Randomized Algorithms

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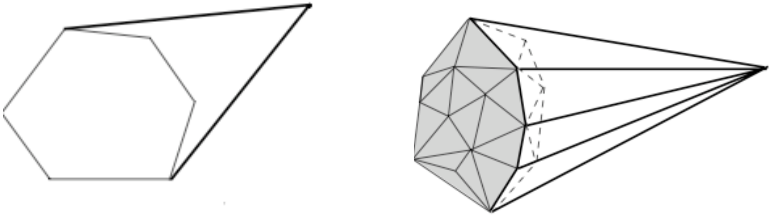
Back to Incremental Convex Hull

$\mathcal{P}$ a set of n points in $\mathbb{R}^d$

1. Sort $\mathcal{P}$ by lexicographic order
   $\mathcal{P} = \{p_1, \ldots, p_n\}$
2. For $i = 1, \ldots n$,
   insert p_{i+1}: $\text{conv}(\mathcal{P}_i) \longrightarrow \text{conv}(\mathcal{P}_i \cup p_{i+1})$

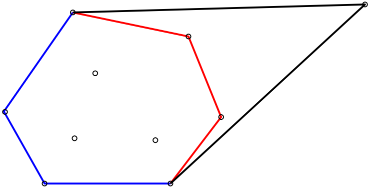


Back to Incremental Convex Hull

$\mathcal{P}_i = \{p_1, \ldots, p_i\}$
Updating $conv(\mathcal{P}_i)$ into $conv(\mathcal{P}_i \cup p_{i+1})$: a color story

Facet A facet f of $conv(\mathcal{P}_i)$ with supporting hyperplan h_f is:
– red iff $conv(\mathcal{P}_i)$ and p_{i+1} are on differents side of h_f.
– blue iff $conv(\mathcal{P}_i)$ and p_{i+1} are on the same side of h_f.

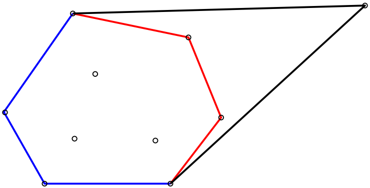
k-**Faces** with $k < d - 1$ of $conv(\mathcal{P}_i)$ is:
– red iff it is the intersection of red facets
– blue iff it is the intersection of blue facets
– purple iff it is included in red and blue facets



Back to Incremental Convex Hull

Updating $conv(\mathcal{P}_i)$ into $conv(\mathcal{P}_i \cup p_{i+1})$:

1. find a first red facets (there is one incident to p_i)
2. find the set of red facets (they form a connected set)
3. delete red faces
   install new faces = $\text{conv}(e \cup p_{i+1})$
   where e is a purple face of $conv(\mathcal{P}_i)$.

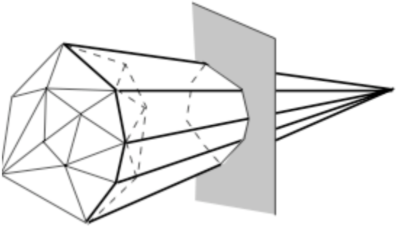


Back to Incremental Convex Hull

Complexity

Sorting + the total number of created facets

$$O\left(n \log n + \Sigma_{i=1}^{n} i^{\left\lfloor \frac{d-1}{2} \right\rfloor}\right) = O\left(n \log n + n^{\left\lfloor \frac{d+1}{2} \right\rfloor}\right)$$

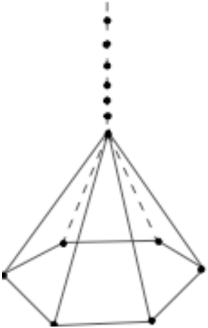


Randomized incremental convex hull: $O\left(n \log n + n^{\left\lfloor \frac{d}{2} \right\rfloor}\right)$

Optimal convex hull: $O\left(n \log n + n^{\left\lfloor \frac{d}{2} \right\rfloor}\right)$

Back to Incremental Convex Hull

A bad case in 3D



The bad case arises with a special set of points
inserted in a special order.
Hence the idea of randomization.

