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Abstract: We present the first efficient algorithm for computing an exact parametric representation of the intersection of two quadrics in three-dimensional real space given by implicit equations with rational coefficients. The output functions parameterizing the intersection are rational functions whenever it is possible, which is the case when the intersection is not a smooth quartic (for example, a singular quartic, a cubic and a line, or two conics). Furthermore, the parameterization is near-optimal in the sense that the number of square roots appearing in the coefficients of these functions is minimal except in a small number of cases where there may be an extra square root. In addition, the algorithm is practical: a complete, robust and efficient C++ implementation is described in Part IV [12] of this paper.

In Part I, we present an algorithm for computing a parameterization of the intersection of two arbitrary quadrics which we prove to be near-optimal in the generic, smooth quartic, case. Parts II and III [4, 5] treat the singular cases. We present in Part II the first classification of pencils of quadrics according to the real type of the intersection and we show how this classification can be used to efficiently determine the type of the real part of the intersection of two arbitrary quadrics. This classification is at the core of the design of our algorithms for computing near-optimal parameterizations of the real part of the intersection in all singular cases. We present these algorithms in Part III and give examples covering all the possible situations in terms of both the real type of intersection and the number and depth of square roots appearing in the coefficients.

Key-words: Intersection of surfaces, quadrics, pencils of quadrics, curve parameterization.

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Paramétrisation quasi-optimale de l'intersection de quadriques : I. L'algorithme générique

Résumé : Nous présentons le premier algorithme efficace pour le calcul d'une représentation paramétrique exacte de l'intersection de deux quadriques de l'espace tridimensionnel réel données par leurs équations implicites à coefficients rationnels. Le paramétrage calculé est polynomial chaque fois que cela est possible, c'est-à-dire quand l'intersection n'est pas une quartique lisse (par exemple, une quartique singulière, une cubique et une droite, ou deux coniques). De plus, le paramétrage est *quasi-optimal* dans le sens où le nombre de racines carrées apparaissant dans les coefficients des polynômes du paramétrage est minimal sauf dans un petit nombre de cas où ils contiennent une racine carrée possiblement inutile.

Dans la partie I, nous présentons un algorithme pour le calcul du paramétrage de l'intersection de deux quadriques arbitraires dont nous prouvons qu'il est quasi-optimal dans le cas générique d'une intersection constituée d'une quartique lisse.

Nous présentons dans la partie II [4] une nouvelle classification des faisceaux de quadriques selon le type réel de l'intersection et montrons comment utiliser cette classification pour déterminer efficacement le type de la partie réelle de l'intersection de deux quadriques arbitraires.

Dans la partie III [5], nous montrons comment utiliser l'information obtenue sur le type réel de l'intersection pour diriger le paramétrage de l'intersection. Dans chaque cas possible, nous donnons la trame d'un algorithme quasi-optimal pour paramétrer la partie réelle de l'intersection et donnons des exemples couvrant toutes les situations possibles en terme de nombre et de profondeur de radicaux impliqués.

Enfin, nous présentons dans la partie IV [12] une implantation complète, robuste et efficace de notre algorithme en C++.

Mots-clés : Intersection de surfaces, quadriques, faisceaux de quadriques, paramétrisation.

1 Introduction

The simplest of all the curved surfaces, quadrics (i.e., algebraic surfaces of degree two), are fundamental geometric objects, arising in such diverse contexts as geometric modeling, statistical classification, pattern recognition, and computational geometry. In geometric modeling, for instance, they play an important role in the design of mechanical parts; patches of natural quadrics (planes, cones, spheres and cylinders) and tori make up to 95 % of all mechanical pieces according to Requicha and Voelcker [22].

Computing the intersection of two general quadrics is a fundamental problem and an explicit parametric representation of the intersection is desirable for most applications. Indeed, computing intersections is at the basis of many more complex geometric operations such as computing convex hulls of quadric patches [9], arrangements of sets of quadrics [1, 18, 25, 36], and boundary representations of quadric-based solid models [10, 24].

Until recently, the only known general method for computing a parametric representation of the intersection between two arbitrary quadrics was due to J. Levin [13, 14]. It is based on an analysis of the pencil generated by the two quadrics, i.e. the set of linear combinations of the two quadrics.

Though useful, Levin's method has serious limitations. When the intersection is singular or reducible, a parameterization by rational functions is known to exist, but Levin's pencil method fails to find it and generates a parameterization that involves the square root of some polynomial. In addition, when a floating point representation of numbers is used, Levin's method sometimes outputs results that are topologically wrong and it may even fail to produce any parameterization at all and crash. On the other hand a correct implementation using exact arithmetic is essentially out of reach because Levin's method introduces algebraic numbers of fairly high degree. A good indication of this impracticality is that even for the simple generic example of Section 8.2, an exact parametric form output by Levin's algorithm (computed by hand with Maple [15]) fills up over 100 megabytes of space!

Over the years, Levin's seminal work has been extended and refined in several different directions. Wilf and Manor [35] use a classification of quadric intersections by the Segre characteristic (see [2]) to drive the parameterization of the intersection by the pencil method. Recently, Wang, Goldman and Tu [32] further improved the method by making it capable of computing structural information on the intersection and its various algebraic components and able to produce a parameterization by rational functions when it exists. Whether their refined algorithm is numerically robust is open to question.

Another method of algebraic flavor was introduced by Farouki, Neff and O'Connor [6] when the intersection is degenerate. In such cases, using a combination of classical concepts (Segre characteristic) and algebraic tools (factorization of multivariate polynomials), the authors show that explicit information on the morphological type of the intersection curve can be reliably obtained. A notable feature of this method is that it can output an exact parameterization of the intersection in simple cases, when the input quadrics have rational coefficients. No implementation is however reported.

Rather than restricting the type of the intersection, others have sought to restrict the type of the input quadrics, taking advantage of the fact that geometric insights can then help compute the intersection curve [8, 16, 17, 26, 27, 28]. Specialized routines are devised to compute the intersection curve in each particular case. Even though such geometric approaches are numerically more stable than the algebraic ones, they are essentially limited to the class of so-called natural quadrics (i.e., the planes, right cones, circular cylinders and spheres) and planar intersections.

Perhaps the most interesting of the known algorithms for computing an explicit representation of the intersection of two arbitrary quadrics is the method of Wang, Joe and Goldman [34]. This algebraic method is based on a birational mapping between the intersection curve and a plane cubic curve. The cubic curve is obtained by projection from a point lying on the intersection. Then the classification and parameterization of the intersection are obtained by invoking classical results on plane cubics. The authors claim that their algorithm is the first to produce a complete topological classification of the intersection (singularities, number and types of algebraic components, etc.). Numerical robustness issues have however not been studied and the intersection may not be correctly classified. Also, the center of projection is currently computed using Levin's (enhanced) method: with floating point arithmetic, it will in general not exactly lie on the curve, which is another source of numerical instability.

1.1 Contributions

In this paper, we present the first exact and efficient algorithm for computing a parametric representation of the intersection of two quadric surfaces in three-dimensional real space given by implicit equations with rational coefficients. (A preliminary version of this paper was presented in [3].)

Our algorithm, as well as its implementation, has the following main features:

- it computes an exact parameterization of the intersection of two quadrics with rational coefficients of arbitrary size;
- it places no restriction of any kind on the type of the intersection or the type of the input quadrics;
- it correctly identifies, separates and parameterizes all the algebraic components of the intersection and gives all the information on the incidence between the components, that is where and how (e.g., tangentially or not) two components intersect;
- the parameterization is rational when one exists; otherwise the intersection is a smooth quartic and the parameterization involves the square root of a polynomial;
- the parameterizations are either optimal in the degree of the extension of $\mathbb{Q}$ on which their coefficients are defined or, in a small number of well-identified cases, involve one extra possibly unnecessary square root.

Moreover, our implementation of this algorithm, which uses arbitrary-precision integer arithmetic, can routinely compute parameterizations of the intersection of quadrics with input integer coefficients having ten digits in less than 50 milliseconds on a mainstream PC (see Part IV [12]).

The above features imply in particular that the output parameterization of the intersection is almost as "simple" as possible, meaning that the parameterization is rational if one exists, and that the coefficients appearing in the parameterization are almost as rational as possible. This "simplicity" is, in itself, a key factor for making the parameterization process both feasible and efficient (by contrast, an implementation of Levin's method using exact arithmetic is essentially out of reach). It is also crucial for the easy and efficient processing of parameterizations in further applications.

Formally, we prove the following.

Theorem 1.1. *In three-dimensional real space, given two quadrics in implicit form with rational coefficients, our algorithm first tests if their intersection is a smooth quartic or not. If it is a smooth quartic, there does not exist any rational parameterization of the intersection and our algorithm computes a parameterization such that, in projective space, each coordinate belongs to* $\mathbb{K}[\xi, \sqrt{\Delta}]$ *(the ring of polynomials in* ξ *and* $\sqrt{\Delta}$ *with coefficients in* $\mathbb{K}$*), where* ξ *is the (real) parameter,* $\Delta \in \mathbb{K}[\xi]$ *is a polynomial in* ξ*, and* $\mathbb{K}$ *is either the field of the rationals or an extension of* $\mathbb{Q}$ *by the square root of an integer. If the intersection is not a smooth quartic, our algorithm computes a rational parameterization of each component of the intersection over a field* $\mathbb{K}$ *of coefficients which is* $\mathbb{Q}$ *or an extension of* $\mathbb{Q}$ *of degree 2 or 4; this means that each projective coordinate of the component of the intersection is a polynomial in* $\mathbb{K}[\xi]$*.*

In all cases, either $\mathbb{K}$ *is a field of smallest possible degree*[1] *over which there exists such a parameterization or* $\mathbb{K}$ *is an extension of such a smallest field by the square root of an integer. In the latter situation, testing if this extra square root is unnecessary and, if so, finding an optimal parameterization are equivalent to finding a rational point on a curve or a surface (which is computationally hard and can even be undecidable).*

Due to the number of contributions and results of this work, this paper has been broken down into four parts. In Part I, we present a first and major improvement to Levin's pencil method and the accompanying theoretical tools. This simple algorithm, referred to from now on as the "generic algorithm", outputs a near-optimal parameterization when the intersection is a smooth quartic, i.e. the generic case. However, the generic algorithm ceases to be optimal (both from the point of view of the functions used in the parameterizations and the size of their coefficient field) in several singular situations. Parts II and III [4, 5] refine the generic algorithm by considering in turn all the possible types of intersection. Part II focuses on the classification of pencils of quadrics over the reals and on the detection of each case. Part III then gives an optimal or near-optimal algorithm in each possible situation. In Part IV [12], we present a complete, robust, and efficient C++ implementation of our algorithm.

1.2 Overview of Part I

Part I is organized as follows. In Section 2, we present basic definitions, notations and useful known results. Section 3 summarizes the ideas on which the pencil method of Levin for intersecting quadrics is based and discusses its shortcomings. In Section 4 we present our generic algorithm. Among the results of independent interest presented in this section are the almost always existence

[1] Recall that, if $\mathbb{K}$ is a field extension of $\mathbb{Q}$, the *degree* of the extension is defined as the dimension of $\mathbb{K}$ as a vector space over $\mathbb{Q}$. For instance, if $\mathbb{Q}(\rho)$ is a field extension of $\mathbb{Q}$ (distinct from $\mathbb{Q}$), then its degree is 2 since there is a one-to-one correspondence between any element $x \in \mathbb{Q}(\rho)$ and $(\alpha_1, \alpha_2) \in \mathbb{Q}^2$ such that $x = \alpha_1 + \alpha_2\,\rho$. Similarly, if $\mathbb{Q}$ and two field extensions $\mathbb{Q}(\rho)$ and $\mathbb{Q}(\rho')$ are pairwise distinct, then the degree of $\mathbb{Q}(\rho, \rho')$ is 4 since there is a one-to-one correspondence between any element $x \in \mathbb{Q}(\rho)$ and $(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \in \mathbb{Q}^2$ such that $x = \alpha_1 + \alpha_2\,\rho + \alpha_3\,\rho' + \alpha_4\,\rho\,\rho'$.

of a ruled quadric with rational coefficients in a pencil (proved in Section 5) and new parameterizations of ruled projective quadrics involving an optimal number of radicals in the worst case (a fact proved in Section 6). In Section 7, we prove the near-optimality of the output parameterization in the generic case, that is when the intersection curve is a smooth quartic, and show that the parameterization is optimal in the worst case, meaning that there are examples in which the possibly extra square root is indeed needed. Then, in Section 8, we give several examples and show the result of our implementation on these examples, before concluding.

2 Notations and preliminaries

In what follows, all the matrices considered are real square matrices. Given a real symmetric matrix S of size $n+1$, the upper left submatrix of size n, denoted S_u, is called the *principal submatrix* of S and the determinant of S_u the *principal subdeterminant* of S.

We call a *quadric* associated to S the set

$$Q_S = \{\mathbf{x} \in \mathbb{P}^n \mid \mathbf{x}^T S \mathbf{x} = 0\},$$

where $\mathbb{P}^n = \mathbb{P}(\mathbb{R})^n$ denotes the real projective space of dimension n. (Note that every matrix of the form αS, where $\alpha \in \mathbb{R} \setminus \{0\}$, represents the same quadric Q_S.) When the ambient space is $\mathbb{R}^n$ instead of $\mathbb{P}(\mathbb{R})^n$, the quadric is simply Q_S minus its points at infinity.

In the rest of this paper, geometric objects and parameterizations are assumed to live in projective space. For instance, a point of $\mathbb{P}^3$ has four coordinates. An object (point, line, plane, cone, quadric, etc.) given by its implicit equation(s) is said to be *rational* over a field $\mathbb{K}$ if the coefficients of its equation(s) live in the field $\mathbb{K}$. Note that, when talking about parameterizations, some confusion can arise between two different notions: the rationality of the coefficients and the rationality of the defining functions (a quotient of two polynomial functions is often called a rational function). The meaning should be clear depending on the context.

Matrix S being symmetric, all of its eigenvalues are real. Let σ^+ and σ^- be the numbers of positive and negative eigenvalues of S, respectively. The *rank* of S is the sum of σ^+ and σ^-. We define the *inertia* of S and Q_S as the pair

$$(\max(\sigma^+, \sigma^-), \min(\sigma^+, \sigma^-)).$$

(Note that it is more usual to define the inertia as the pair (σ^+, σ^-), but our definition, in a sense, reflects the fact that Q_S and Q_{-S} are one and the same quadric.) A matrix of inertia $(n, 0)$ is called *definite*. It is *positive definite* if $\sigma^- = 0$, *negative definite* otherwise. Matrix S and quadric Q_S are called *singular* if the determinant of S is zero; otherwise they are called *nonsingular*.

The inertia of a quadric in $\mathbb{P}^3$ is a fundamental concept which somehow replaces the usual type of a quadric in $\mathbb{R}^3$. For the convenience of the reader we recall in Table 1 the correspondence between inertias in $\mathbb{P}^3$ and types in $\mathbb{R}^3$.

