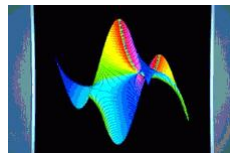


Chebyshev Expansions for Solutions of Linear Differential Equations

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Joint work with Bruno Salvy

INRIA

October 28, 2009



I Introduction

How to evaluate a function f in $[-1, 1]$?

Two representations of f :

- in Taylor series

$$f = \sum_{n=0}^{+\infty} c_n x^n, \quad c_n = \frac{f^{(n)}(0)}{n!},$$

- or in Chebyshev series

$$f = \sum_{n=0}^{+\infty} t_n T_n(x),$$

$$t_n = \frac{2}{\pi} \int_{-1}^1 T_n(t) \frac{f(t)}{\sqrt{1-t^2}} dt.$$

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Basic properties of Chebyshev polynomials

$$T_n(\cos(\theta)) = \cos(n\theta)$$

$$\int_{-1}^1 \frac{T_n(x) T_m(x)}{\sqrt{1-x^2}} dx = \begin{cases} 0 & \text{if } m \neq n \\ \pi & \text{if } m = 0 \\ \frac{\pi}{2} & \text{otherwise} \end{cases}$$

$$T_{n+1} = 2xT_n - T_{n-1}$$

$$T_0(x) = 1$$

$$T_1(x) = x$$

$$T_2(x) = 2x^2 - 1$$

$$T_3(x) = 4x^3 - 3x$$

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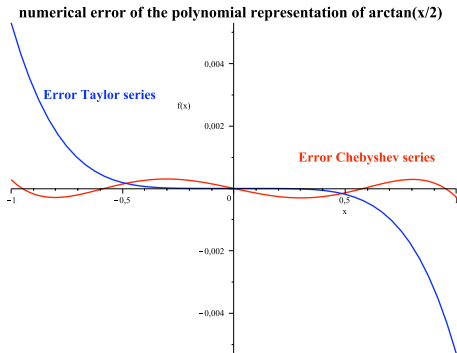
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Projects using Chebyshev series to represent functions in Matlab :
Chebfun, Miscfun.



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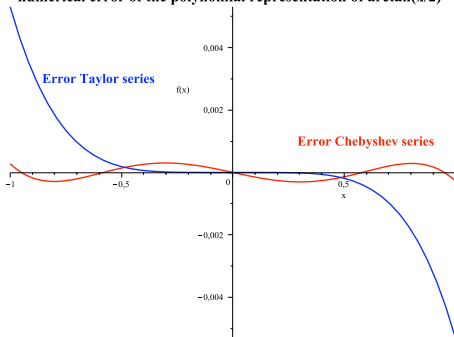
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numerical error of the polynomial representation of $\arctan(x/2)$



How to compute t_n ?

General case: numerical computation of the integral. [Slow](#).

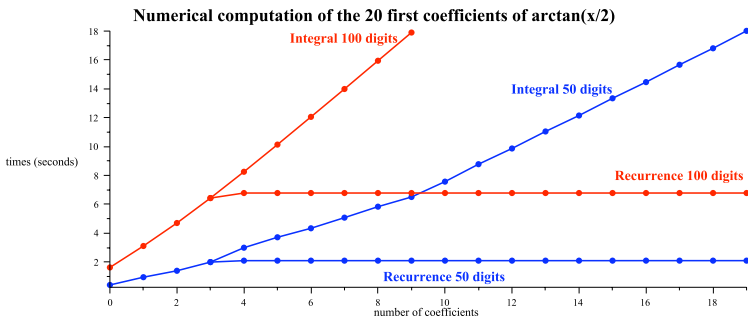
Computation of Coefficients with Recurrences

Theorem (60's)

If f is solution of a **linear differential** equation with polynomial coefficients, then the Chebyshev coefficients are cancelled by a **linear recurrence** with polynomial coefficients.

Applications:

- Numerical computation of the coefficients.