Randomized Algorithms

What is a randomized algorithms

- A randomized algorithm computes the exact solution of a deterministic problem
- The algorithm performs internal random choices.
  that influence the algorithm's behaviour but not the output.
- Analysis of the algorithm: expectation over the random choices.

Randomized incremental algorithms

random choices: random insertion order of the data

Randomized analysis of on line algorithms

- An on line algorithm handles data according to the given input order.
- Expected complexity with respect to a random input order.

The Formalism of Regions and Conflicts

Data : a finite set $\mathcal{P}$ of objects in $\mathcal{O}$

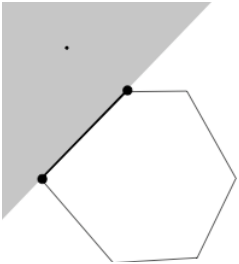
Region - a subset of $i < b$ objects that defines the region
- a subset of $\mathcal{O}$ in conflict with the region

Problem of interest build the set of regions defined by $\mathcal{P}$ without conflict on $\mathcal{P}$

Regions and Conflicts for the Convex hull

– objects are points in $\mathbb{R}^d$
– regions are halfspaces of $\mathbb{R}^d$
– each region is defined by d-points
– p in conflict with h^+ iff $p \in h^+$

$$\text{facets of conv}(\mathcal{P}) = \left\{ \begin{array}{l} \text{set of regions} \\ \text{defined by } \mathcal{P} \\ \text{with no conflict on } \mathcal{P} \end{array} \right.$$



Randomized Incremental Construction

Notations

Let $\mathcal{P}$ be a finite set of objects.
$\mathcal{F}(\mathcal{P})$ the set of regions defined by objects in $\mathcal{P}$
$\mathcal{F}_j(\mathcal{P})$ the set of regions defined by objects in $\mathcal{P}$
in conflict with j objects in $\mathcal{P}$.

Incremental construction of $\mathcal{F}_0(\mathcal{P})$.

At each step:
- a new point p of $\mathcal{P}$ is added in a subset $\mathcal{R} \subset \mathcal{P}$
- the set $\mathcal{F}_0(\mathcal{R})$ is updated.

Randomized hypothesis

The order of insertion of objects in $\mathcal{P}$ is random.
At each step r, $\mathcal{R}$ is a r-sample of $\mathcal{P}$, i.e. a random subset with size r.

Randomized Incremental Construction

$\mathcal{R}$ random subset of $\mathcal{P}$ with size r
$f_0(r)$ expected number of regions defined on $\mathcal{R}$ and without conflict on $\mathcal{R}$

Theorem (First RIC theorem)

The expected total number of regions constructed by the RIC is:
$O\left(\sum_{r=1}^{n} \frac{f_0(r)}{r}\right)$

Proof.

probability for $f \in \mathcal{F}(\mathcal{P})$ to be in $\mathcal{F}_0(\mathcal{R})$: $p_f(r)$
Probability for $f \in \mathcal{F}(\mathcal{P})$ to appear in $\mathcal{F}_0(\mathcal{R})$ at step r : $\frac{b}{r} p_f(r)$

$$f_0(r) = \sum_{f \in \mathcal{F}(\mathcal{P})} p_f(r)$$

Expected total number of regions constructed by the RIC $= \sum_r \sum_f \frac{b}{r} p_f(r) = O\left(\sum_{r=1}^{n} \frac{f_0(r)}{r}\right)$

Randomized Incremental Convex Hull

Expected number of constructed regions

In the convex hull case $f_0(r) = O(r^{\lfloor \frac{d}{2} \rfloor})$

Expected total number of regions constructed by the convex hull RIC $= O\left(\sum_{r=1}^{n} \frac{f_0(r)}{r}\right) = O\left(n^{\lfloor \frac{d}{2} \rfloor}\right)$

This not the end of the story for the RIC of convex hulls:
since points are inserted in random order
we can no more rely on the lexicographic order
to find the first *red* facet at each insertion.