In $\mathbb{P}^3$, any quadric not of inertia $(3,1)$ is either a ruled surface or not a surface. Also, the quadrics of inertia $(3,1)$ are the only ones with a strictly negative determinant. The nonsingular quadrics are those of rank 4, i.e. those of inertia $(4,0), (3,1)$ and $(2,2)$. Quadrics of inertia $(4,0)$ are however

Inertia of Q_S	Inertia of S_u	Euclidean canonical equation	Euclidean type of Q_S
$(4,0)$	$(3,0)$	$x^2+y^2+z^2+1$	$\emptyset$ (imaginary ellipsoid)
$(3,1)$	$(3,0)$	$x^2+y^2+z^2-1$	ellipsoid
	$(2,1)$	$x^2+y^2-z^2+1$	hyperboloid of two sheets
	$(2,0)$	x^2+y^2+z	elliptic paraboloid
$(3,0)$	$(3,0)$	$x^2+y^2+z^2$	point
	$(2,0)$	x^2+y^2+1	$\emptyset$ (imaginary elliptic cylinder)
$(2,2)$	$(2,1)$	$x^2+y^2-z^2-1$	hyperboloid of one sheet
	$(1,1)$	x^2-y^2+z	hyperbolic paraboloid
$(2,1)$	$(2,1)$	$x^2+y^2-z^2$	cone
	$(2,0)$	x^2+y^2-1	elliptic cylinder
	$(1,1)$	x^2-y^2+1	hyperbolic cylinder
	$(1,0)$	x^2+y	parabolic cylinder
$(2,0)$	$(2,0)$	x^2+y^2	line
	$(1,0)$	x^2+1	$\emptyset$ (imaginary parallel planes)
$(1,1)$	$(1,1)$	x^2-y^2	intersecting planes
	$(1,0)$	x^2-1	parallel planes
	$(0,0)$	x	simple plane
$(1,0)$	$(1,0)$	x^2	double plane
	$(0,0)$	1	$\emptyset$ (double plane at infinity)

Table 1: Correspondence between quadric inertias and Euclidean types.

empty of real points. A quadric of rank 3 is called a *cone*. The cone is said to be *real* if its inertia is $(2,1)$. It is said to be *imaginary* otherwise, in which case its real projective locus is limited to its singular point. A quadric of rank 2 is a *pair of planes*. The pair of planes is real if its inertia is $(1,1)$. It is called imaginary if its inertia is $(2,0)$, in which case its real projective locus consists of its singular line, i.e. the line of intersection of the two planes. A quadric of inertia $(1,0)$ is called a *double plane* and is necessarily real.

Two real symmetric matrices S and S' of size n are said to be *similar* if and only if there exists a nonsingular matrix P such that

$$S' = P^{-1}SP.$$

Note that two similar matrices have the same characteristic polynomial, and thus the same eigenvalues. Two matrices are said to be *congruent* or *projectively equivalent* if and only if there exists a nonsingular matrix P with real coefficients such that

$$S' = P^T SP.$$

Sylvester's Inertia Law asserts that the inertia is invariant under a congruence transformation [11], i.e. S and S' have the same inertia. Note also that the determinant of S is invariant by a congruence transformation, up to a square factor (the square of the determinant of the transformation matrix).

Let S and T be two real symmetric matrices of the same size and let $R(\lambda,\mu) = \lambda S + \mu T$. The set

$$\{R(\lambda,\mu) \mid (\lambda,\mu) \in \mathbb{P}^1\}$$

is called the *pencil* of matrices generated by S and T. For the sake of simplicity, we sometimes write a member of the pencil $R(\lambda) = \lambda S - T, \quad \lambda \in \overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$. Associated to it is a pencil of quadrics $\{Q_{R(\lambda,\mu)} \mid (\lambda,\mu) \in \mathbb{P}^1\}$. Recall that the intersection of two distinct quadrics of a pencil is independent of the choice of the two quadrics. We call the binary form

$$\mathcal{D}(\lambda,\mu) = \det R(\lambda,\mu)$$

the *determinantal equation* of the pencil.

3 Levin's pencil method

Since our solution to quadric surface intersection builds upon the pencil method of J. Levin, we start by recalling the main steps of the algorithm described in [13, 14] for computing a parameterized representation of the intersection of two distinct implicit quadrics Q_S and Q_T of $\mathbb{R}^3$. Starting from this short description, we then identify where this algorithm introduces high-degree algebraic numbers and why this is a problem.

The high-level idea behind Levin's algorithm is this: if (say) Q_S is of some "good" type, then Q_S admits a parameterization which is linear in one of its parameters and plugging this parameterization in the implicit equation of Q_T yields a degree 2 equation in one of the parameters (instead of a degree 4 equation) which can be easily solved to get a parametric representation of $Q_S \cap Q_T$. When neither Q_S nor Q_T has a "good" type, then one can find a quadric Q_R of "good" type in the pencil generated by Q_S and Q_T, and we are back to the previous case replacing Q_S by Q_R.

The definition of a "good" type is embodied in Levin's notion of simple ruled quadric[2] and the existence of such a quadric Q_R is Levin's key result:

Theorem 3.1 ([13]). *The pencil generated by any two distinct quadrics contains at least one simple ruled quadric, i.e., one of the quadrics listed in Table 2, or the empty set.*

In more details, Levin's method is as follows:

1. Find a simple ruled quadric in the pencil $\{Q_{R(\lambda)=\lambda S-T} \mid \lambda \in \overline{\mathbb{R}}\}$ generated by Q_S and Q_T, or report an empty intersection. Since simple ruled quadrics have a vanishing principal subdeterminant, this is achieved by searching for a $\lambda_0 \in \overline{\mathbb{R}}$ such that $\det(R_u(\lambda_0)) = 0$ and $Q_R = Q_{R(\lambda_0)}$ is simple ruled; by Theorem 3.1, such a quadric exists or the pencil contains the empty set. Assume, for the sake of simplicity, that the intersection is not empty and that Q_R and Q_S are distinct. Then $Q_S \cap Q_T = Q_S \cap Q_R$.

[2] In [13, 14], Levin refers to these quadrics as to nonelliptic paras.

quadric	canonical equation $(a,b>0)$	parameterization $\mathbf{X}=[x,y,z]$, $u,v\in\mathbb{R}$
simple plane	$x=0$	$\mathbf{X}(u,v)=[0,u,v]$
double plane	$x^2=0$	$\mathbf{X}(u,v)=[0,u,v]$
parallel planes	$ax^2=1$	$\mathbf{X}(u,v)=[\frac{1}{\sqrt{a}},u,v],\quad \mathbf{X}(u,v)=[-\frac{1}{\sqrt{a}},u,v]$
intersecting planes	$ax^2-by^2=0$	$\mathbf{X}(u,v)=[\frac{u}{\sqrt{a}},\frac{u}{\sqrt{b}},v],\quad \mathbf{X}(u,v)=[\frac{u}{\sqrt{a}},-\frac{u}{\sqrt{b}},v]$
hyperbolic paraboloid	$ax^2-by^2-z=0$	$\mathbf{X}(u,v)=[\frac{u+v}{2\sqrt{a}},\frac{u-v}{2\sqrt{b}},uv]$
parabolic cylinder	$ax^2-y=0$	$\mathbf{X}(u,v)=[u,au^2,v]$
hyperbolic cylinder	$ax^2-by^2=1$	$\mathbf{X}(u,v)=[\frac{1}{2\sqrt{a}}(u+\frac{1}{u}),\frac{1}{2\sqrt{b}}(u+\frac{1}{u}),v]$

Table 2: Parameterizations of canonical simple ruled quadrics.

2. Determine the orthonormal transformation matrix P_u which sends R_u in diagonal form by computing the eigenvalues and the normalized eigenvectors of R_u. Deduce the transformation matrix P which sends Q_R into canonical form. In the orthonormal frame in which it is canonical, Q_R admits one of the parameterizations $\mathbf{X}$ of Table 2.

3. Compute the matrix $S'=P^TSP$ of the quadric Q_S in the canonical frame of Q_R and consider the equation
$$\mathbf{X}^T S'\mathbf{X}=a(u)v^2+b(u)v+c(u)=0, \tag{1}$$
where $\mathbf{X}$ has been augmented by a fourth coordinate set to 1. (The parameterizations of Table 2 are such that $a(u),b(u)$ and $c(u)$ are polynomials of degree two in u.)

   Solve (1) for v in terms of u and determine the corresponding domain of validity of u on which the solutions are defined, i.e., the set of u such that $\Delta(u)=b^2(u)-4a(u)c(u)\geqslant 0$. Substituting v by its expression in terms of u in $\mathbf{X}$, we have a parameterization of $Q_S\cap Q_T=Q_S\cap Q_R$ in the orthonormal coordinate system in which Q_R is canonical.

4. Output $P\mathbf{X}(u)$, the parameterized equation of $Q_S\cap Q_T$ in the global coordinate frame, and the domain of $u\in\mathbb{R}$ on which it is valid.

This method is very nice and powerful since it gives an explicit representation of the intersection of two general quadrics. However, it is far from being ideal from the point of view of precision and robustness since it introduces non-rational numbers at several different places. Thus, if a floating point representation of numbers is used, the result may be wrong (geometrically and topologically) or, worse, the program may crash (especially in Step 1 when the type of the quadrics $Q_{R(\lambda_0)}$ are incorrectly computed). In theory, exact arithmetic would do, except that it would highly slow down the computations. In practice, however, a correct implementation using exact arithmetic seems out of reach because of the high degree of the algebraic numbers involved.

Let us examine more closely the potential sources of numerical instability in Levin's algorithm.

inertia of S	canonical equation $(a,b,c,d>0)$	parameterization $\mathbf{X}=[x,y,z,w]$
$(4,0)$	$ax^2+by^2+cz^2+dw^2=0$	$Q_S=\emptyset$
$(3,0)$	$ax^2+by^2+cz^2=0$	Q_S is point $(0,0,0,1)$
$(2,2)$	$ax^2+by^2-cz^2-dw^2=0$	$\mathbf{X}=[\frac{ut+avs}{a},\frac{us-bvt}{b},\frac{ut-avs}{\sqrt{ac}},\frac{us+bvt}{\sqrt{bd}}],\ (u,v),(s,t)\in\mathbb{P}^1$
$(2,1)$	$ax^2+by^2-cz^2=0$	$\mathbf{X}=[uv,\frac{u^2-abv^2}{2b},\frac{u^2+abv^2}{2\sqrt{bc}},s],\ (u,v,s)\in\mathbb{P}^{\star 2}$
$(2,0)$	$ax^2+by^2=0$	$\mathbf{X}=[0,0,u,v],\ (u,v)\in\mathbb{P}^1$
$(1,1)$	$ax^2-by^2=0$	$\mathbf{X}_1=[u,\frac{\sqrt{ab}}{b}u,v,s],\ \mathbf{X}_2=[u,-\frac{\sqrt{ab}}{b}u,v,s],\ (u,v,s)\in\mathbb{P}^2$
$(1,0)$	$ax^2=0$	$\mathbf{X}=[0,u,v,s],\ (u,v,s)\in\mathbb{P}^2$

Table 3: Parameterization of projective quadrics of inertia different from $(3,1)$. In the parameterization of projective cones, $\mathbb{P}^{\star 2}$ stands for the 2-dimensional real quasi-projective space defined as the quotient of $\mathbb{R}^3\setminus\{0,0,0\}$ by the equivalence relation $\sim$ where $(x,y,z)\sim(y_1,y_2,y_3)$ iff $\exists\lambda\in\mathbb{R}\setminus\{0\}$ such that $(x,y,z)=(\lambda y_1,\lambda y_2,\lambda^2 y_3)$.

- Step 1: λ_0 is the root of a third degree polynomial with rational coefficients. In the worst case, it is thus expressed with nested radicals of depth two. Since determining if $Q_{R(\lambda_0)}$ is simple ruled involves computing its Euclidean type (not an easy task considering that $Q_{R(\lambda_0-\varepsilon)}$ and $Q_{R(\lambda_0+\varepsilon)}$ may be and often are of different types), this is probably the biggest source of non-robustness.

- Step 2: Since Q_R is simple ruled, the characteristic polynomial of R_u is a degree three polynomial having zero as a root and whose coefficients are in the field extension $\mathbb{Q}(\lambda_0)$. Thus, the nonzero eigenvalues of R_u may involve nested radicals of depth three. Since the corresponding eigenvectors have to be normalized, the coefficients of the transformation matrix P are expressed with radicals of nesting depth four in the worst case.

  Since the coefficients of the parameterization $\mathbf{X}$ of Q_R are expressed as square roots of the coefficients of the canonical equation Q_{P^TRP} (as in Table 2), the coefficients of the parameterization of $Q_S\cap Q_T$ can involve *nested radicals of depth five* in the worst case.

- Step 3: Computing the domain of $\mathbf{X}$ amounts to solving the fourth degree equation $\Delta(u)=0$ whose coefficients are nested radicals of worst-case depth five in $\mathbb{Q}$.

Note that this worst-case picture is the generic case. Indeed, given two arbitrary quadrics with rational coefficients, the polynomial $\det(R_u(\lambda))$ will generically have no rational root (a consequence of Hilbert's Irreducibility Theorem).

4 Generic algorithm

We now present a first but major improvement to Levin's pencil method for computing parametric representations of the intersection of quadrics.

This so-called "generic algorithm" removes most of the sources of radicals in Levin's algorithm. We prove in Section 7 that it is near-optimal in the generic, smooth quartic case. It is however not optimal for all the possible types of intersection and will need later refinements (see the comments in Section 9, and Parts II and III). But it is sufficiently simple, robust and efficient to be of interest to many.

We start by introducing the projective framework underlying our approach and stating the main theorem on which the generic approach rests. We then outline our algorithm and detail particular steps in ensuing sections.

From now on, all the input quadrics considered have their coefficients (i.e., the entries of the corresponding matrices) in $\mathbb{Q}$.

4.1 Key ingredients

The first ingredient of our approach is to work not just over $\mathbb{R}^3$ but over the real projective space $\mathbb{P}^3$. Recall that, in projective space, quadrics are entirely characterized by their inertia (i.e., two quadrics with the same inertia are projectively equivalent), while in Euclidean space they are characterized by their inertia and the inertia of their principal submatrix.

In our algorithm, quadrics of inertia different from $(3,1)$ (i.e., ruled quadrics) play the role of simple ruled quadrics in Levin's method. In Table 3, we present a new set of parameterizations of ruled projective quadrics that are both linear in one of their parameters and involve, in the worst case, a minimal number of square roots[3], which we prove in Section 6. That these parameterizations are faithful parameterizations of the projective quadrics (i.e., there is a one-to-one correspondence between the points of the quadric and the parameters) is proved in the appendix.

Another key ingredient of our approach is encapsulated in the following theorem, which mirrors, in the projective setting, Levin's theorem on the existence of ruled quadrics in a pencil.

Theorem 4.1. *In a pencil generated by any two distinct quadrics, the set $\mathcal{S}$ of quadrics of inertia different from $(3,1)$ is not empty. Furthermore, if no quadric in $\mathcal{S}$ has rational coefficients, then the intersection of the two initial quadrics is reduced to two distinct points.*

This theorem, which is proved in Section 5.2, generalizes Theorem 3.1. Indeed, it ensures that the two quadrics we end up intersecting have rational coefficients, except in one very specific situation. This is how we remove the main source of nested radicals in Levin's algorithm.

The last basic ingredient of our approach is the use of Gauss reduction of quadratic forms for diagonalizing a symmetric matrix and computing the canonical form of the associated projective quadric, instead of the traditional eigenvalues/eigenvectors approach used by Levin. Since Gauss transformation is rational (the elements of the matrix P which sends S into canonical form are rational), this removes some layers of nested radicals from Levin's algorithm. Note, also, that there is no difficulty parameterizing the reduced quadric $S' = P^T SP$ since, by Sylvester's Inertia Law, S and S' have the same inertia.

[3]Note that there is necessarily a trade-off between the minimal degree of a parameterization in one of its parameters and the degree of its coefficient field. For instance, Wang, Joe and Goldman [33] give parameterizations of quadrics that have rational coefficients but are quadratic in all of their parameters.

4.2 Algorithm outline

Armed with these ingredients, we are now in a position to outline our generic algorithm.