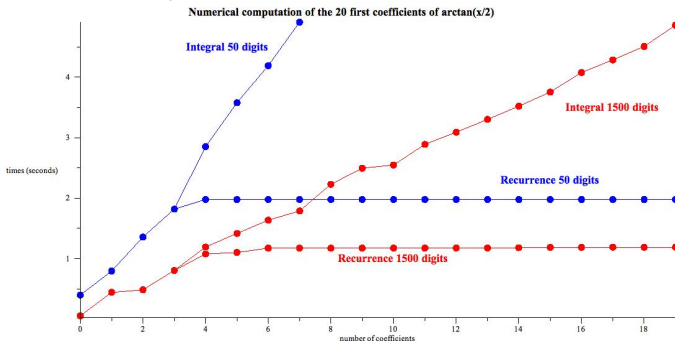
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Applications:

- Numerical computation of the coefficients.
- Computation of closed-form for coefficients.

Example ($f(x) = \arctan(x/2)$)

```
deg := gfun[holexpertodiffeq](arctan(x/2), y(x));
```

$$deg := \left\{ \left(\frac{d}{dx} y(x) \right) (x^2 + 4) - 2, y(0) = 0 \right\}$$

```
rec := diffeqtorecchebyshev(deg, y(x), u(n));
```

$$rec := \left\{ -n u(n) + (-18n - 36) u(n+2) + (-4 - n) u(n+4), u(0) = 0, u(1) = \frac{2}{2 + \sqrt{5}}, u(2) = 0, u(3) = -\frac{2}{3(2 + \sqrt{5})^3} \right\}$$

```
simplify(evalc(allvalues(rsolve(rec, u(n)))) assuming n::integer;
```

$$\begin{cases} 0 & n=0 \\ \frac{2(-2 + \sqrt{5})^{n+1} (2 + \sqrt{5}) \sin\left(\frac{1}{2} n \pi\right)}{n} & \text{otherwise} \end{cases}$$

State of the Art

- Clenshaw (1957): **numerical scheme** to compute the Chebyshev coefficients without computing all these integrals.
- Fox and Parker (1968): method for the computation of the **Chebyshev recurrence relations** for differential equations of small orders.
- Paszkowski (1975): **algorithm** for computing the Chebyshev recurrence relation.
- Lewanowicz (1976): algorithm for computing a **smaller order** Chebyshev recurrence relation in some cases.
- Rebillard (1998): **new algorithm** for computing the Chebyshev recurrence relation.
- Rebillard and Zakrajšek (2006): algorithm for computing a **smaller order** Chebyshev recurrence relation compared with Lewanowicz algorithm.

New Results (2009)

- Simple unified presentation of these algorithms using **fractions of recurrence operators**.
- **Complexity analysis** of the existing algorithms (order k , degree k)
 - Paszkowski's and Lewanowicz's algorithms: $O(k^4)$ arithmetic operations in \mathbb{Q} .
 - Rebillard's algorithm: $O(k^5)$ arithmetic operations in \mathbb{Q} .
- **New fast algorithm**: $O(k^\omega)$ arithmetic operations. Here, ω is a feasible exponent for matrix multiplication with coefficients in \mathbb{Q} ($\omega \leq 3$).
- **Implementation** of algorithm in Maple.

II Fractions of Recurrence Operators

Morphisms of Rings of Operators ($S \cdot u_n = u_{n+1}$)Taylor series ($f := \sum c_n x^n$)

$$xf = \sum c_n x^{n+1} = \sum c_{n-1} x^n,$$

$$f' = \sum n c_n x^{n-1} = \sum (n+1) c_{n+1} x^n$$

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$$(4 + x^2) \left(\frac{d}{dx} \right)^2 + 2x \frac{d}{dx}$$

$$\mapsto (4 + S^{-2})(n+1)(n+2)S^2 + 2S^{-1}(n+1)S$$

$$= (n+1)(4(n+2)S^2 + n)$$

$$4(n+2)c_{n+2} + nc_n = 0$$

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Application to Chebyshev recurrences relations

Definition

Let φ be “the Chebyshev morphism”:

$$\varphi(x) = \frac{1}{2} (S + S^{-1}) \text{ et } \varphi \left(\frac{d}{dx} \right) = \frac{2n}{-S + S^{-1}}.$$

Theorem (BenoitSalvy2009)

$f \in \mathcal{C}^k$, L is a differential operator of order k such that $L \cdot f = 0$.