The conflict graph

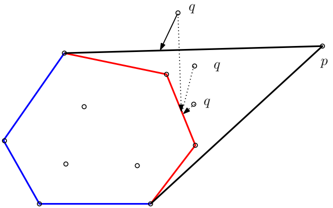
A bipartite graph on $\mathcal{P} \times \mathcal{F}_0(\mathcal{R})$:
$\forall p \in \mathcal{P} \setminus \mathcal{R}$, an edge (p, f) where $f \in \mathcal{F}_0(\mathcal{R})$ conflicts with p.

Using conflict graph

$\text{conv}(\mathcal{R}) \longrightarrow \text{conv}(\mathcal{R} \cup p)$
- the conflict graph edge of p provides the first red facet.
- Find all the red facets by walking on facets of $\text{conv}(\mathcal{R})$ in conflict with p
- Update the convex hull
- Update the conflict graph
  for each $q \neq p$ in $\mathcal{P} \setminus \mathcal{R}$, find a new conflict graph edge
  by walking on facets of $\text{conv}(\mathcal{R})$ conflicting with p and q

Randomized Incremental Convex hull

Updating the conflict graph



Update the conflict graph

Insertion of p : conv($\mathcal{R}$) $\longrightarrow$ conv($\mathcal{R} \cup p$)
Let $q \in \mathcal{P} \setminus \mathcal{R},\ q \neq p$.
Walk on the facets f of $conv(\mathcal{R})$ in conflict with p and q
- if a neighbor g of f conflicts with q but not with p : edge(q,g)
- if a neighbor g of f conflicts neither with q nor f
  f' neighbor of g in $conv(\mathcal{R} \cup p)$ s.t. $g \cap f = g \cap f'$
  edge (q, f') if q and f' conflict.
- if neither happens, discard q.

Randomized Incremental Constructions

Conflict graph complexity

Total complexity of convex hull

- Update of convex hulls : total number of constructed regions
- Finding conflicts + updating the conflict graphs :
  total number of conflicts with constructed regions

Theorem (Second RIC theorem)

The expected total number of conflicts with constructed regions is:

$$O\left(\sum_{r=1}^{n} \frac{n-r}{r^2} f_0(r)\right)$$

Expected complexity of convex hull RIC

$f_0(r) = O\left(r^{\lfloor \frac{d}{2} \rfloor}\right), \quad O\left(\sum_r \frac{n-r}{r^2} f_0(r)\right) = O\left(n \log n + n^{\lfloor \frac{d}{2} \rfloor}\right)$

Proof of 2d RIC Theorem

Statistics on regions

$\mathcal{R} \subset \mathcal{P}$
$\mathcal{F}_j(\mathcal{P})$ the set of regions defined by $\mathcal{P}$ with j conflicts in $\mathcal{P}$.
$\mathcal{F}_j(\mathcal{R})$ the set of regions defined by $\mathcal{R} \subset \mathcal{P}$ with j conflicts in $\mathcal{R}$.
$f_j(r)$ expected size of $\mathcal{F}_j(\mathcal{R})$.

$\mathcal{R}$ being a random sample
probability for $f \in \mathcal{F}_j(\mathcal{P})$
to be in $\mathcal{F}_k(\mathcal{R})$

$$p_{j,k}(r) = \frac{\binom{j}{k}\binom{n-b-j}{r-b-k}}{\binom{n}{r}}$$

$$f_k(r) = \sum_j |\mathcal{F}_j(\mathcal{P})| \; p_{j,k}(r)$$

$$f_0(r) = \sum_j |\mathcal{F}_j(\mathcal{P})| \; p_j(r)$$

$\mathcal{R} \subset \mathcal{P}$
$\mathcal{P}(f)$ set of objects in $\mathcal{P}$ in conflict with region f

Moment of order k of $\mathcal{R}$ with respect to $\mathcal{P}$

$m_k(\mathcal{R}, \mathcal{P}) = \sum_{f \in \mathcal{F}_0(\mathcal{R})} \binom{|\mathcal{P}(f)|}{k}$.