Let $R(\lambda) = \lambda S - T$ be the pencil generated by the quadrics Q_S and Q_T of $\mathbb{P}^3$ and $\mathcal{D}(\lambda) = \det(R(\lambda))$ be the determinantal equation of the pencil. Recall that, although working in all cases, our generic algorithm is best designed when $\mathcal{D}(\lambda)$ is not identically zero and does not have any multiple root. In the other case, a better algorithm is described in parts II and III. The outline of our intersection algorithm is as follows (details follow in ensuing sections):

1. Find a quadric Q_R with rational coefficients in the pencil, such that $\det R > 0$ if possible or $\det R = 0$ otherwise. (If no such R exists, the intersection is reduced to two points, which we output.) If the inertia of R is $(4,0)$, output empty intersection. Otherwise, proceed.

   Assume for the sake of simplicity that $Q_S \neq Q_R$, in such a way that $Q_S \cap Q_R = Q_S \cap Q_T$.

2. If the inertia of R is not $(2,2)$, apply Gauss reduction to R and compute a frame in which $P^T RP$ is diagonal.

   If the inertia of R is $(2,2)$, find a rational point close enough to Q_R that the quadric $Q_{R'}$ through this point has the same inertia as Q_R. Replace Q_R by this quadric. Use that rational point to compute a frame in which $P^T RP$ is the diagonal matrix $\operatorname{diag}(1,1,-1,-\delta)$, with $\delta \in \mathbb{Q}$.

   In the local frame, Q_R can be described by one of the parameterizations $\mathbf{X}$ of Table 3. Compute the parameterization $P\mathbf{X}$ of Q_R in the global frame.

3. Consider the equation
   $$\Omega : (P\mathbf{X})^T S(P\mathbf{X}) = 0. \tag{2}$$
   Equation Ω is of degree at most 2 in (at least) one of the parameters. Solve it for this parameter in terms of the other(s) and compute the domain of the solution.

4. Substitute this parameter in $P\mathbf{X}$, giving a parameterization of the intersection of Q_S and Q_T.

4.3 Details of Step 1

The detailed description of Step 1 is as follows. Recall that $\mathcal{D}(\lambda) = \det(R(\lambda))$ is the determinantal equation of the pencil.

1. a. If $\mathcal{D}(\lambda) \equiv 0$, set $R = S$ and proceed.

   b. Otherwise, compute isolating intervals for the real roots of $\mathcal{D}(\lambda)$ (using for instance a variant of Uspensky's algorithm [23]). Compute a rational number λ_0 in between each of the separating intervals and, for each λ_0 such that $\mathcal{D}(\lambda_0) > 0$, compute the inertia of the corresponding quadrics using Gauss reduction. If one of the inertias is $(4,0)$, output $Q_S \cap Q_T = \emptyset$. Otherwise, one of these inertias is $(2,2)$ and we proceed with the corresponding quadric.

c. Otherwise (i.e. $\mathcal{D}(\lambda) \not\equiv 0$ and $\mathcal{D}(\lambda) \leqslant 0$ for all λ), compute the greatest common divisor $\gcd(\lambda)$ of $\mathcal{D}(\lambda)$ and its derivative with respect to λ. If $\gcd(\lambda)$ has a rational root λ_0, proceed with the corresponding quadric $Q_{R(\lambda_0)}$.

d. Otherwise (i.e. $\mathcal{D}(\lambda)$ has two non-rational double real roots), $Q_S \cap Q_T$ is reduced to two points. The quadric corresponding to one of these two roots is of inertia $(2,0)$ (an imaginary pair of planes). The singular line of this pair of planes is real and can be parameterized easily, even though it is not rational. Intersecting that line with any of the input quadrics gives the two points.

To assert the correctness of this algorithm, we have several things to prove. First, we make clear why, when looking for a quadric in the pencil (S,T) with inertia different from those of S and T, the right polynomial to consider is $\mathcal{D}(\lambda)$:

Lemma 4.2. *The inertia of $R(\lambda)$ is invariant on any interval of λ not containing a root of $\mathcal{D}(\lambda)$.*

Proof. It suffices to realize that the eigenvalues of $R(\lambda)$ are continuous functions of λ and that the characteristic polynomial of $R(\lambda)$

$$\det(R(\lambda) - lI)$$

is a polynomial in l whose constant coefficient is $\mathcal{D}(\lambda)$, where I is the identity matrix of size 4. Thus the eigenvalues of $R(\lambda)$ may change of sign only at a zero of $\det(R(\lambda))$. □

Let us now show that Step 1 of our algorithm always outputs empty intersection when $Q_S \cap Q_T = \emptyset$. This, in fact, is a direct consequence of Lemma 4.2 and of the following theorem proved in 1936/1937 by the German mathematician Paul Finsler.

Theorem 4.3 ([7]). *Assume $n \geqslant 3$ and let S,T be real symmetric matrices of size n. Then $Q_S \cap Q_T = \emptyset$ if and only if the pencil of matrices generated by S and T contains a definite matrix.*

In Step 1.d, Q_S and Q_T intersect in two points by Theorem 4.1. Furthermore, the quadric corresponding to one of two roots of $\mathcal{D}(\lambda)$ is a real line by the proof of Theorem 4.1.

Finally, note that we can further refine Step 1.b by computing the inertia of the quadrics $Q_{R(\lambda_0)}$ with positive determinant only when the determinantal equation has four real roots counted with multiplicities. Indeed, in view of the following proposition, testing for the presence of a definite matrix in the pencil needs to be done only in that case.

Proposition 4.4. *Assume $n \geqslant 3$ and let S,T be real symmetric matrices of size n. Then $Q_S \cap Q_T = \emptyset$ implies that $\det(\lambda S + \mu T)$ does not identically vanish and that all its roots are real.*

Proof. We use the equivalence provided by Theorem 4.3 of the emptiness of the intersection and the existence of a definite matrix in the pencil. Let U be a definite matrix of the pencil which we choose positive (a similar proof goes for negative definite).

Since U is positive definite, we can apply to it a Cholesky factorization: $U = HH^T$, where H is a lower triangular matrix. Consider the matrix $C = (H^{-1})S(H^{-1})^T$. Since C is real symmetric, it has n pairs of real eigenvalues and eigenvectors $(\nu_i, \mathbf{x}_i)$. Let $\mathbf{y}_i = (H^{-1})^T \mathbf{x}_i$. Then we have

$$H(C\mathbf{x}_i) = H(\nu_i \mathbf{x}_i) \quad \Longrightarrow \quad S\mathbf{y}_i = \nu_i U \mathbf{y}_i.$$

Hence all the roots of the characteristic polynomial of $U^{-1}S$ are real, which implies that all the roots of $\det(\lambda S+\mu U)=0$ are real. It follows that all the roots of $\det(\lambda S+\mu T)=0$ are also real. □

4.4 Details of Step 2

There are two cases, according to the inertia of R.

4.4.1 The inertia of R is not $(2,2)$

When the inertia of R is different from $(2,2)$, we use Gauss reduction of quadratic forms and parameterize the resulting quadric, whose associated matrix P^TRP is diagonal. In view of Sylvester's Inertia Law, the reduced quadric Q_{P^TRP} has the same inertia as Q_R. Thus it can be parameterized with at most one square root by one of the parameterizations $\mathbf{X}$ of Table 3. Since Gauss reduction is rational (i.e. P is a matrix with rational coefficients), the parameterization $P\mathbf{X}$ of Q_R contains at most one square root.

4.4.2 The inertia of R is $(2,2)$

When the inertia of R is $(2,2)$, the coefficients of the parameterization of Q_R can live, in the worst case, in an extension $\mathbb{Q}(\sqrt{m},\sqrt{n})$ of degree 4 of $\mathbb{Q}$ (see Table 3). We show here that there exists, in the neighborhood of Q_R, a quadric $Q_{R'}$ with rational coefficients such that

$$Q_S\cap Q_{R'}=Q_S\cap Q_R=Q_S\cap Q_T$$

and the coefficients of the parameterization of $Q_{R'}$ are in $\mathbb{Q}(\sqrt{\det R'})$.

First, apply Gauss reduction to Q_R. If any of $\sqrt{ac}$ or $\sqrt{bd}$ is rational in the parameterization of Q_R (as in Table 3), we are done. Otherwise, compute an arbitrary point $\mathbf{p}\in\mathbb{P}^3(\mathbb{R})$ on Q_R by taking any value of the parameters like, say, $(u,v)=(0,1)$ and $(s,t)=(0,1)$. Approximate $\mathbf{p}$ by a point $\mathbf{p}'\in\mathbb{P}^3(\mathbb{Q})$ not on $Q_S\cap Q_T$. Then compute $\lambda_0'\in\mathbb{Q}$ such that $\mathbf{p}'$ belongs to the quadric $Q_{R(\lambda_0')}$ of the pencil. This is easy to achieve in view of the following lemma.

Lemma 4.5. *In a pencil generated by two quadrics Q_S,Q_T with rational coefficients, there is exactly one quadric going through a given point $\mathbf{p}'$ that is not on $Q_S\cap Q_T$. If $\mathbf{p}'$ is rational, this quadric is rational.*

Proof. In the pencil generated by Q_S and Q_T, a quadric $Q_{R(\lambda,\mu)}$ contains $\mathbf{p}'$ if and only if $\mathbf{p}'^T(\lambda S+\mu T)\mathbf{p}'=0$, that is if and only if $\lambda(\mathbf{p}'^TS\mathbf{p}')+\mu(\mathbf{p}'^TT\mathbf{p}')=0$. If $\mathbf{p}'$ is not on $Q_S\cap Q_T$, this equation is linear in $(\lambda,\mu)\in\mathbb{P}^1$ and thus admits a unique solution. Moreover, if $\mathbf{p}'$ is rational, the equation has rational coefficients and thus the quadric of the pencil containing $\mathbf{p}'$ is rational. □

Note that λ_0' and the λ_0 such that $R=R(\lambda_0)$ get arbitrarily close to one another as $\mathbf{p}'$ gets close to $\mathbf{p}$. Thus if $\mathbf{p}'$ is close enough to $\mathbf{p}$, $R'=R(\lambda_0')$ has the same inertia $(2,2)$ as R, by Lemma 4.2. We refine the approximation $\mathbf{p}'$ of $\mathbf{p}$ until R' has inertia $(2,2)$.

We now have a quadric $Q_{R'}$ of inertia $(2,2)$ and a rational point on $Q_{R'}$. Consider any rational line through $\mathbf{p}'$ that is not in the plane tangent to $Q_{R'}$ at $\mathbf{p}'$. This line further intersects $Q_{R'}$ in another point $\mathbf{p}''$. Point $\mathbf{p}''$ is rational because otherwise $\mathbf{p}'$ and $\mathbf{p}''$ would be conjugate in the field extension of $\mathbf{p}''$ (since $Q_{R'}$ and the line are both rational) and thus $\mathbf{p}'$ would not be rational. Compute the rational transformation P sending $\mathbf{p}', \mathbf{p}''$ onto $(1, \pm 1, 0, 0)$. Apply this transformation to R' and then apply Gauss reduction of quadratic forms. In the local frame, $Q_{R'}$ has equation (up to a constant factor)

$$x^2 - y^2 + \alpha z^2 + \beta w^2 = 0, \tag{3}$$

with $\alpha\beta < 0$. Now consider the linear transformation whose matrix is P'

$$P' = \frac{1}{2}\begin{pmatrix} 1+\alpha & 0 & 1-\alpha & 0 \\ 1-\alpha & 0 & 1+\alpha & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2\alpha \end{pmatrix}.$$

Applying P' to the already reduced quadric of Eq. (3) gives the equation

$$x^2 + y^2 - z^2 - \delta w^2 = 0, \tag{4}$$

where $\delta = -\alpha\beta > 0$. The quadric of Eq. (4) can be parameterized by

$$\mathbf{X}((u,v),(s,t)) = \left(ut + vs, us - vt, ut - vs, \frac{us + vt}{\sqrt{\delta}}\right),$$

with $(u,v),(s,t) \in \mathbb{P}^1$ (see Table 3).

The three consecutive transformation matrices have rational coefficients thus $\mathbb{Q}(\sqrt{\delta}) = \mathbb{Q}(\sqrt{\det R'})$ and the product of these transformation matrices with $\mathbf{X}$ is a polynomial parameterization of $Q_{R'}$ with coefficients in $\mathbb{Q}(\sqrt{\delta})$, $\delta \in \mathbb{Q}$.

4.5 Details of Step 3

Solving Equation (2) can be done as follows. Recall that the content in the variable x of a multivariate polynomial is the gcd of the coefficients of the x^i.

Equation (2) may be seen as a quadratic equation in one of the parameters. For instance, if R has inertia $(2,2)$, Eq. (2) is a homogeneous biquadratic equation in the variables $\xi = (u,v)$ and $\tau = (s,t)$. Using only gcd computations, we can factor it in its content in ξ (which is a polynomial in τ or a constant), its content in τ, and a remaining factor. If the content in ξ (or in τ) is not constant, solve it in τ (in ξ); substituting the obtained real values in $\mathbf{X}$, we have a parameterization of some components of $Q_S \cap Q_T = Q_S \cap Q_R$ in the frame in which Q_R is canonical. If the remaining factor is not constant, solve it in a parameter in which it is linear, if any, or in τ. Substituting the result in $\mathbf{X}$, we have a parameterization of the last component of the intersection. If the equation which is solved is not linear, the domain of the parameterization is the set of ξ such that the degree 4 polynomial $\Delta(\xi) = b^2(\xi) - 4a(\xi)c(\xi)$ is positive, where $a(\xi), b(\xi)$ and $c(\xi)$ are the coefficients of τ^2, τ and 1 in (2), respectively.

5 Canonical forms and proof of Theorem 4.1

We now prove Theorem 4.1, the key result stated in the previous section. We start by recalling some preliminary results.

5.1 Canonical form for a nonsingular pair of symmetric matrices

We state results, proved by Uhlig [30, 31], we need for computing the canonical form of a pair of real symmetric matrices. Though only part of this theory is required for the proof of Theorem 4.1 (Section 5.2), we will need its full power in Part II of this paper for characterizing real pencils of quadrics, so we explicit it entirely.

Let us start by defining the notion of Jordan blocks.

Definition 5.1. Let M be a square matrix of the form

$$(\ell) \quad \text{or} \quad \begin{pmatrix} \ell & e & & 0 \\ & \ddots & \ddots & \\ & & \ddots & e \\ 0 & & & \ell \end{pmatrix}.$$

If $l \in \mathbb{R}$ and $e = 1$, M is called a *real Jordan block* associated with ℓ. If

$$\ell = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}, \quad a, b \in \mathbb{R},\ b \neq 0, \qquad e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

M is called a *complex Jordan block* associated with $a + \mathrm{i}b$.

Now we can state the real Jordan normal form theorem for real square matrices.

Theorem 5.2 (Real Jordan normal form). *Every real square matrix A is similar over the reals to a block diagonal matrix* $\operatorname{diag}(A_1, \dots, A_k)$*, called real Jordan normal form of A, in which each A_j is a (real or complex) Jordan block associated with an eigenvalue of A.*

The Canonical Pair Form Theorem then goes as follows:

Theorem 5.3 (Canonical Pair Form). *Let S and T be two real symmetric matrices of size n, with S nonsingular. Let $S^{-1}T$ have real Jordan normal form* $\operatorname{diag}(J_1, \dots, J_r, J_{r+1}, \dots, J_m)$*, where $J_1, \dots, J_r$ are real Jordan blocks corresponding to real eigenvalues of $S^{-1}T$ and $J_{r+1}, \dots, J_m$ are complex Jordan blocks corresponding to pairs of complex conjugate eigenvalues of $S^{-1}T$. Then:*

(a) The characteristic polynomial of $S^{-1}T$ and $\det(\lambda S - T)$ *have the same roots λ_j with the same (algebraic) multiplicities m_j.*

(b) S and T are simultaneously congruent by a real congruence transformation to

$$\operatorname{diag}(\varepsilon_1 E_1, \dots, \varepsilon_r E_r, E_{r+1}, \dots, E_m)$$

and

$$\operatorname{diag}(\varepsilon_1 E_1 J_1, \dots, \varepsilon_r E_r J_r, E_{r+1} J_{r+1}, \dots, E_m J_m),$$

respectively, where $\varepsilon_i = \pm 1$ *and* E_i *denotes the square matrix*

$$\begin{pmatrix} 0 & & 1 \\ & \iddots & \\ 1 & & 0 \end{pmatrix}$$

of the same size as J_i *for* $i = 1, \ldots, m$. *The signs* ε_i *are unique (up to permutations) for each set of indices* i *that are associated with a set of identical real Jordan blocks* J_i.