Suppose that either of the following holds:

- (i). $\int_{-1}^1 \frac{f^{(k)}(x)}{\sqrt{1-x^2}} dx$ is convergent;
- (ii). $\int_{-1}^1 \frac{(1-x^2)^k f^{(k)}(x)}{\sqrt{1-x^2}} dx$ is convergent and $(1-x^2)^i |p_i$, $i = 0, \dots, k$.

Then, the Chebyshev coefficients of f are cancelled by a **numerator of $\varphi(L)$** .

Ore Polynomials: Framework for Recurrence Operators

- $\sum a_i(n)u_{n+i}$ is represented by $\sum a_i(n)S^i$.
- These polynomials are non-commutative.
- Multiplication defined by: $Sn = (n+1)S$.
- Ring denoted $\mathbb{Q}(n)\langle S \rangle$.

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- Multiplication defined by: $Sn = (n + 1)S$.
- Ring denoted $\mathbb{Q}(n)\langle S \rangle$.
- **Main property:** the degree in S of a product is the sum of the degrees of its factors.
 - Algorithm for (left or right) euclidian division.

gcd algorithm (Ore 1933)

INPUT recurrence operators A and B
 OUTPUT The “greatest” G such that
 $A = G\tilde{A}$ and $B = G\tilde{B}$

lcm algorithm (Ore 1933)

INPUT recurrence operators A and B
 OUTPUT The “smallest” U and V
 such that $UA = VB$

Fractions of Recurrence Operators (Ore 1933)

- Field of fractions of $\mathbb{Q}(n)\langle S \rangle$ defined by:

$$\frac{A}{B} = \frac{C}{D} \Leftrightarrow \exists (U, V) \text{ such that } UA = VC \text{ and } UB = VD.$$

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- Addition:

$$\frac{A}{B} + \frac{C}{D} = \frac{UA}{UB} + \frac{VC}{VD} = \frac{UA + VC}{UB},$$

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$$\frac{A}{B} + \frac{C}{D} = \frac{UA}{UB} + \frac{VC}{VD} = \frac{UA + VC}{UB},$$

- Multiplication:

$$\frac{D}{C} \cdot \frac{A}{B} = \frac{VD}{VC} \cdot \frac{UA}{UB} = \frac{UA}{VC}.$$

Example: $\sqrt{1-x^2}$

$f = \sqrt{1-x^2}$ is cancelled by the differential operator : $x + (1-x^2)\frac{d}{dx}$.

$$\begin{aligned} \varphi \left(x + (1-x^2)\frac{d}{dx} \right) &= \frac{S+S^{-1}}{2} + \left(1 - \frac{S^2+2+S^{-2}}{4} \right) \frac{2n}{-S+S^{-1}} \\ &= \frac{(-S+S^{-1})(S+S^{-1})}{2(-S+S^{-1})} - \frac{(n+2)S^2+2n-(n-2)S^{-2}}{2(-S+S^{-1})} \\ &= \frac{-(n+3)S^2+2n-(n-3)S^{-2}}{2(-S+S^{-1})} \end{aligned}$$

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The Chebyshev coefficients c_n satisfy :

$$(n+3)c_{n+2} - 2nc_n + (n-3)c_{n-2} = 0.$$

Normalization

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Example: Normalized fraction for $\sqrt{1-x^2}$

we have:

$$\begin{aligned} \varphi \left(-x + (-1 + x^2) \frac{d}{dx} \right) &= \frac{-(n+3)S^2 + 2n - (n-3)S^{-2}}{2(-S + S^{-1})} \\ &= \frac{(-S + S^{-1})((n+2)S - (n-2)S^{-1})}{2(-S + S^{-1})}. \end{aligned}$$

Smaller order

$$\Rightarrow (n+2)c_{n+1} - (n-2)c_{n-1} = 0.$$

III Algorithms

Lewanowicz's algorithm (1976)

Horner+Normalize at each step.