Expectation for a r-sample

$m_k(r) = \sum_f \binom{|\mathcal{P}(f)|}{k} \text{proba}(f \in \mathcal{F}_0(\mathcal{R})) = \sum_j |\mathcal{F}_j(\mathcal{P})| \binom{j}{k} p_j(r)$.

RIC and first order moment

Expected total number of conflicts with constructed regions

$$\sum_r \sum_j |\mathcal{F}_j(\mathcal{P})| \binom{j}{1} \frac{b}{r} p_j(r) = \sum_r b \frac{m_1(r)}{r}$$

Theorem (Moments theorem)

$$m_k(\mathcal{R}) \leq f_k(\mathcal{R}) \frac{(n-r+k)!}{(n-r)!} \frac{(r-b-k)!}{(r-b)!}$$

Proof

$m_k(r) =$

$$\sum_j |\mathcal{F}_j(\mathcal{P})| \binom{j}{k} \frac{\binom{n-b-j}{r-b}}{\binom{n}{r}}$$

$f_k(r) =$

$$\sum_j |\mathcal{F}_j(\mathcal{P})| \frac{\binom{j}{k}\binom{n-b-j}{r-b-k}}{\binom{n}{r}}$$

First order moment : $m_1(r) = f_1(r)\frac{n-r+1}{r-b} = O\left(\frac{n-r}{r} f_1(r)\right)$

Theorem

$$f_1(r) = O(f_0(r))$$

$\mathcal{R}$ a subset of $\mathcal{P}$ of size r
Consider a random sample $\mathcal{R}'$ of $\mathcal{R}$ of size $r-1$

$$\begin{aligned} Exp(|\mathcal{F}_0(\mathcal{R}')|) &= \frac{1}{r}|\mathcal{F}_1(\mathcal{R})| + \frac{r-b}{r}|\mathcal{F}_0(\mathcal{R})| \\ f_0(r-1) &= \frac{1}{r}f_1(r) + \frac{r-b}{r}f_0(r) \\ \frac{b}{r}f_0(r) &\geq \frac{1}{r}f_1(r) \end{aligned}$$

(assuming that $f_0(r)$ is a growing function of r). □

RIC and first order moment

Expected total number of conflicts with constructed regions

$$\sum_r \sum_j |\mathcal{F}_j(\mathcal{P})| \begin{pmatrix} j \\ 1 \end{pmatrix} \frac{b}{r} p_j(r) = \sum_r b \frac{m_1(r)}{r}$$

First order moment theorem

$m_1(r) = f_1(r) \frac{n-r+1}{r-b} = O\left(\frac{n-r}{r} f_1(r)\right)$

Backward Analysis

$f_1(r) = O(f_0(r))$

Conclusion

The expected total number of conflicts with constructed regions is:

$$O\left(\sum_{r=1}^{n} \frac{n-r}{r^2} f_0(r)\right)$$

On Line Algorithms

The influence graph

- A directed acyclic connected graph,
- with one node for each constructed region.
- The region of a node is included in the union of the regions of its parents

Localisation

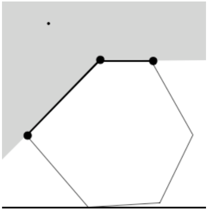
To find the conflicts when inserting a new object p
visit all the nodes in the influence graph in conflict with p

Randomised complexity: $O\left(\sum_{r=1}^{n} \frac{n-r}{r^2} f_0(r)\right)$
provided that the outdegree of each node is bounded.