(c) The sum of the sizes of the blocks corresponding to one of the λ_j *is the multiplicity* m_j *if* λ_j *is real or twice this multiplicity if* λ_j *is complex. The number of the corresponding blocks (the geometric multiplicity of* λ_j*) is* $t_j = n - \operatorname{rank}(\lambda_j S - T)$, *and* $1 \leqslant t_j \leqslant m_j$.

Note that the canonical pair form of Theorem 5.3 can be considered the finest simultaneous block diagonal structure that can be obtained by real congruence for a given pair of real symmetric matrices, in the sense that it maximizes the number of blocks in the diagonalization of S and T.

5.2 Proof of Theorem 4.1

To prove Theorem 4.1, we consider a pencil of real symmetric 4×4 matrices generated by two symmetric matrices S and T of inertia $(3,1)$. We may suppose that they have the block diagonal form of the above theorem.

If all the blocks had an even size, the determinant of S would be positive, contradicting our hypothesis. Thus, there is a block of odd size in the canonical form of S. It follows that $\det(\lambda S - T)$ has at least one real root and the matrix of the pencil corresponding to this root has an inertia different from $(3,1)$. This proves the first part.

If $\det(\lambda S - T)$ has a simple real root, there is an interval of values for λ for which $\det(\lambda S - T) > 0$, and we are done with any rational value of λ in this interval. If $\det(\lambda S - T)$ has either a double real root and two complex roots, two rational double real roots or a quadruple real root, the quadrics corresponding to the real root(s) have rational coefficients and have inertia different from $(3,1)$.

Thus we are left with the case where $\det(\lambda S - T)$ has two non rational double real roots, which are algebraically conjugate. In other words,

$$\det(\lambda S - T) = c(\lambda - \lambda_1)^2(\lambda - \lambda_2)^2,$$

with $\lambda_1, \lambda_2 \in \mathbb{R} \setminus \mathbb{Q}$ and $\lambda_2 = \overline{\lambda_1}$ its (real algebraic) conjugate. Following the notations of Theorem 5.3, we have $m_1 = m_2 = 2$ and $1 \leqslant t_i \leqslant 2$, for $i = 1,2$. In other words, $(t_1, t_2) \in \{(1,1),(1,2),(2,1),(2,2)\}$.

We can quickly get rid of the case $(t_1,t_2) = (1,1)$. Indeed, in this case the blocks have an even size and S is not of inertia $(3,1)$. We can also eliminate the cases $(t_1,t_2) \in \{(1,2),(2,1)\}$, because the matrices $\lambda_1 S - T$ and $\lambda_2 S - T$ are algebraically conjugate, and so must have the same rank and the same number of blocks.

We are thus left with the case $(t_1,t_2) = (2,2)$. In this situation, S and T have four blocks, i.e., they are diagonal:

$$\begin{cases} S = \operatorname{diag}(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4), \\ T = \operatorname{diag}(\varepsilon_1\lambda_1, \varepsilon_2\lambda_1, \varepsilon_3\lambda_2, \varepsilon_4\lambda_2). \end{cases}$$

The pencil $\lambda S - T$ is generated by the two quadrics of rank 2

$$\begin{cases} S' = \lambda_1 S - T = \operatorname{diag}(0, 0, \varepsilon_3(\lambda_1 - \lambda_2), \varepsilon_4(\lambda_1 - \lambda_2)), \\ T' = \lambda_2 S - T = \operatorname{diag}(\varepsilon_1(\lambda_2 - \lambda_1), \varepsilon_2(\lambda_2 - \lambda_1), 0, 0). \end{cases}$$

We have that

$$\det(S' + T') = \varepsilon_1 \varepsilon_2 \varepsilon_3 \varepsilon_4 (\lambda_1 - \lambda_2)^4$$

is negative since all the quadrics of the pencil have negative determinant except $Q_{S'}$ and $Q_{T'}$. Thus $\varepsilon_1\varepsilon_2$ and $\varepsilon_3\varepsilon_4$ have opposite signs. It follows that one of S' and T' has inertia $(2,0)$ (say S') and the other has inertia $(1,1)$. Thus $Q_{S'}$ is a straight line, which intersects the real pair of planes $Q_{T'}$. Since $Q_{S'} \cap Q_{T'}$ is contained in all the quadrics of the pencil and since the pencil has quadrics of inertia $(3,1)$ (which are not ruled), the line $Q_{S'}$ is not included in $Q_{T'}$ and the intersection is reduced to two real points. Since the equations of $Q_{S'}$ and $Q_{T'}$ are $z^2 + w^2 = 0$ and $x^2 - y^2 = 0$ respectively, the two points have coordinates $(1,1,0,0)$ and $(-1,1,0,0)$. They are thus distinct. □

Remark 5.4. Pencils generated by two quadrics of inertia $(3,1)$ and having no quadric with rational coefficients of inertia different from $(3,1)$ do exist. Consider for instance

$$Q_S : 2x^2 - 2xz - 2yw + z^2 + w^2 = 0,$$
$$Q_T : 4x^2 + 2y^2 - 2yw + z^2 - 6xz + 3w^2 = 0.$$

Then, $\det(\lambda S - T) = -(\lambda^2 - 5)^2$.

6 Optimality of the parameterizations

We now prove that, among the parameterizations of projective quadrics linear in one of the parameters, the ones of Table 3 have, in the worst case, an optimal number of radicals. In other words, for each type of projective quadric, there are examples of surfaces for which the number of square roots of the parameterizations of Table 3 is required.

More precisely, we prove the following theorem, which will be crucial in asserting the near-optimality of our algorithm for parameterizing quadrics intersection.

Theorem 6.1. *In the set of parameterizations linear in one of the parameters, the parameterizations of Table 3 are worst-case optimal in the degree of the extension of $\mathbb{Q}$ on which they are defined.*

For a quadric Q of equation $ax^2 + by^2 - cz^2 - dw^2 = 0$ $(a,b,c,d > 0)$, the parameterization of Table 3 is optimal if Q has no rational point, which is the case for some quadrics. Knowing a rational point on Q (if any), we can compute a rational congruent transformation sending Q into the quadric of equation $x^2 + y^2 - z^2 - abcd\, w^2 = 0$, for which the parameterization of Table 3 is optimal.

For a quadric Q of equation $ax^2 + by^2 - cz^2 = 0$ $(a,b,c > 0)$, the parameterization of Table 3 is optimal if Q has no rational point other than its singular point $(0,0,0,1)$, which is the case for some quadrics. Knowing such a rational point on Q (if any), we can compute a rational congruent transformation sending Q into the quadric of equation $x^2 + y^2 - z^2 = 0$, for which the parameterization of Table 3 is rational (and thus optimal).

For the other types of projective quadrics, the parameterizations of Table 3 are optimal in all cases.

We prove this theorem by splitting it into four more detailed propositions: Proposition 6.2 for inertia $(1,1)$, Proposition 6.3 for inertia $(2,1)$ and Propositions 6.4 and 6.6 for inertia $(2,2)$.

Proposition 6.2. *A projective quadric Q of equation $ax^2 - by^2 = 0$ $(a,b > 0)$ admits a rational parameterization in $\mathbb{Q}$ if and only if it has a rational point outside the singular line $x = y = 0$, or equivalently iff ab is a square in $\mathbb{Q}$. If ab is a square in $\mathbb{Q}$, then the parameterization of Table 3 is rational.*

Proof. A point (x,y,z,w) on Q not on its singular line $x = y = 0$ is rational if and only if y/x, z/x, and w/x are rational. Since $(y/x)^2 = \frac{ab}{b^2}$ and z and w are not constrained, there exists such a rational point if and only if ab is a square.

If there exists a parameterization which is rational over $\mathbb{Q}$, then there exists some rational point outside the line $x = y = 0$, showing *a contrario* that there is no rational parameterization if ab is not a square.

Finally, if ab is the square of a rational number, then the parameterization of Table 3 is rational. □

Proposition 6.3. *A projective quadric Q of equation $ax^2 + by^2 - cz^2 = 0$ $(a,b,c > 0)$ admits a rational parameterization in $\mathbb{Q}$ if and only it contains a rational point other than the singular point $(0,0,0,1)$. Knowing such a rational point, we can compute a rational congruent transformation P sending Q into the quadric of equation $x^2 + y^2 - z^2 = 0$ for which the parameterization of Table 3 is rational; lifting this parameterization to the original space by multiplying by matrix P, we have a rational parameterization of Q.*

On the other hand, there are such quadrics without rational point and thus without rational parameterization, for example the quadric of equation $x^2 + y^2 - 3z^2 = 0$.

Proof. If Q has a rational point other than $(x = y = z = 0)$, any rational line passing through this point and not included in Q cuts Q in another rational point. Compute the rational congruent transformation sending these points onto $(\pm 1,1,0,0)$. Applying this transformation to Q gives a quadric of equation $x^2 - y^2 + r$, where r is a polynomial of degree at most one in x and y. Thus Gauss reduction algorithm leads to the form $x^2 - y^2 + dz^2 = (X^2 + Y^2 - Z^2)/d$ where $X = (1+d)x/2 + (1-d)y/2$, $Y = dz$ and $Z = (1-d)x/2 + (1+d)y/2$. The parameterization of Table 3 applied to equation $X^2 + Y^2 - Z^2$ is clearly rational. Lifting this parameterization back to the original space, we obtain a rational parameterization of Q.

Reciprocally, if Q has no rational point, then Q does not admit a rational parameterization.

Now, suppose for a contradiction that the quadric with equation $x^2 + y^2 - 3z^2 = 0$ has a rational point (x,y,z,w) different from $(0,0,0,1)$. By multiplying x, y, and z by a common denominator and dividing them by their gcd, we obtain another rational point on the quadric for which x, y and z are integers that are not all even. Note that x^2 is equal, modulo 4, to 0 if x is even and 1 otherwise (indeed, modulo 4, $0^2 = 0$, $1^2 = 1$, $2^2 = 0$ and $3^2 = 1$). Thus, $x^2 + y^2 - 3z^2 \equiv x^2 + y^2 + z^2 \pmod 4$ is equal to the number of odd numbers in x, y, z, i.e. $1, 2$ or 3. Thus $x^2 + y^2 - 3z^2 \neq 0$, contradicting the hypothesis that (x,y,z,w) is a point on the quadric. □

Proposition 6.4. *Let Q be the quadric of equation $ax^2+by^2-cz^2-dw^2=0$ $(a,b,c,d>0)$. Any field $\mathbb{K}$ in which Q admits a rational parameterization, linear in one of its parameters, contains $\sqrt{abcd}$.*

Proof. Let $\mathbb{K}$ be a field in which Q admits a rational parameterization, linear in the parameter $(u,v)\in\mathbb{P}(\mathbb{R})$. Fixing the value of the other parameter $(s,t)\in\mathbb{P}(\mathbb{K})$ defines a rational line L (in $\mathbb{K}$) contained in Q. L cuts any plane (in possibly infinitely many points) in projective space. In particular, L cuts the plane of equation $z=0$. Since $L\subseteq Q$, L cuts the conic of equation $ax^2+by^2-dw^2=z=0$ in a point $\mathbf{p}=(x_0,y_0,0,1)$. Moreover, $\mathbf{p}$ is rational in $\mathbb{K}$ (i.e., $x_0,y_0\in\mathbb{K}$) because it is the intersection of a rational line and the plane $z=0$.

The plane tangent to Q at $\mathbf{p}$ has equation $ax_0x+by_0y-dw=0$. We now compute the intersection of Q with this plane. Since $ax_0^2+by_0^2=d$ and $a,b,d>0$, x_0 or y_0 is nonzero; assume for instance that $x_0\neq 0$. Squaring the equation of the tangent plane yields $(ax_0x)^2=(by_0y-dw)^2$. By eliminating x^2 between this equation and the equation of Q, we get

$$(by_0y-dw)^2+ax_0^2(by^2-cz^2-dw^2)=0$$

or

$$dw^2(d-ax_0^2)+by^2(ax_0^2+by_0^2)-2bdy_0yw-acx_0^2z^2=0.$$

It follows from $ax_0^2+by_0^2=d$ that $bd(y-y_0w)^2-acx_0^2z^2=0$ or also

$$b^2d^2(y-y_0w)^2-abcd\,x_0^2z^2=0. \tag{5}$$

The intersection of Q and its tangent plane at $\mathbf{p}$ contains the line L which is rational in $\mathbb{K}$. Thus, Equation (5) can be factored over $\mathbb{K}$ into two linear terms. Hence, $\sqrt{abcd}$ belongs to $\mathbb{K}$. □

Remark 6.5. *abcd* is the discriminant of the quadric, i.e., the determinant of the associated matrix, so it is invariant by a change of coordinates (up to a square factor). Thus, if R and R' are two matrices representing the same quadric in different frames, the fields $\mathbb{Q}(\sqrt{\det R})$ and $\mathbb{Q}(\sqrt{\det R'})$ are equal.

Proposition 6.6. *A projective quadric Q of equation $ax^2+by^2-cz^2-dw^2=0$ $(a,b,c,d>0)$ admits a rational parameterization in $\mathbb{Q}(\sqrt{abcd})$ if and only it contains a rational point. Knowing such a rational point, we can compute a rational congruent transformation P sending Q into the quadric of equation $x^2+y^2-z^2-abcd\,w^2=0$ for which the parameterization of Table 3 is rational over $\mathbb{Q}(\sqrt{abcd})$; lifting this parameterization to the original space by multiplying by matrix P, we have a rational parameterization of Q over $\mathbb{Q}(\sqrt{abcd})$.*

On the other hand, there are such quadrics with no rational point and thus without rational parameterization in $\mathbb{Q}(\sqrt{abcd})$, for example the quadric of equation $x^2+y^2-3z^2-11w^2=0$.

Proof. If Q admits a rational parameterization in $\mathbb{Q}(\sqrt{abcd})$, then it has infinitely many rational points over this field. If Q has a point (x,y,z,w) that is rational over $\mathbb{Q}(\sqrt{abcd})$, but not rational over $\mathbb{Q}$, we may suppose without loss of generality that $x=1$, by permuting the variables in order that $x\neq 0$ and then by dividing all coordinates by x. The conjugate point $(1,y',z',w')$ over $\mathbb{Q}(\sqrt{abcd})$ belongs also to Q. The line passing through these points is rational (over $\mathbb{Q}$), as is the point $(1,(y+y')/2,(z+$

$z')/2,(w+w')/2)$. Choose a rational frame transformation such that this line becomes the line $z=w=0$ and this point becomes $(1,0,0,0)$. In this new frame the coordinates of the conjugate points are $(1,\pm e\sqrt{abcd},0,0)$ for some rational number e, and the equation of Q is $abcd\,e^2x^2-y^2+r=0$ where r is a polynomial of degree at most 1 in x and y. Gauss reduction thus provides an equation of the form $abcd\,e^2x^2-y^2+fz^2-gw^2=0$, and the invariance of the determinant (Remark 6.5) shows that fg is the square of a rational number h. Thus $(0,0,g,h)$ is a rational point of Q over $\mathbb{Q}$.

