \) with circumradius \( r_t \)
triggers the insertion of point \( q \) in the refinement region \( (\nu, \delta r_\nu) \)
of either the original sliver \( t \), or a constrained facet, or a constrained edge.

\[
r_q \geq (1 - \delta) r_\nu \geq \begin{vmatrix}
(1 - \delta) r_t \\
(1 - \delta)^2 \frac{r_t}{\sqrt{2}} \\
(1 - \delta)^3 \frac{r_t}{2}
\end{vmatrix}
\]
\[
r_t \geq \frac{l_1}{2}
\]
Sliver elimination
Proof of termination

Insertion radius

Flow diagram

\[ r_q \geq (1 - \delta)r_v \]
\[ r_p \geq (1 - \delta)r_c \]
\[ r_p \leq \min(||pa||, ||pb||, ||pd||) \leq \sqrt{2}r_v \]
\[ r_q \geq \frac{(1 - \delta)^2}{\sqrt{2}}r_c \]
Lemma

If \(\frac{(1-\delta)^3 \rho_0}{2} \geq 1\) and \(\frac{(1-\delta)^3 \gamma}{4} \geq 1\)
any vertex inserted during the sliver elimination phase
has an insertion radius at least \(\ell_2 = \frac{(1-\delta)^3 \ell_1}{4}\).

Proof.
By induction, let \(t\) be the tetrahedron that triggers the insertion of \(p\)
- done if \(t\) is an original sliver
- otherwise

\[
r_p \geq (1 - \delta) r_v \geq \left\{ \begin{array}{l}
(1 - \delta)^3 \frac{r_t}{2} \\
(1 - \delta)^2 \frac{r_t}{\sqrt{2}} \\
(1 - \delta)^3 r_t
\end{array} \right.
\]

with \(r_t \geq \rho_0 \ell_2\)
\(r_t \geq \gamma r'_t \geq \gamma \ell_2\)