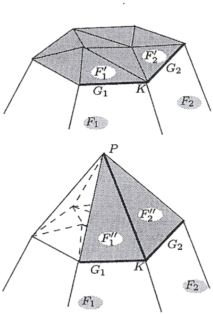
On Line Convex Hull

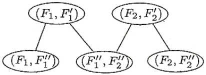
Regions are now defined by $d+1$ points: $\{p_0, p_1, \ldots p_d\}$
h_0^+ halfspace bounded hy hyperplan through $\{p_1, \ldots p_d\}$
and not including p_0
h_d^+ halfspace bounded hy hyperplan through $\{p_0, p_1, \ldots p_{d-1}\}$
and not including p_d
region: $h_0^+ \cup h_d^+$

Regions defined by $\mathcal{P}$
and without conflict in $\mathcal{P}$
are in bijection with the
$(d-2)$-faces of conv($\mathcal{P}$)



Updating the influence graph



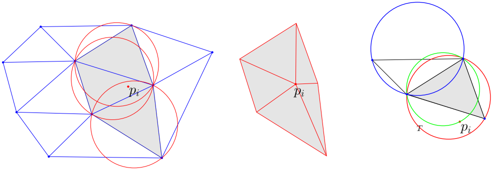


- Each node is attached to one or two parents
- Each node received $1 + (d - 1) = d$ children

Incremental Delaunay Triangulation

$\mathcal{P}$ set of points in $\mathbb{R}^d$
$\mathrm{Del}(\mathcal{P})$ can be obtained from a convex hull in $\mathbb{R}^{d+1}$
Inserting a new point p_i:

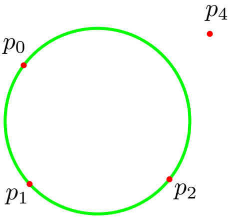
1. Location : Find all current cells
   whose circumball includes p_i
2. Update the Delaunay triangulation :
   star the hole from p_i



Incremental Delaunay Triangulation

conflict graph

Ojects : point in $\mathbb{R}^d$
Regions : d-simplex in $\mathbb{R}^d$
p conflicts with τ iff
$p \in$ circumball(τ)



Conflict test

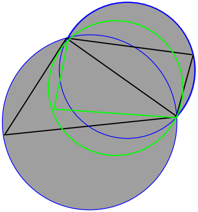
en 2D `incircle`(p_i, p_j, p_k, p_l)

$$\texttt{sign}\left(\begin{vmatrix} 1 & 1 & 1 & 1 \\ x_i & x_j & x_k & x_l \\ y_i & y_j & y_k & y_l \\ x_i^2 + y_i^2 & x_j^2 + y_j^2 & x_k^2 + y_k^2 & x_l^2 + y_l^2 \end{vmatrix} * \begin{vmatrix} 1 & 1 & 1 \\ x_i & x_j & x_k \\ y_i & y_j & y_k \end{vmatrix} \right)$$

Incremental Delaunay Triangulation

Influence graph: the Delaunay tree

Ojects : points in $\mathbb{R}^d$
Regions : union of two balls circumscribed
to adjacents d-simplex,
defined by $d+2$ points

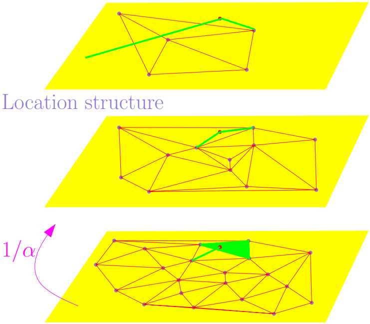


Updating the Delaunay tree

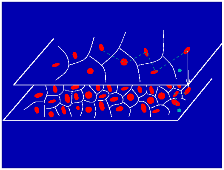
Each constructed region is attached to one or two parents in the Delaunay tree

The Delaunay hierarchy
A location data structure widely used.