Now, if Q has a rational point over $\mathbb{Q}$, one may get another rational point as the intersection of the quadric and any line passing through the point and not tangent to the quadric. One can compute a rational congruent transformation such that these points become $(1,\pm 1,0,0)$. In this new frame the equation of Q has the form x^2-y^2-r where r is a polynomial of degree at most 1 in x and y. Gauss reduction provides thus an equation of the form $x^2-y^2+ez^2-fw^2=(X^2+Y^2-Z^2-efw^2)/e$, with $X=(1+e)x/2+(1-e)y/2$, $Y=ez$ and $Z=(1-e)x/2+(1+e)y/2$. By the invariance of the determinant, $ef=g^2abcd$ for some rational number g. Putting $W=gw$, we get the equation $X^2+Y^2-Z^2-abcd\,W^2=0$ for Q, and the parameterization of Table 3 is rational over $\mathbb{Q}(\sqrt{abcd})$.

It follows from this proof that, if a quadric of inertia $(2,2)$ has a rational point, it has a parameterization in $\mathbb{Q}(\sqrt{abcd})$, which is linear in one of the parameters. Conversely, for proving that such a parameterization does not always exist, it suffices to prove that there are quadrics of inertia $(2,2)$ having no rational point over $\mathbb{Q}$. Let us consider the quadric of equation $x^2+y^2-3z^2-11w^2=0$. If it has a rational point (x,y,z,w), then by multiplying x, y, z and w by some common denominator and by dividing them by their gcd, we may suppose that x, y, z and w are integers which are not all even. As in the proof of Proposition 6.3, $x^2+y^2-3z^2-11w^2$ is equal modulo 4 to the number of odd numbers in x,y,z,w. Thus all of them are odd. It is straightforward that the square of an odd number is equal to 1 modulo 8. It follows that $x^2+y^2-3z^2-11w^2$ is equal to 4 modulo 8, a contradiction with $x^2+y^2-3z^2-11w^2=0$. $\square$

7 Near-optimality in the smooth quartic case

In this section, we prove that the algorithm given in Section 4 outputs, in the generic (smooth quartic) case, a parameterization of the intersection that is optimal in the number of radicals up to one possibly unnecessary square root. We also show that deciding whether this extra square root can be avoided or not is hard. Moreover, we give examples where the extra square root cannot be eliminated, for the three possible morphologies of a real smooth quartic.

7.1 Algebraic preliminaries

First recall that, as is well known from the classification of quadric pencils by invariant factors (see [2] and Part II for more), the intersection of two quadrics is a nonsingular quartic exactly when $\mathcal{D}(\lambda,\mu)=\det R(\lambda,\mu)$ has no multiple root. Otherwise the intersection is singular. Note that the intersection is nonsingular exactly when $\gcd(\frac{\partial\mathcal{D}}{\partial\lambda},\frac{\partial\mathcal{D}}{\partial\mu})=1$.

Moreover, when the intersection is nonsingular, the rank of any quadric in the pencil is at least three; indeed, all the roots of $\mathcal{D}(\lambda,\mu)$ are simple and thus, in Theorem 5.3(c), $m_j=1$, thus $t_j=1$, hence the quadrics associated with the roots of $\mathcal{D}(\lambda,\mu)$ have rank 3.

Whether the intersection of two quadrics admits a parameterization with rational functions directly follows from classical results:

Proposition 7.1. *The intersection of two quadrics admits a parameterization with rational functions if and only if the intersection is singular.*

Proof. First recall that a curve admits a parameterization with rational functions if and only if it has zero genus [20].

Assume first that the intersection of the two quadrics is irreducible. In $\mathbb{P}^3(\mathbb{C})$, if two algebraic surfaces of degree d_1 and d_2 intersect in an irreducible curve, its genus is

$$\frac{1}{2}d_1 d_2(d_1+d_2-4)+1-\sum_{i=1}^{k}\frac{q_i(q_i-1)}{2},$$

where k is the number of singular points and $q_{i,i=1,\ldots,k}$ their respective multiplicity [19]. The intersection curve has thus genus 1 when it is smooth, 0 otherwise. The result follows.

Assume now that the intersection of the two quadrics is reducible. If the intersection contains only points, lines and conics, which are classically rational, we are done. For the remaining case (cubic and line), we use the following result. In $\mathbb{P}^3(\mathbb{C})$, if two algebraic surfaces of degree d_1 and d_2 intersect in two irreducible curves of degree d and d' and of genus g and g', then [20]

$$g'-g=\left(\frac{1}{2}(d_1+d_2)-2\right)(d'-d).$$

For quadrics, $d_1+d_2=4$, so we get $g=g'$. So the genus of the cubic is that of the line, i.e. 0. $\square$

Finally consider the equation $\Omega : \mathbf{X}^T S'\mathbf{X}=0$, obtained in Step 3 of our algorithm, where $\mathbf{X}$ is the parameterization of Q_R and S' is the matrix of Q_S in the canonical frame of Q_R. Let C_Ω be the curve zero-set of Ω. Depending on the projective type of Q_R, C_Ω is a bidegree $(2,2)$ curve in $\mathbb{P}^1\times\mathbb{P}^1$ (inertia $(2,2)$ or $(2,0)$), a quartic curve in $\mathbb{P}^{\star 2}$ (inertia $(2,1)$) or a quartic curve in $\mathbb{P}^2$ (inertia $(1,1)$ or $(1,0)$). Let C denote the curve of intersection of the two given quadrics Q_S and Q_T. We have the following classical result.

Fact 7.2. *The parameterization of Q_R defines an isomorphism between C and C_Ω. In particular, C and C_Ω have the same genus, irreducibility, and factorization.*

7.2 Optimality

Assume the intersection is a real nonsingular quartic. Then $\mathcal{D}(\lambda,\mu)$ has no multiple root, and thus Q_R is necessarily a quadric of inertia $(2,2)$. After Step 2 of our algorithm, Q_R has a parameterization in $\mathbb{Q}(\sqrt{\delta})$ that is bilinear in $\xi=(u,v)$ and $\tau=(s,t)$. After resolution of Ω and substitution in Q_R, we get a parameterization in $\mathbb{Q}(\sqrt{\delta})[\xi,\sqrt{\Delta}]$ with $\Delta\in\mathbb{Q}(\sqrt{\delta})[\xi]$ of degree 4.

Proposition 7.1 implies that it cannot be parameterized by rational functions, so $\sqrt{\Delta}$ cannot be avoided. The question now is: can $\sqrt{\delta}$ be avoided? The answer is twofold:

1. deciding whether $\sqrt{\delta}$ can be avoided amounts, in the general case, to finding a rational point on a surface of degree 8,
2. there are cases in which $\sqrt{\delta}$ cannot be avoided.

We prove these results in the following two sections.

7.2.1 Optimality test

We first prove two preliminary lemmas.

Lemma 7.3. *If the intersection of two given quadrics has a parameterization involving only one square root, there exists a quadric with rational coefficients in the pencil that contains a rational line.*

Proof. In what follows, call *degree* of a point the degree of the smallest field extension of $\mathbb{Q}$ containing the coordinates of this point.

If the parameterization of the intersection involves only one square root, the intersection contains infinitely many points of degree at most 2, one for any rational value of the parameters. Now we have several cases according to the type of points contained in the intersection.

If the intersection contains a point $\mathbf{p}$ of degree 2, it contains also its algebraic conjugate $\overline{\mathbf{p}}$. The line passing through $\mathbf{p}$ and $\overline{\mathbf{p}}$ is invariant by conjugation, so is rational. Let $\mathbf{q}$ be a rational point on this line. The quadric of the pencil passing through $\mathbf{q}$ is rational (Lemma 4.5). Since it also contains $\mathbf{p}$ and $\overline{\mathbf{p}}$ (the intersection is contained in any quadric of the pencil), this quadric cuts the line in at least 3 points and thus contains it.

If the intersection contains a regular rational point (i.e. a rational point which is not a singular point of the intersection), then the line tangent to the intersection at this point is rational, and is tangent to any quadric of the pencil. The quadric of the pencil passing through a rational point of this tangent line contains the contact point; thus it contains the tangent line.

If the intersection contains a singular rational point $\mathbf{p}$, then all the quadrics of the pencil which are not singular at $\mathbf{p}$ have the same tangent plane at $\mathbf{p}$. Let us consider the quadric of the pencil passing through a rational point $\mathbf{q}$ of this tangent plane (or through any rational point, if none of the quadrics is regular at $\mathbf{p}$). Similarly as above, this quadric contains the rational line $\mathbf{pq}$. ☐

Lemma 7.4. *If a quadric contains a rational line, its discriminant is a square in $\mathbb{Q}$.*

Proof. If the quadric has rank less than 4, its discriminant is zero. We may thus suppose that the discriminant is not 0 and that the equation of the quadric is $ax^2+by^2-cz^2-dw^2=0$. Since this quadric contains a rational line L, and thus a rational point, there is a rational change of frames such that the quadric has equation $x^2+y^2-z^2-abcd\,w^2=0$, by Proposition 6.6. Cut the quadric by the plane $z=0$. Since the intersection of the plane $z=0$ and the rational L is a rational point, the cone $x^2+y^2-abcd\,w^2=0$ contains a rational point outside it singular locus. By Proposition 6.3, there is a rational congruent transformation P sending this cone into the cone of equation $x^2+y^2-w^2=0$. These two cones can be seen as conics in $\mathbb{P}^2(\mathbb{Q})$ and P can be seen as a rational transformation in $\mathbb{P}^2(\mathbb{Q})$. The discriminant $-abcd$ of the conic $x^2+y^2-abcd\,w^2=0$ is thus equal to $(\det P)^2$ times -1, the discriminant of the conic $x^2+y^2-w^2=0$. Hence $abcd$ is a square in $\mathbb{Q}$. ☐

From these two technical results and the results of Section 6, we obtain the following equivalence.

Proposition 7.5. *When the intersection is a nonsingular quartic, it can be parameterized in* $\mathbb{Q}[\xi, \sqrt{\Delta}]$ *with* $\Delta \in \mathbb{Q}[\xi]$ *if and only if there exists a quadric of the pencil with rational coefficients having a nonsingular rational point and whose discriminant is a square in* $\mathbb{Q}$*.*

Proof. If $\sqrt{\delta}$ can be avoided, there exists, by Lemma 7.3, a quadric of the pencil with rational coefficients containing a rational line. By Lemma 7.4, the discriminant of this quadric is thus a square in $\mathbb{Q}$. Moreover, since the quadrics of the pencil have rank at least three, the rational line is not the singular line of some quadric (see Table 1) and thus contains a nonsingular point.

Conversely, if there exists a quadric of the pencil with rational coefficients having a rational nonsingular point and whose discriminant is a square, then it has a rational parameterization by Theorem 6.1 and thus $\sqrt{\delta}$ can be avoided. □

Mirroring Proposition 7.5, we can devise a general test for deciding, in the smooth quartic case, whether the square root $\sqrt{\delta}$ can be avoided or not. Consider the equation

$$\sigma^2 = \det\left((\mathbf{x}^T T\mathbf{x})\, S - (\mathbf{x}^T S\mathbf{x})\, T\right), \quad \mathbf{x} = (x, y, z, c)^T, \tag{6}$$

where $c \in \mathbb{Q}$ is some constant such that plane $w = c \in \mathbb{Q}$ contains the vertex of no cone (inertia $(2,1)$) of the pencil. Note that (6) has degree 8 in the worst case.

Theorem 7.6. *When the intersection is a nonsingular quartic, it can be parameterized in* $\mathbb{Q}[\xi, \sqrt{\Delta}]$ *with* $\Delta \in \mathbb{Q}[\xi]$ *if and only Equation* (6) *has a rational solution.*

Proof. Suppose first that (6) has a rational solution $(x_0, y_0, z_0, \sigma_0)$ and let $\mathbf{x}_0 = (x_0, y_0, z_0, c)^T$ and $(\lambda_0, \mu_0) = (\mathbf{x}_0^T T\mathbf{x}_0, -\mathbf{x}_0^T S\mathbf{x}_0)$. The quadric $Q = \lambda_0 Q_S + \mu_0 Q_T$ of the pencil has rational coefficients, contains the rational point $\mathbf{x}_0 = (x_0, y_0, z_0, c)^T$ and its discriminant is a square, equal to σ_0^2. Moreover, if Q has inertia $(2,1)$, then $\mathbf{x}_0$ is not its apex because, by assumption, the plane $w = c$ contains the vertex of no cone of the pencil. It then follows from Theorem 6.1 that our algorithm produces a rational parameterization of Q, and thus a parameterization of the curve of intersection with rational coefficients.

Conversely, if the curve of intersection can be parameterized in $\mathbb{Q}[\xi, \sqrt{\Delta}]$ (with $\Delta \in \mathbb{Q}[\xi]$) there exists a quadric Q of the pencil with rational coefficients containing a rational line and whose discriminant is a square in $\mathbb{Q}$, by Lemmas 7.3 and 7.4. The quadric Q contains a line and thus intersects any plane. Consider any plane $w = c \in \mathbb{Q}$. Since the intersection of a rational line with a rational plane is (or contains) a rational point, the intersection of Q with plane $w = c$ contains a rational point $\mathbf{x} = (x, y, z, c)^T$. The quadric (Q) of the pencil containing that point has associated matrix $(\mathbf{x}^T T\mathbf{x})\, S - (\mathbf{x}^T S\mathbf{x})\, T$ and its determinant is a square. Hence Equation (6) admits a rational solution. □

Unfortunately, the question underlying the above optimality test is not within the range of problems that can currently been answered by algebraic number theory. Indeed, it is not known whether the general problem of determining if an algebraic set contains rational points (known, over $\mathbb{Z}$, as

Hilbert's 10th problem) is decidable [21]. It is known that this problem is decidable for genus zero curves and, under certain conditions, for genus one curves [21], but, for varieties of dimension two or more, very little has been proved on the problem of computing rational points.

The above theorem thus implies that computing parameterizations of the intersections of two arbitrary quadrics that are always optimal in the number of radicals is currently out of reach.

However, in some particular cases, we can use the following corollary to Theorem 7.6 to prove that $\sqrt{\delta}$ cannot be avoided.

Corollary 7.7. *If the intersection C of Q_S and Q_T is a nonsingular quartic and the rational hyperelliptic quartic curve $\sigma^2 = \det(S + \lambda T)$ has no rational point, then the parameterization of C in $\mathbb{Q}(\sqrt{\delta})[\xi, \sqrt{\Delta}]$ with $\Delta \in \mathbb{Q}(\sqrt{\delta})[\xi]$ is optimal in the number of radicals.*

We use this corollary in the next section.

7.3 Worst case examples

We prove here that there are pairs of quadrics, intersecting in the different types of real smooth quartic, such that (6) has no rational solution.

Recall that a set of points L of $\mathbb{P}^3$ is called *affinely finite* if there exists a projective plane P such that $P \cap L = \emptyset$. L is called *affinely infinite* otherwise. In [29], Tu, Wang and Wang prove that a real smooth quartic can be of three different morphologies according to the number of real roots of the determinantal equation. This result, completed with the distinction between the two possible cases when the four roots of the determinantal equation are real, is embodied in the following theorem.

Theorem 7.8. *Let Q_S and Q_T be two quadrics intersecting in $\mathbb{C}$ in a smooth quartic C. C can be classified as follows:*

- *If $\mathcal{D}(\lambda, \mu)$ has four real roots, then C has either two real affinely finite connected components or is empty.*
- *If $\mathcal{D}(\lambda, \mu)$ has two real roots and two complex roots, then C has one real affinely finite connected component.*
- *If $\mathcal{D}(\lambda, \mu)$ has four complex roots, then C has two real affinely infinite connected components.*

7.3.1 Two real affinely finite components

We first look at the case where the quartic has two real affinely finite components and start with a preliminary lemma.