Example with $f = \sqrt{1 - x^2}$

$$(1 - x^2) \frac{d}{dx} + x.$$

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$$\begin{aligned} \varphi(1-x^2)\varphi\left(\frac{d}{dx}\right) + \varphi(x) &= \frac{((n+1)S - (n-1)S^{-1})}{2} + \frac{S + S^{-1}}{2} \\ &= \frac{(n+2)S - (n-2)S^{-1}}{2} \end{aligned}$$

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A recurrence verified by the Chebyshev coefficients of f is:

$$(n+2)c_{n+1} - (n-2)c_{n-1} = 0$$

Algorithms of Paszkowski (1975) and Rebillard (1998)

Observation: if $D = \varphi\left(\frac{d}{dx}\right) = \frac{2n}{-s+s-1}$ then D^{-1} is a polynomial.

- INPUT :

$$L = \sum_{i=0}^k p_i(x) \left(\frac{d}{dx}\right)^i$$

- OUTPUT : A numerator of $\varphi(L)$

Computation with polynomials only.

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$$\sum_{i=0}^k p_i(x) \left(\frac{d}{dx}\right)^i = \sum_{i=0}^k \left(\frac{d}{dx}\right)^i q_i(x).$$

$$\sum_{i=0}^k p_i(X) D^i = \frac{\sum_{i=0}^k D^{-k+i} q_i(X)}{D^{-k}}.$$

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Rebillard

$$X_k := D^{-k} X D^k.$$

$$\sum_{i=0}^k p_i(X) D^i = \frac{\sum_{i=0}^k p_i(X_k) D^{-k+i}}{D^{-k}}.$$

Our algorithm: Divide and conquer

D^{-i} is of bidegree $(2i, 2i)$.

New, fast algorithm

Step 1: Compute $q_i(x)$ such that

$$\sum_{i=0}^k p_i(x) \left(\frac{d}{dx}\right)^i = \sum_{i=0}^k \left(\frac{d}{dx}\right)^i q_i(x).$$

Step 2 : Divide and conquer

$$\sum_{i=0}^k D^{-k+i} q_i(X) =$$

$$D^{-\frac{k}{2}} \sum_{i=0}^{\frac{k}{2}} D^{-\frac{k}{2}+i} q_i(X) + \sum_{i=\frac{k}{2}+1}^k D^{-k+i} q_i(X).$$

Balanced products.

Our algorithm: Divide and conquer

D^{-i} is of bidegree $(2i, 2i)$.

New, fast algorithm

Step 1: Compute $q_i(x)$ such that

$$\sum_{i=0}^k p_i(x) \left(\frac{d}{dx}\right)^i = \sum_{i=0}^k \left(\frac{d}{dx}\right)^i q_i(x).$$

Step 2 : Divide and conquer

$$\sum_{i=0}^k D^{-k+i} q_i(X) =$$

$$D^{-\frac{k}{2}} \sum_{i=0}^{k/2} D^{-\frac{k}{2}+i} q_i(X) + \sum_{i=k/2+1}^k D^{-k+i} q_i(X).$$

Balanced products.

Theorem

If the degrees of p_i are at most k ,

- New: $O(k^\omega)$ arithmetic operations.
- Paszkowski and Lewanowicz algorithms : $O(k^4)$ arithmetic operations.
- Rebillard : $O(k^5)$ arithmetic operations.

IV Conclusion and Future Works

Conclusion and Future works

Contributions:

- Use of fractions of recurrence operators.
- New algorithm.
- Maple code.
- Available in <http://ddmf.msr-inria.inria.fr/>

Perspectives:

- Numerical computation of the coefficients.
- Compare our algorithm with the algorithm of Rebillard and Zakrajšek.
- Recurrence in other bases (Jacobi, Hermite and Laguerre polynomials, Bessel functions)

> **diffeqtorecgegenbauer(y(x)-diff(y(x),x),y(x),alpha,u(n));**
 $(-n-2-\alpha)u(n) + (2n^2+4n+4\alpha n+4\alpha+2\alpha^2)u(n+1) + (n+\alpha)u(n+2)$