Level 0 is $\mathrm{Del}(\mathcal{P})$
Each data point p in level l is introduced in level $l+1$ with probability $\beta = \frac{1}{\alpha}$



Location of point q:
find the nearest neighbor of q in $\mathcal{P}$
$n_l(q)$: nearest neighbor of q in $\mathcal{P}_l$



Locate q in highest level
From $n_{l+1}(q)$ to $n_l(q)$:
- use the pointer of $n_{l+1}(q)$ to level l
- walk in level l from $n_{l+1}(q)$ to $n_l(q)$

The number of steps performed at $level(l)$: m_l
at most k if $n_{l+1}(p)$ is the kth neighbor of q in $\mathcal{P}_l$

$$
\begin{aligned}
\text{Exp}(m_l) &\leq \sum_{k=1}^{n_l} k(1-\beta)^{k-1}\beta \\
&\leq \beta\left[-\frac{\partial}{\partial\beta}\sum_k (1-\beta)^k\right] = \frac{1}{\beta}
\end{aligned}
$$

Expected total number of steps: $O(\log n)$.

Theorem (The sampling theorem)

$\mathcal{P}$ a set of n objects
$\mathcal{F}_{\leq k}(\mathcal{P})$ regions defined by $\mathcal{P}$ with at most k conflicts on $\mathcal{P}$
b the number of objects to define a region
$f_0(r)$ the expected number of regions defined and without conflict on a random r-sample.
For $2 \leq k \leq \frac{n}{b+1}$,

$$|\mathcal{F}_{\leq k}(\mathcal{P})| \leq 4(b+1)^b k^b f_0(\left\lfloor \frac{n}{k} \right\rfloor).$$

$$f_0(r) = \sum_j |\mathcal{F}_j(\mathcal{P})| \frac{\binom{n-b-j}{r-b}}{\binom{n}{r}} \geq |\mathcal{F}_{\leq k}(\mathcal{P})| \frac{\binom{n-b-k}{r-b}}{\binom{n}{r}}$$

then, we prove that $\dfrac{\binom{n-b-k}{r-b}}{\binom{n}{r}} \geq \dfrac{1}{4(b-1)^b k^b}$

$$\frac{\binom{n-b-k}{r-b}}{\binom{n}{r}} = \underbrace{\frac{r!}{(r-b)!}\frac{(n-b)!}{n!}}_{\geq \frac{1}{4}} \underbrace{\frac{(n-r)!}{(n-r-k)!}\frac{(n-b-k)!}{(n-b)!}}_{\geq \frac{1}{(b-1)^b k^b}}$$

$$\frac{(n-r)!}{(n-r-k)!}\frac{(n-b-k)!}{(n-b)!} \geq \left(\frac{n-r-k+1}{n-b-k+1}\right)^k$$
$$\geq \left(\frac{n-n/k-k+1}{n-k}\right)^k$$
$$\geq (1-1/k)^k \geq 1/4 \ \text{ pour } (2 \leq k),$$

$$\frac{r!}{(r-b)!}\frac{(n-b)!}{n!} = \prod_{l=0}^{b-1}\frac{r-l}{n-l} \geq \prod_{l=1}^{b}\frac{r+1-b}{n}$$
$$\geq \prod_{l=1}^{b}\frac{n/k-b}{n}$$
$$\geq 1/k^b(1-\frac{bk}{n})^b \geq \frac{1}{k^b(b+1)^b} \ \text{ pour } (k \leq \frac{n}{b+1}).$$

Bound on the number of k-sets

using randomization

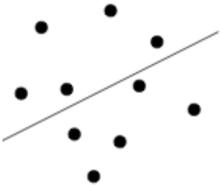
k-sets

$\mathcal{P}$ a set of n points in $\mathbb{R}^d$.
A k-set of $\mathcal{P}$ is a subset $\mathcal{P}'$ of $\mathcal{P}$ with size k
that can be separated from $\mathcal{P} \setminus \mathcal{P}'$ by a hyperplan.