Lemma 7.9. *The equation*

$$y^2 = ax^4 + bx^2 + c + d(x^3 + x) \tag{7}$$

has no rational solution if $a, c \equiv 3 \pmod 8$, $b \equiv 7 \pmod 8$ and $d \equiv 4 \pmod 8$.

Proof. Assume for a contradiction that (x, y) is a rational solution to (7). We can write $x = X/Z$ and $y = Y/Z^2$, where X, Y, Z are integers, $Z \neq 0$ and X, Z are mutually prime (so are not both even).

Consider first the reduction of Equation (7) modulo 8:

$$Y^2 \equiv 3X^4 + 7X^2Z^2 + 3Z^4 + 4XZ(X^2 + Z^2) \pmod 8.$$

If both X and Z are odd, X^2 and Z^2 are equal to $1 \pmod 8$. Thus $4(X^2 + Z^2) \equiv 0 \pmod 8$ and $Y^2 \equiv 3 + 7 + 3 \equiv 5 \pmod 8$, contradicting the fact that $Y^2 \equiv 0, 1$ or $4 \pmod 8$, for all integers Y.

If X and Z are not both odd, one of X^2 and Z^2 is equal to $0 \pmod 4$ and the other is equal to $1 \pmod 4$. The reduction of Equation (7) modulo 4 thus gives $Y^2 \equiv 3 \pmod 4$, contradicting the fact that $Y^2 \equiv 0$ or $1 \pmod 4$, for all integers Y. □

Proposition 7.10. *Consider the following pair of quadrics intersecting in a smooth quartic with two real affinely finite components:*

$$Q_S : 5y^2 + 6xy + 2z^2 - w^2 + 6zw = 0,$$
$$Q_T : 3x^2 + y^2 - z^2 - w^2 = 0.$$

Then the square root $\sqrt{\delta}$ is necessary to parameterize the curve of intersection.

Proof. The determinantal equation has four simple real roots and we find a quadric of inertia $(2, 2)$ in each of the intervals on which it is positive (in fact Q_S and Q_T are representative quadrics in these intervals). Thus, by Theorem 7.8, the intersection of Q_S and Q_T is a real smooth quartic with two affinely finite components.

We now apply Corollary 7.7 and show that the square root $\sqrt{\delta}$ is necessary to parameterize the curve of intersection. We have:

$$\begin{aligned}\sigma^2 &= \det(S + \lambda T),\\ &= 3\lambda^4 + 12\lambda^3 - 57\lambda^2 - 156\lambda + 99,\\ &\equiv 3\lambda^4 + 7\lambda^2 + 3 + 4(\lambda^3 + \lambda) \pmod 8,\end{aligned}$$

which has no rational solution by Lemma 7.9, so $\sqrt{\delta}$ cannot be avoided. □

7.3.2 One real affinely finite component

As above, we prove a preliminary lemma.

Lemma 7.11. *The equation*

$$y^2 = ax^4 + bx^3 + cx^2 + dx + e \tag{8}$$

has no rational solution if $a, e \equiv 2 \pmod 4$, $b, d \equiv 0 \pmod 4$ and $c \equiv 3 \pmod 4$.

Proof. As before, we assume for a contradiction that (8) has a rational solution (x, y) and write $x = X/Z$ and $y = Y/Z^2$, where X, Y, Z are integers, $Z \neq 0$ and X, Z are mutually prime (so are not both even). We consider the reduction of Equation (8) modulo 4:

$$Y^2 = 2X^4 + 3X^2Z^2 + 2Z^4.$$

If X and Z are not both odd, then $Y^2 \equiv 2 \pmod 4$. If both X and Z are odd, then $Y^2 \equiv 3 \pmod 4$. In both cases, we have a contradiction since $Y^2 \equiv 0$ or $1 \pmod 4$, for all integers Y. □

We can now prove the following.

Proposition 7.12. *Consider the following pair of quadrics intersecting in a smooth quartic with one real affinely finite component:*

$$Q_S : 2x^2 - 2xy + 2xz - 2xw + y^2 + 4yz - 4yw + 2z^2 - 4zw = 0,$$
$$Q_T : x^2 - 2xy + 4xz + 4xw - y^2 + 2yz + 4yw + 4zw - 2w^2 = 0.$$

Then the square root $\sqrt{\delta}$ is necessary to parameterize the curve of intersection.

Proof. The determinantal equation has two simple real roots so it is immediate that the intersection of Q_S and Q_T is a real smooth quartic with one affinely finite component, by Theorem 7.8.

We again apply Corollary 7.7 and show that the square root $\sqrt{\delta}$ is necessary to parameterize the curve of intersection. We have:

$$\begin{aligned}\sigma^2 &= \det(S + \lambda T),\\ &= 22\lambda^4 + 48\lambda^3 - 9\lambda^2 + 60\lambda + 30,\\ &\equiv 2\lambda^4 + 3\lambda^2 + 2 \pmod 4,\end{aligned}$$

which has no rational solution by Lemma 7.11, so $\sqrt{\delta}$ cannot be avoided. □

7.3.3 Two real affinely infinite components

We again prove a preliminary result.

Lemma 7.13. *The equation*

$$y^2 = a(x^4 + x + 1) + bx^3 + cx^2 \tag{9}$$

has no rational solution if $a \equiv 2 \pmod 4$, $b \equiv 0 \pmod 4$ and $c \equiv 1 \pmod 4$.

Proof. We proceed as in Lemmas 7.9 and 7.11, and consider the reduction of Equation (9) modulo 4:

$$Y^2 = 2X^4 + X^2Z^2 + 2XZ^3 + 2Z^4.$$

If X is even and Z is odd, the equation reduces to $Y^2 = 2XZ + 2 \equiv 2 \pmod 4$. If X is odd and Z is even, we also have $Y^2 \equiv 2 \pmod 4$. Finally, if both X and Z are odd, (9) reduces to $Y^2 = 1 + 2XZ \equiv 3 \pmod 4$. In all cases, we have a contradiction since $Y^2 \equiv 0$ or $1 \pmod 4$, for all integers Y. □

This is enough to prove the following.

Proposition 7.14. *Consider the following pair of quadrics intersecting in a smooth quartic with two real affinely infinite components:*

$$Q_S : x^2 - 2y^2 + 4zw = 0,$$
$$Q_T : xy + z^2 + 2zw - w^2 = 0.$$

Then the square root $\sqrt{\delta}$ is necessary to parameterize the curve of intersection.

Proof. The determinantal equation has four simple complex roots so it is immediate that the intersection of Q_S and Q_T is a real smooth quartic with two affinely infinite components, by Theorem 7.8.

We again apply Corollary 7.7. We have:

$$\begin{aligned}\sigma^2 &= \det(S + \lambda T),\\ &= 2\lambda^4 + 4\lambda^3 + 5\lambda^2 + 2\lambda + 2,\\ &\equiv 2\lambda^4 + \lambda^2 + 2\lambda + 2 \pmod 4,\end{aligned}$$

which has no rational solution by Lemma 7.13, so $\sqrt{\delta}$ cannot be avoided. □

8 Examples

We now give several examples of computing a parameterization of the intersection in case the intersection of two quadrics is a smooth quartic. The examples presented cover the range of morphologies discussed in the previous section and illustrate all aspects of optimality and near-optimality. For more examples, see Part IV [12]. All parameterizations have been computed with a C++ implementation of our intersection software (see Part IV).

8.1 Example 1

Our first example consists of the quadrics given in Output 1. The gcd of the partial derivatives of the determinantal equation is 1, so the intersection consists of a (possibly complex) smooth quartic. Since the determinantal equation is found to have four real roots, the intersection, over the reals, is either empty or made of two real affinely finite components (Theorem 7.8). We find a sample quadric in each of the intervals on which $\mathcal{D}(\lambda,\mu)$ is positive and compute its inertia. In the first interval, we find a quadric of inertia $(2,2)$ so we proceed. In the second interval, we find a quadric of inertia $(4,0)$. By Theorem 4.3, we conclude the intersection is empty of real points.

8.2 Example 2

Our second example is as in Output 2. The gcd of the two partial derivatives of the determinantal equation is 1, so the intersection (over $\mathbb{C}$) is a smooth quartic. The fact that the determinantal

Output 1 Execution trace for Example 1.

```
>> quadric 1: 6*x*y + 5*y^2 + 2*z^2 + 6*z*w - w^2
>> quadric 2: 3*x^2 + y^2 - z^2 + 11*w^2

>> launching intersection
>> determinantal equation: 33*l^4 - 124*l^3*m + 137*l^2*m^2 - 32*l*m^3 - 11*m^4
>> gcd of derivatives of determinantal equation: 1
>> number of real roots: 4
>> intervals: ]-4, 0[, ]0, 1[, ]2/2^1, 3/2^1[, ]3/2^1, 4/2^1[
>> picked test point 1 at [ -4 1 ], sign > 0 -- inertia [ 2 2 ] found
>> picked test point 2 at [ 1 1 ], sign > 0 -- inertia [ 4 0 ] found
>> complex intersection: smooth quartic
>> real intersection: empty
>> end of intersection

>> time spent: 10 ms
```

Output 2 Execution trace for Example 2.

```
>> quadric 1: x^2 - x*y - y^2 - y*w + z^2 + w^2
>> quadric 2: 2*x^2 - x*y + y^2 - y*z + y*w + z^2

>> launching intersection
>> determinantal equation: - 6*l^4 - 12*l^3*m + 3*l^2*m^2 + 6*l*m^3 - 2*m^4
>> gcd of derivatives of determinantal equation: 1
>> complex intersection: smooth quartic
>> real intersection: smooth quartic, one real affinely finite component
>> number of real roots: 2
>> intervals: ]-2, -1[, ]-1, 0[
>> picked test point 1 at [ -1 1 ], sign > 0 -- inertia [ 2 2 ] found
>> quadric (2,2) found: x^2 + 2*y^2 - y*z + 2*y*w - w^2
>> decomposition of its determinant [a,b] (det = a^2*b): [ 2 1 ]
>> a point on the quadric: [ 0 0 1 0 ]
>> param of quadric (2,2): [- s*u + t*v, - 2*s*v, (2*s + 2*t)*u + (- 4*s + 2*t)*v, s*u + t*v]
>> status of smooth quartic param: optimal
>> end of intersection

>> parameterization of smooth quartic, branch 1:
[- 4*u^3 + u^2*v + 6*u*v^2 + 2*v^3 - u*sqrt(Delta), - 6*u^3 - 8*u^2*v - 4*u*v^2, - 4*u^3
 + 2*u^2*v + (2*u + 2*v)*sqrt(Delta), 4*u^3 + 5*u^2*v + 2*u*v^2 + 2*v^3 + u*sqrt(Delta)]
>> parameterization of smooth quartic, branch 2:
[- 4*u^3 + u^2*v + 6*u*v^2 + 2*v^3 + u*sqrt(Delta), - 6*u^3 - 8*u^2*v - 4*u*v^2, - 4*u^3
 + 2*u^2*v + (- 2*u - 2*v)*sqrt(Delta), 4*u^3 + 5*u^2*v + 2*u*v^2 + 2*v^3 - u*sqrt(Delta)]
Delta = - 2*u^4 + 10*u^3*v - 9*u^2*v^2 - 8*u*v^3 - 2*v^4

>> time spent: 10 ms
```

equation has two real roots implies that the smooth quartic is real and that it consists of one affinely finite component (Theorem 7.8). Here, the two input quadrics have inertia $(3,1)$ and a first quadric Q_R of inertia $(2,2)$ is found in the pencil between the two roots of $\mathcal{D}$. A point is taken on Q_R and then approximated by a point with integer coordinates. It turns out that the approximation, i.e. $(0,0,1,0)$, also lies on Q_R. We thus use this quadric to parameterize the intersection. Since the determinant of Q_R is a square, it can be rationally parameterized (Proposition 6.6). The end of the calculation is as in Section 4.

Output 3 Execution trace for Example 3.

```
>> quadric 1: 19*x^2 + 22*y^2 + 21*z^2 - 20*w^2
>> quadric 2: x^2 + y^2 + z^2 - w^2

>> launching intersection
>> determinantal equation: - 175560*l^4 - 34358*l^3*m - 2519*l^2*m^2 - 82*l*m^3 - m^4
>> gcd of derivatives of determinantal equation: 1
>> number of real roots: 4
>> intervals: ]-14/2^8, -13/2^8[, ]-26/2^9, -25/2^9[, ]-25/2^9, -24/2^9[, ]-3/2^6, -2/2^6[
>> picked test point 1 at [ -13 256 ], sign > 0 -- inertia [ 2 2 ] found
>> picked test point 2 at [ -3 64 ], sign > 0 -- inertia [ 2 2 ] found
>> complex intersection: smooth quartic
>> real intersection: smooth quartic, two real affinely finite components
>> quadric (2,2) found: - 16*x^2 + 5*y^2 - 2*z^2 + 9*w^2
>> decomposition of its determinant [a,b] (det = a^2*b): [ 12 10 ]
>> a point on the quadric: [ 3 0 0 4 ]
>> param of quadric (2,2): [0, - 24*s*u - 24*t*v, 0, 0] + sqrt(10)*[3*t*u + 6*s*v, 0,
   12*s*u - 12*t*v, - 4*t*u + 8*s*v]
>> status of smooth quartic param: near-optimal
>> end of intersection

>> parameterization of smooth quartic, branch 1:
[(72*u^3 + 4*u*v^2)*sqrt(10) + 3*v*sqrt(10)*sqrt(Delta), - 340*u^2*v + 10*v^3
 - 24*u*sqrt(Delta), (- 118*u^2*v + 5*v^3)*sqrt(10) + 12*u*sqrt(10)*sqrt(Delta),
 (96*u^3 - 12*u*v^2)*sqrt(10) - 4*v*sqrt(10)*sqrt(Delta)]
>> parameterization of smooth quartic, branch 2:
[(72*u^3 + 4*u*v^2)*sqrt(10) - 3*v*sqrt(10)*sqrt(Delta), - 340*u^2*v + 10*v^3
 + 24*u*sqrt(Delta), (- 118*u^2*v + 5*v^3)*sqrt(10) - 12*u*sqrt(10)*sqrt(Delta),
 (96*u^3 - 12*u*v^2)*sqrt(10) + 4*v*sqrt(10)*sqrt(Delta)]
Delta = 20*u^4 - 140*u^2*v^2 + 5*v^4

>> time spent: 10 ms
```

8.3 Example 3

Our third example is Example 5 from [34]. It is the intersection of a sphere and an ellipsoid that are very close to one another. The output of our implementation on that example is shown in Output 3. Since the determinantal equation has four simple real roots, the intersection is either empty or made of two real affinely finite components (Theorem 7.8). Picking a sample quadric in each of the intervals on which $\det R(\lambda,\mu)$ is positive shows that the pencil contains no quadric of inertia $(4,0)$, so the quartic is real. Here, the determinant of the quadric of inertia $(2,2)$ used to parameterize the intersection is not a square, so the parameterization of the quartic contains the square root of some integer. It is thus only near-optimal in the sense that this square root can possibly be avoided.