Bound on the number of k-sets

For a set of n points in $\mathbb{R}^d$,
the number of l-sets with $l \leq k$ is:

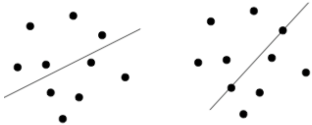
$$O\left(k^{\lceil \frac{d}{2} \rceil} n^{\lfloor \frac{d}{2} \rfloor}\right)$$



Bound on the number of k-sets

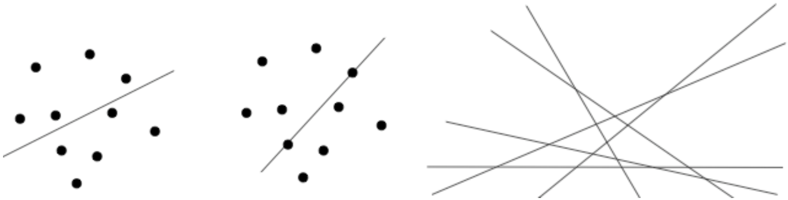
using randomization

$c_k(\mathcal{P})$ number of k-sets of $\mathcal{P}$
$c'_k(\mathcal{P})$ number of k-sets separated by a hyperplan through points of $\mathcal{P}$
$c'_{\leq k}(\mathcal{P}) = \sum_{l \leq k} c'_l(\mathcal{P})$
$c'_{\leq k}(n) = \sup_{|\mathcal{P}|=n} c'_{\leq k}(\mathcal{P})$



Objects : points of $\mathbb{R}^d$
Regions : halfspaces in $\mathbb{R}^d$, $b = d$
Conflict between p and h^+ : $p \in h^+$

$$\left.\begin{array}{l} \text{Sampling th: } c'_{\leq k}(\mathcal{P}) \leq 4(b+1)^b k^b f_0\left(\left\lfloor \frac{n}{k} \right\rfloor\right) \\ \text{Upper bound th: } f_0(\left\lfloor \frac{n}{k} \right\rfloor) = O\left(\frac{n^{\left\lfloor \frac{d}{2} \right\rfloor}}{k^{\left\lfloor \frac{d}{2} \right\rfloor}}\right) \end{array}\right\} \Rightarrow c'_{\leq k}(n) = O\left(k^{\left\lceil \frac{d}{2} \right\rceil} n^{\left\lfloor \frac{d}{2} \right\rfloor}\right)$$



Duality:

$$p \in \mathbb{R}^d \quad \longrightarrow \quad p^* : x_d + p_d - 2\sum_{i=1}^{d-1} p_i \cdot x_i = 0$$

$$\mathcal{P} \in \mathbb{R}^d \quad \longrightarrow \quad \mathcal{P}^* = \{p^* : p \in \mathcal{P}\}$$

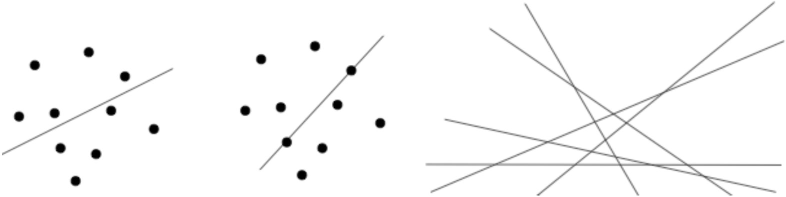
Arrangement : $\mathcal{A}(\mathcal{P}^*)$

$c_k(\mathcal{P}) \quad \leftrightarrow \quad$ cells with level k or $n-k$

$c'_k(\mathcal{P}) \quad \leftrightarrow \quad$ vertices of $\mathcal{A}(\mathcal{P}^*)$ with level k or $n-k$

Bound on the number of k-sets

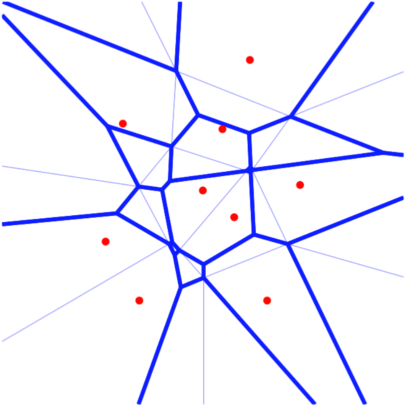
End



Each cell in $\mathcal{A}(\mathcal{P}^*)$ has at most one leftmost and one rightmost vertex.
Each vertex v in $\mathcal{A}(\mathcal{P}^*)$ is the rightmost or leftmost vertex of a single cell f.
If the level of f is k, the level of $v \in [k-d+1, k-1]$
hence, $c_k(n) \leq 2\sum_{l=k-d+1}^{k-1} c'_l(n)$

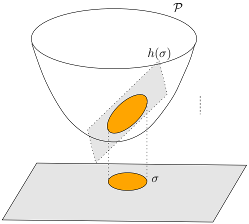
$$\left.\begin{array}{l} c_k(n) \leq 2\sum_{l=k-d+1}^{k-1} c'_l(n) \\ c'_{\leq k}(n) = O\left(k^{\lceil \frac{d}{2} \rceil} n^{\lfloor \frac{d}{2} \rfloor}\right) \end{array}\right\} \Rightarrow c_{\leq k}(n) = O\left(k^{\lceil \frac{d}{2} \rceil} n^{\lfloor \frac{d}{2} \rfloor}\right)$$

k-order Voronoi Diagrams



Each cell is the locus of points with the same set of k-nearest neighbors.

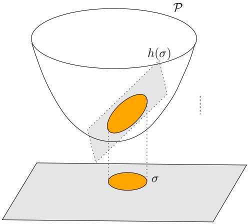
Back to Space of Spheres



p point in $\mathbb{R}^d$	$\rightarrow$	$\phi(p) = (p, p^2) \in \mathbb{R}^{d+1}$
σ sphere de $\mathbb{R}^d$	$\rightarrow$	$\phi(\sigma) \in \mathbb{R}^{d+1}$
$x^2 - 2c \cdot x + s = 0$		$\phi(\sigma) = (c, s)$,
$s = c^2 - r^2$		hyperplane $\phi(\sigma)^*$
		$x_{d+1} - 2c \cdot x + s = 0$

1. Si $\sigma = \{p\}$, $\phi(\sigma) \in \mathcal{P}$, $\phi(\sigma)^*$ hyperplan tangent à $\mathcal{P}$
2. L'intersection $\phi(\sigma)^* \cap \mathcal{P}$ se projette sur $x_{d+1} = 0$ selon σ

$$\begin{aligned}
\sigma(x) = 0 &\iff x^2 - 2c \cdot x + s = 0 \iff \phi(x) \in \phi(\sigma)^* \\
\sigma(x) < 0 &\iff x^2 - 2c \cdot x + s < 0 \iff \phi(x) \in \phi(\sigma)^{*-} \\
\sigma(x) > 0 &\iff x^2 - 2c \cdot x + s > 0 \iff \phi(x) \in \phi(\sigma)^{*+}
\end{aligned}$$



Complexity of k-order Voronoi diagram

Let $\mathcal{P}$ be a set of points in $\mathbb{R}^d$. The total number of faces in Voronoi diagrams of $\mathcal{P}$ of order up to k is

$$O\left(k^{\lceil \frac{d+1}{2} \rceil} n^{\lfloor \frac{d+1}{2} \rfloor}\right)$$

Proof. Each cell in the k-order Voronoi diagram of $\mathcal{P}$ corresponds to :
- a k-set of $\phi(\mathcal{P})$ in $\mathbb{R}^{d+1}$.
- a cell of level k in the arrangement $\mathcal{A}(\phi^*(\mathcal{P}))$ in $\mathbb{R}^{d+1}$.