It turns out that in this particular example it can be avoided. Consider the cone Q_R corresponding to the rational root $(\lambda_0,\mu_0)=(-1,21)$ of the determinantal equation:

$$Q_R : -Q_S + 21\,Q_T = 2x^2 - y^2 - w^2.$$

Q_R contains the obvious rational point $(1,1,0,1)$, which is not its singular point. It implies that it can be rationally parameterized by Proposition 6.3. Plugging this parameterization in the equation

Output 4 Execution trace for Example 4.

```
>> quadric 1: x^2 - 2*y^2 + 4*z*w
>> quadric 2: x*y + z^2 + 2*z*w - w^2

>> launching intersection
>> determinantal equation: 2*l^4 + 4*l^3*m + 5*l^2*m^2 + 2*l*m^3 + 2*m^4
>> gcd of derivatives of determinantal equation: 1
>> number of real roots: 0
>> complex intersection: smooth quartic
>> real intersection: smooth quartic, two real affinely infinite components
>> quadric (2,2) found: x*y + z^2 + 2*z*w - w^2
>> decomposition of its determinant [a,b] (det = a^2*b): [ 2 2 ]
>> a point on the quadric: [ 1 0 0 0 ]
>> param of quadric (2,2): [4*t*u, - 2*s*v, s*u + t*v, s*u + t*v]
   + sqrt(2)*[0, 0, 0, - s*u + t*v]
>> status of smooth quartic param: near-optimal
>> end of intersection

>> parameterization of smooth quartic, branch 1:
[- 4*u*v^2 + 4*v*sqrt(Delta), - 2*u^3 - 8*u*v^2 + 2*u^3*sqrt(2), 4*v^3 - u^2*v*sqrt(2)
 + u*sqrt(Delta), - 2*u^2*v + 4*v^3 + (u^2*v + 4*v^3)*sqrt(2) + (u - u*sqrt(2))*sqrt(Delta)]
>> parameterization of smooth quartic, branch 2:
[- 4*u*v^2 - 4*v*sqrt(Delta), - 2*u^3 - 8*u*v^2 + 2*u^3*sqrt(2), 4*v^3 - u^2*v*sqrt(2)
 - u*sqrt(Delta), - 2*u^2*v + 4*v^3 + (u^2*v + 4*v^3)*sqrt(2) - (u - u*sqrt(2))*sqrt(Delta)]
Delta = 2*u^4 + 10*u^2*v^2 - 4*v^4 + (- 2*u^4 - 4*v^4)*sqrt(2)

>> time spent: 10 ms
```

of Q_S or Q_T gives a simple parameterization for the smooth quartic:

$$\mathbf{X}(u,v) = \begin{pmatrix} u^2+2v^2 \\ 2uv \\ u^2-2v^2 \\ 0 \end{pmatrix} \pm \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \sqrt{2u^4+4u^2v^2+8v^4}.$$

8.4 Example 4

Our last example is the one of Proposition 7.14. The result in shown in Output 4. Here, again, the gcd of the partial derivatives of the determinantal equation is 1, so the intersection curve is, over $\mathbb{C}$, a smooth quartic. But since $\mathcal{D}(\lambda,\mu)$ has in fact no real root, we know by Theorem 7.8 that the smooth quartic is real and has two affinely infinite components. Here, the intermediate quadric Q_R of inertia $(2,2)$ found (which is in fact Q_T) is such that its determinant is not a square. So the parameterization of the quartic contains a square root. Our implementation cannot decide whether this square root is needed or not, so outputs that the parameterization is near-optimal. In this particular example, we know in fact that the parameterization is optimal, by Proposition 7.14.

9 Conclusion

The generic algorithm introduced in Section 4 already represents a substantial improvement over Levin's pencil method and its subsequent refinements. Indeed, we proved that, when the intersection is a smooth quartic (the generic case) our algorithm computes a parameterization which is optimal in the number of radicals involved up to one possibly unnecessary square root. We also showed that deciding (in all cases) whether this extra square root can be avoided is out of reach, and that the parameterization is optimal in some cases. Moreover, for the first time, our algorithms enable to compute in practice an exact form of the parameterization of two arbitrary quadrics with rational coefficients.

Even though this first part of our paper has focused on the generic, smooth quartic case, this algorithm can also be used when the intersection is singular. Assume the intermediate quadric Q_R has inertia $(2,2)$. When the curve of intersection consists of a cubic and a line, the equation Ω in the parameters has a cubic factor of bidegree $(2,1)$ and a linear factor of bidegree $(0,1)$, in view of Fact 7.2. Similarly, when the curve of intersection consists of a conic and two lines, Ω factors in a quadratic factor of bidegree $(1,1)$ and two linear factors of bidegree $(1,0)$ and $(0,1)$. Thus, assuming we know how to factor Ω, we have a way to parameterize each component of the intersection.

Unfortunately, this does not always lead to a parameterization of the intersection that involves only rational functions. When the intersection C is a singular quartic, Ω is irreducible since C itself is, and solving Ω for s in terms of u (or the converse) introduces the square root of a polynomial, while we know that there exists a parameterization of C with rational functions (the genus of the curve is 0).

Always computing parameterizations with rational functions when such parameterizations are known to exist will necessitate rethinking the basic philosophy of our algorithm. Essentially, while the idea of the generic algorithm is to use the rational quadric with *largest* rank as intermediate quadric for parameterizing the intersection, the refined method will instead use the rational quadric with *smallest* rank as intermediate quadric.

Proceeding that way will have the double benefit of always computing the simplest possible parameterizations and much better controlling the size of their coefficients. The price to pay is a multiplication of the cases and the writing of a dedicated piece of software for each (real projective) type of intersection. This is the subject of Parts II and III of this paper.

A The parameterizations of Table 3 are proper

We prove in this section that the parameterizations of Table 3 are not only proper parameterizations of the projective quadrics (in the sense that they define one-to-one correspondences between a dense open subset of the space of the parameters and a dense open subset of the quadric) but they are bijections between the space of the parameters and the quadric. The following two lemmas deal with the parameterizations of quadrics of inertia $(2,2)$ and $(2,1)$. For other types of quadrics, it is straightforward to show that the parameterizations of Table 3 are bijections.

Lemma A.1. $(u,v),(s,t) \mapsto (\frac{ut+a_1vs}{a_1}, \frac{us-a_2vt}{a_2}, \frac{ut-a_1vs}{\sqrt{a_1a_3}}, \frac{us+a_2vt}{\sqrt{a_2a_4}})$ *is a bijection from* $\mathbb{P}^1 \times \mathbb{P}^1$ *onto the surface* $\{(x_1,x_2,x_3,x_4) \in \mathbb{P}^3 \mid a_1x_1^2 + a_2x_2^2 - a_3x_3^2 - a_4x_4^2 = 0\}$*, where* a_1,a_2,a_3,a_4 *are positive.*

Proof. To prove this lemma, we apply the change of coordinates in $\mathbb{P}^3$

$$X = \frac{a_1x_1 + \sqrt{a_1a_3}x_3}{2},\ Y = \frac{a_1x_1 - \sqrt{a_1a_3}x_3}{2a_1},\ Z = \frac{a_2x_2 + \sqrt{a_2a_4}x_4}{2},\ W = \frac{-a_2x_2 + \sqrt{a_2a_4}x_4}{2a_2},$$

or equivalently

$$x_1 = \frac{X + a_1 Y}{a_1},\ x_3 = \frac{X - a_1 Y}{\sqrt{a_1 a_3}},\ x_2 = \frac{Z - a_2 W}{a_2},\ x_4 = \frac{Z + a_2 W}{\sqrt{a_2 a_4}}.$$

In the new frame, the equation of the surface is $XY - ZW = 0$ and the map becomes

$$\Phi : (u,v),(s,t) \mapsto (X,Y,Z,W) = (ut, vs, us, vt).$$

The map Φ is clearly a map from $\mathbb{P}^1 \times \mathbb{P}^1$ into $\mathbb{P}^3$ because $\Phi((\lambda u, \lambda v),(\mu s, \mu t)) = \lambda\mu\Phi((u,v),(s,t))$ and $\Phi((u,v),(s,t)) = (0,0,0,0)$ if and only if $(u,v) = (0,0)$ or $(s,t) = (0,0)$. Moreover, the image of Φ is clearly included in the surface of equation $XY - ZW = 0$. Conversely, if (X,Y,Z,W) is a point of this surface, at least one of its coordinates is non zero (we are in a projective space), and by symmetry we may suppose that $X \neq 0$. Considering $(X,Z,W) = (ut,us,vt)$, we have $ut \neq 0$, $\frac{Z}{X} = \frac{s}{t}$, and $\frac{W}{X} = \frac{v}{u}$. Thus $\frac{Z}{X}$ uniquely defines (s,t) up to a constant factor and similarly for $\frac{W}{X}$ and (u,v), which shows the injectivity of Φ. Furthermore, $XY - ZW = 0$ implies $Y = \frac{ZW}{X} = \frac{us\ vt}{ut} = vs$ which shows that Φ is surjective. □

Recall that $\mathbb{P}^{\star 2}$ denotes the quasi-projective space defined as the quotient of $\mathbb{R}^3 \setminus \{0,0,0\}$ by the equivalence relation $\sim$ where $(x_1,x_2,x_3) \sim (y_1,y_2,y_3)$ if and only if $\exists \lambda \in \mathbb{R} \setminus \{0\}$ such that $(x_1,x_2,x_3) = (\lambda y_1, \lambda y_2, \lambda^2 y_3)$.

Lemma A.2. *$(u,v,s) \mapsto (uv, \frac{u^2 - a_1 a_2 v^2}{2a_2}, \frac{u^2 + a_1 a_2 v^2}{2\sqrt{a_2 a_3}}, s)$ is a bijection from $\mathbb{P}^{\star 2}$ onto the surface $\{(x_1,x_2,x_3,x_4) \in \mathbb{P}^3 \mid a_1 x_1^2 + a_2 x_2^2 - a_3 x_3^2 = 0\}$, where a_1, a_2, a_3 are positive.*

Proof. For this lemma, we consider the change of coordinates in $\mathbb{P}^3$

$$X = x_1,\ Y = \sqrt{a_2 a_3} x_3 + a_2 x_2,\ Z = \frac{\sqrt{a_2 a_3} x_3 - a_2 x_2}{a_1 a_2},\ W = x_4,$$

or equivalently

$$x_1 = X,\ x_2 = \frac{Y - a_1 a_2 Z}{2a_2},\ x_3 = \frac{Y + a_1 a_2 Z}{2\sqrt{a_2 a_3}},\ x_4 = W.$$

In the new frame, the equation of the surface is $X^2 - YZ = 0$ and the map becomes

$$\Psi : (u,v,s) \mapsto (X,Y,Z,W) = (uv, u^2, v^2, s).$$

The map Ψ is clearly a map from $\mathbb{P}^{\star 2}$ into $\mathbb{P}^3$ because $\Psi(\lambda u, \lambda v, \lambda^2 s) = \lambda^2 \Psi(u,v,s)$ and $\Psi(u,v,s) = (0,0,0,0)$ if and only if $(u,v,s) = (0,0,0)$. Moreover, the image of Ψ is clearly included in the surface of equation $X^2 - YZ = 0$. Conversely, if (X,Y,Z,W) is a point of this surface, then we have to prove that its preimage consists in exactly one point of $\mathbb{P}^{\star 2}$. If $Y = Z = 0$, we have also $X = 0$ and a point of the preimage should satisfy $u = v = 0$; it is therefore unique (in $\mathbb{P}^{\star 2}$) an it exists by $\Psi(0,0,W) = (0,0,0,W)$.

If Y or Z is nonzero, we may suppose by symmetry that $Y \neq 0$. Considering $(X,Y,W) = (uv, u^2, s)$ we have $u \neq 0$, $\frac{X}{Y} = \frac{v}{u}$ and $\frac{W}{Y} = \frac{s}{u^2}$. Thus $\frac{X}{Y}$ and $\frac{W}{Y}$ uniquely define $(u,v,s) \in \mathbb{P}^{\star 2}$ which implies that Ψ is injective. Furthermore, $YZ = X^2$ implies $Z = \frac{X^2}{Y} = \frac{(uv)^2}{u^2} = v^2$ which shows that Ψ is surjective. □

Remark A.3. Although the statements and the proofs of Lemma A.1 and A.2 are very similar, there is a big difference between the two bijections: the bijection is an isomorphism and a diffeomorpism in Lemma A.1 but not in Lemma A.2 where the space of the parameters is smooth while the surface is singular at (0,0,0,1).

References

[1] E. Berberich, M. Hemmer, L. Kettner, E. Schömer, and N. Wolpert. An exact, complete and efficient implementation for computing planar maps of quadric intersection curves. In *Proc. of SoCG (ACM Symposium on Computational Geometry), San Diego*, pages 99–115, 2005.

[2] T. Bromwich. *Quadratic Forms and Their Classification by Means of Invariant Factors*. Cambridge Tracts in Mathematics and Mathematical Physics, 1906.

[3] L. Dupont, D. Lazard, S. Lazard, and S. Petitjean. Near-optimal parameterization of the intersection of quadrics. In *Proc. of SoCG (ACM Symposium on Computational Geometry), San Diego*, pages 246–255, 2003.

[4] L. Dupont, D. Lazard, S. Lazard, and S. Petitjean. Near-optimal parameterization of the intersection of quadrics: II. A classification of pencils. Research Report n^{o} 5668, INRIA, September 2005.

[5] L. Dupont, D. Lazard, S. Lazard, and S. Petitjean. Near-optimal parameterization of the intersection of quadrics: III. Parameterizing singular intersections. Research Report n^{o} 5669, INRIA, September 2005.

[6] R. Farouki, C. Neff, and M. O'Connor. Automatic parsing of degenerate quadric-surface intersections. *ACM Transactions on Graphics*, 8(3):174–203, 1989.

[7] P. Finsler. Über das Vorkommen definiter und semidefiniter Formen in Scharen quadratischer Formen. *Comment. Math. Helv.*, 9:188–192, 1936/1937.

[8] R. Goldman and J. Miller. Combining algebraic rigor with geometric robustness for the detection and calculation of conic sections in the intersection of two natural quadric surfaces. In *Proc. of ACM Symposium on Solid Modeling Foundations and CAD/CAM Applications*, pages 221–231, 1991.

[9] C.-K. Hung and D. Ierardi. Constructing convex hulls of quadratic surface patches. In *Proceedings of 7th CCCG (Canadian Conference on Computational Geometry), Québec, Canada*, pages 255–260, 1995.

[10] J. Keyser, T. Culver, M. Foskey, S. Krishnan, and D. Manocha. ESOLID: A system for exact boundary evaluation. *Computer-Aided Design*, 36(2):175–193, 2004.

[11] T. Lam. *The Algebraic Theory of Quadratic Forms*. W.A. Benjamin, Reading, MA, 1973.

[12] S. Lazard, L. M. Peñaranda, and S. Petitjean. Near-optimal parameterization of the intersection of quadrics: IV. An efficient and exact implementation. Research Report n^o 5670, INRIA, September 2005.

[13] J. Levin. A parametric algorithm for drawing pictures of solid objects composed of quadric surfaces. *Communications of the ACM*, 19(10):555–563, 1976.

[14] J. Levin. Mathematical models for determining the intersections of quadric surfaces. *Computer Graphics and Image Processing*, 11(1):73–87, 1979.

[15] The Maple System. Waterloo Maple Software. `http://www.maplesoft.com`.

[16] J. Miller. Geometric approaches to nonplanar quadric surface intersection curves. *ACM Transactions on Graphics*, 6(4):274–307, 1987.

[17] J. Miller and R. Goldman. Geometric algorithms for detecting and calculating all conic sections in the intersection of any two natural quadric surfaces. *Graphical Models and Image Processing*, 57(1):55–66, 1995.

[18] B. Mourrain, J.-P. Técourt, and M. Teillaud. Sweeping of an arrangement of quadrics in 3D. *Computational Geometry: Theory and Applications*, 30:145–164, 2005. Special issue, 19th European Workshop on Computational Geometry.

[19] M. Namba. *Geometry of Projective Algebraic Curves*. Marcel Dekker, Inc., 1984.

[20] D. Perrin. *Géométrie algébrique : une introduction*. InterEditions, 1995.

[21] B. Poonen. Computing rational points on curves. In *Number Theory for the Millenium*. A. K. Peters, Boston, 2001. Proc. of Millennial Conference on Number Theory, 2000.

[22] A. Requicha and H. Voelcker. Solid modeling: a historical summary and contemporary assessment. *IEEE Computer Graphics and Applications*, 2(1):9–24, 1982.

[23] F. Rouillier and P. Zimmermann. Efficient isolation of polynomial's real roots. *Journal of Computational and Applied Mathematics*, 162(1):33–50, 2004.

[24] R. Sarraga. Algebraic methods for intersections of quadric surfaces in GMSOLID. *Computer Vision, Graphics, and Image Processing*, 22:222–238, 1983.

[25] E. Schömer and N. Wolpert. An exact and efficient approach for computing a cell in an arrangement of quadrics. *Computational Geometry: Theory and Applications*, 2005. Special Issue on Robust Geometric Algorithms and their Implementations, to appear.

[26] C.-K. Shene and J. Johnstone. Computing the intersection of a plane and a natural quadric. *Computer & Graphics*, 12(2):179–186, 1992.

[27] C.-K. Shene and J. Johnstone. On the lower degree intersections of two natural quadrics. *ACM Transactions on Graphics*, 13(4):400–424, 1994.

[28] C.-K. Shene and J. Johnstone. On the lower degree intersections of two natural quadrics. *ACM Transactions on Graphics*, 13(4):400–424, 1994.

[29] C. Tu, W. Wang, and J. Wang. Classifying the nonsingular intersection curve of two quadric surfaces. In *Proc. of GMP'02 (Geometric Modeling and Processing)*, pages 23–32, 2002.

[30] F. Uhlig. Simultaneous block diagonalization of two real symmetric matrices. *Linear Algebra and Its Applications*, 7:281–289, 1973.

[31] F. Uhlig. A canonical form for a pair of real symmetric matrices that generate a nonsingular pencil. *Linear Algebra and Its Applications*, 14:189–209, 1976.

[32] W. Wang, R. Goldman, and C. Tu. Enhancing Levin's method for computing quadric-surface intersections. *Computer-Aided Geometric Design*, 20(7):401–422, 2003.

[33] W. Wang, B. Joe, and R. Goldman. Rational quadratic parameterizations of quadrics. *International Journal of Computational Geometry and Applications*, 7(6):599–619, 1997.

[34] W. Wang, B. Joe, and R. Goldman. Computing quadric surface intersections based on an analysis of plane cubic curves. *Graphical Models*, 64(6):335–367, 2002.

[35] I. Wilf and Y. Manor. Quadric-surface intersection curves: shape and structure. *Computer-Aided Design*, 25(10):633–643, 1993.

[36] N. Wolpert. *An Exact and Efficient Algorithm for Computing a Cell in an Arrangement of Quadrics*. PhD thesis, Universität des Saarlandes, Saarbrücken, 2002.

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Near-Optimal Parameterization of the Intersection of Quadrics: II. A Classification of Pencils

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Near-Optimal Parameterization of the Intersection of Quadrics: II. A Classification of Pencils

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Projets Vegas et Salsa

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Abstract: While Part I [2] of this paper was devoted mainly to quadrics intersecting in a smooth quartic, we now focus on singular intersections. To produce optimal or near-optimal parameterizations in all cases, we first determine the the real type of the intersection before computing the actual parameterization.

In this second part, we present the first classification of pencils of quadrics based on the type of the real intersection and we show how this classification can be used to compute efficiently the type of the real intersection. The near-optimal parameterization algorithms in all singular cases will be given in Part III [3].

Key-words: Intersection of surfaces, quadrics, pencils of quadrics, classification, curve parameterization.

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Paramétrisation quasi-optimale de l'intersection de quadriques : II. Classification des faisceaux

Résumé : Alors que la partie I [2] de cet article s'est principalement focalisée sur les paires de quadriques dont l'intersection est une quartique lisse, nous nous concentrons maintenant sur les intersections singulières. Pour parvenir à l'obtention de paramétrages optimaux ou quasi-optimaux dans tous les cas, nous découplons la détermination du type réel de l'intersection du paramétrage proprement dit.

Dans cette seconde partie de notre article, nous présentons la première classification des faisceaux de quadriques basée sur le type de l'intersection réelle et nous montrons comment cette classification peut être utilisée pour calculer efficacement le type réel de l'intersection. Les algorithmes quasi-optimaux de paramétrage dans tous les cas sont présentés dans la partie III [3].

Mots-clés : Intersection de surfaces, quadriques, faisceaux de quadriques, classification, paramétrisation.

1 Introduction

At the end of Part I [2], we saw that the generic algorithm we introduced, while being simple and giving optimal parameterizations in some cases, fails to achieve the stated goal of computing (near-)optimal parameterizations (both in terms of functions and coefficients) of intersections of arbitrary quadrics.

Unfortunately, it turns out that achieving this goal involves more than simple adaptations to the generic algorithm. Reaching optimality implies looking carefully at *each* type of *real* intersection and designing a dedicated algorithm to handle each situation. For this, we need to understand precisely which situations can happen over the reals and thus classify real pencils of quadrics of $\mathbb{P}^3(\mathbb{R})$.

Classifying pencils of quadrics over the complexes was achieved by Segre in the nineteenth century [6]. Its practical value is however limited since its proper interpretation lies in the complex domain (i.e. points on the intersection might be real or complex), whereas our concern is with real parts of the intersection.

Accordingly, we refine the Segre classification of pencils of $\mathbb{P}^3(\mathbb{R})$ by examining the different cases occurring over the reals. This refinement is, in itself, of partial assistance for the parameterization problem: no more than the Segre classification can it be "reverse engineered" to construct explicit representations of the various intersection components. It is however mandatory for the following two reasons: it allows us to obtain structural information on the intersection curve which we use to drive the algorithm for computing a near-optimal parameterization of the intersection curve (Part III); it is also a prerequisite for proving the (near-)optimality of our parameterization algorithm.

In this second part of our paper, we present a new classification of pencils of quadrics based on the type of the real intersection. A summary of this classification is given in Tables 4 and 5. We then show how to use this classification to compute efficiently the type of the real intersection. In particular we show how computations with non-rational numbers can be avoided for detecting the type of the intersection when the input quadrics have rational coefficients.

It should be stressed that, even though the classification of pencils of the reals is presented here as an intermediate step in a more global process (i.e., parameterization of the intersection), this classification has an interest on its own. It could be used for instance in a collision detection context to predict at which time stamps a collision between two moving objects will happen.

The rest of this part is organized as follows. Section 2 reviews the classical Segre classification of pencils of quadrics over the complexes. We then refine, in Sections 3 and 4, the Segre classification over the reals with a repeated application of the Canonical Pair Form Theorem for pairs of real symmetric matrices introduced in Part I. In Section 3, we consider *regular* pencils, i.e., pencils that contain a non-singular quadric, and, in Section 4, *singular* pencils, i.e., pencils that contain only singular quadrics, or, equivalently, pencils with identically vanishing determinantal equation. In Section 5, we use the results of the classification of pencils over the reals to design an algorithm to quickly and efficiently characterize the complex and real types of the intersection given two input quadrics. Several examples are detailed in Section 6, before concluding.

2 Classification of pencils of quadrics over the complexes

In this section, we review classical material on the classification of pencils of quadrics. It will serve as the starting point for our classification of pencils over the reals in Sections 3 and 4.

In the rest of the paper all quadrics are considered in real projective space $\mathbb{P}^3(\mathbb{R})$; their coefficients as well as the coefficients of the determinantal equations of pencils are thus real. However, we consider the intersection of quadrics both in $\mathbb{P}^3(\mathbb{R})$ and in $\mathbb{P}^3(\mathbb{C})$. Accordingly, the classification of pencils is considered both over the complexes and over the reals.

In the following, the parameters of the quadric parameterizations live in real projective spaces. For simplicity, parameter spaces $\mathbb{P}^n(\mathbb{R})$ are denoted $\mathbb{P}^n$.

We start in Section 2.1 with a proof that the existence of a singularity on the intersection curve is equivalent either to the existence of a multiple root in the determinantal equation or to the fact that the determinantal equation vanishes identically. Then Section 2.2 recalls the basic tenets of the classification of pencils over the complexes. The well-known Segre characteristic is recalled in Section 2.2.1 and its relation with the Canonical Pair Form Theorem for pairs of real symmetric matrices (Part I and [8, 9]) is thoroughly explained in Section 2.2.2.

2.1 Singular intersections and multiple roots

In the ensuing sections, we use the following equivalence for classifying the singular intersections through the multiplicities of the roots of the determinantal equation $\mathcal{D}(\lambda,\mu)$ and the rank of the corresponding quadrics.

Proposition 2.1. *If the intersection of two distinct quadrics Q_S and Q_T has a singular point* **p***, then*

- *either $\mathcal{D}\equiv 0$ and Q_S and Q_T are singular at* **p***,*
- *or $\mathcal{D}\equiv 0$ and there is a unique quadric Q_R of the pencil that is singular at* **p***,*
- *or $\mathcal{D}\not\equiv 0$, there is a unique quadric $Q_R=\lambda_0 Q_S+\mu_0 Q_T$ that is singular at* **p** *and (λ_0,μ_0) is a multiple root of $\mathcal{D}$.*

In the last two cases, all the quadrics of the pencil except Q_R share a common tangent plane at **p***.*

Proof. First recall that a curve C defined by implicit equations $Q_S=Q_T=0$ is singular at **p** if and only if **p** is on C and the rank of the Jacobian matrix J of C is strictly less than 2 when evaluated at **p**. J is the matrix of partial derivatives:

$$J=\begin{pmatrix} \frac{\partial Q_S}{\partial x} & \frac{\partial Q_S}{\partial y} & \frac{\partial Q_S}{\partial z} & \frac{\partial Q_S}{\partial w} \\ \frac{\partial Q_T}{\partial x} & \frac{\partial Q_T}{\partial y} & \frac{\partial Q_T}{\partial z} & \frac{\partial Q_T}{\partial w} \end{pmatrix}. \tag{1}$$

Let J_S and J_T be the first and second rows of J.

If all the coefficients of J vanish at **p**, then **p** is a singular point of both Q_S and Q_T and thus of all quadrics of the pencil, implying that $\mathcal{D}\equiv 0$.

Otherwise, J has rank one and there exists a linear relationship between the rows of J evaluated at $\mathbf{p}$:

$$\lambda_0 J_S|_{\mathbf{p}} + \mu_0 J_T|_{\mathbf{p}} = 0, \quad (\lambda_0, \mu_0) \in \mathbb{P}^1.$$

Also, there is a $\mathbf{v} \in \mathbb{P}^3(\mathbb{R})$ such that $J|_{\mathbf{p}}\mathbf{v}$ is a non-zero multiple of $\mathbf{v}$. This exactly means that the tangent plane at $\mathbf{p}$ of all the quadrics of the pencil is the plane P of equation $\mathbf{v} \cdot \begin{pmatrix} x & y & z & w \end{pmatrix}^T = 0$, except for the quadric $\lambda_0 Q_S + \mu_0 Q_T$. For this last quadric, all the partial derivatives at $\mathbf{p}$ vanish, implying that it is singular at $\mathbf{p}$ and has rank at most 3.

Now, we may change the generators of the pencil by taking $\lambda_0 S + \mu_0 T$ as first generator in place of S. This has the effect of translating (λ_0, μ_0) to $(1,0)$. If we change of frame in order that the coordinates of $\mathbf{p}$ become $(0,0,0,1)$ and that the equation of P becomes $x = 0$, the matrices of the generators of the pencil become

$$S' = \begin{pmatrix} * & * & * & 0 \\ * & \multicolumn{2}{c}{\multirow{2}{*}{A}} & 0 \\ * & & & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad T' = \begin{pmatrix} * & * & * & 1 \\ * & \multicolumn{2}{c}{\multirow{2}{*}{B}} & 0 \\ * & & & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

where A and B are 2×2 matrices and the stars denote any element. It follows immediately that

$$\det(\lambda S' + \mu T') = -\mu^2 \det(\lambda A + \mu B).$$

The case $\det(\lambda A + \mu B) \equiv 0$ proves the second assertion. The case $\det(\lambda A + \mu B) \not\equiv 0$ proves the last assertion. □

2.2 Classification of pencils by elementary divisors

For the reader's convenience, we review, in this section, the classical classification of pencils of quadrics as originally done by the Italian mathematician Corrado Segre [6]. More recent and accessible accounts can be found in [1, 5].

2.2.1 Segre characteristic

Assume we are given a pencil $R(\lambda, \mu) = \lambda S + \mu T$ of symmetric matrices of size n such that $\mathcal{D}(\lambda, \mu) = \det R(\lambda, \mu)$ is not identically zero. In general, $\mathcal{D}$ has n complex roots to which correspond n complex projective cones of the pencil. But there can be exceptions to this when a root (λ_0, μ_0) of $\mathcal{D}$ appears with multiplicity larger than 1. It can also happen that (λ_0, μ_0) makes not only the determinant $\mathcal{D}$ vanish but also all its subdeterminants of order $n - t + 1$ say, $t > 0$. This means that the corresponding quadric has as singular set a linear space of dimension $t - 1$.

Let the $(\lambda_i, \mu_i), i = 1, \ldots, p$, be the roots of $\mathcal{D}$ and the m_i their respective multiplicities. Indicate by m_i^j the minimum multiplicity with which the root (λ_i, μ_i) appears in the subdeterminants of order $n - j$ of $\mathcal{D}$. Let $t_i \geqslant 1$ be the smallest integer such that $m_i^{t_i} = 0$. We see that $m_i^j \geqslant m_i^{j+1}$ for all j. Define a sequence of indices e_i^j as follows:

$$e_i^j = m_i^{j-1} - m_i^j, \quad j = 1, \ldots, t_i,$$

with $m_i^0 = m_i$. The multiplicity m_i of (λ_i, μ_i) is the sum $e_i^1 + \cdots + e_i^{t_j}$. We have therefore:

$$\mathcal{D}(\lambda, \mu) = (\lambda\mu_i - \mu\lambda_i)^{m_i} \mathcal{D}^*(\lambda, \mu) = (\lambda\mu_i - \mu\lambda_i)^{e_i^1} \cdots (\lambda\mu_i - \mu\lambda_i)^{e_i^{t_i}} \mathcal{D}^*(\lambda, \mu),$$

where $\mathcal{D}^*(\lambda_i, \mu_i) \neq 0$.

The factors $(\lambda\mu_i - \mu\lambda_i)^{e_i^j}$ are called the *elementary divisors* and the exponents e_i^j the *characteristic numbers*, associated with the root (λ_i, μ_i). Their study goes back to Karl Weierstrass [10]. Segre introduced the following notation to denote the various characteristic numbers associated with the degenerate quadrics that appear in a pencil:

$$\sigma_n = [(e_1^1, \ldots, e_1^{t_1}), (e_2^1, \ldots, e_2^{t_2}), \ldots, (e_p^1, \ldots, e_p^{t_p})],$$

with the convention that the parentheses enclosing the characteristic numbers of (λ_i, μ_i) are dropped when $t_i = 1$. This is known as the *Segre characteristic* or *Segre symbol* of the pencil.

The following theorem, essentially due to Weierstrass [10], proves that a pencil of quadrics and the intersection it defines are uniquely and entirely characterized, over the complexes, by its Segre symbol.

Theorem 2.2 (Characterization by Segre symbol). *Consider two pencils of quadrics $R(\lambda_1, \mu_1) = \lambda_1 S_1 + \mu_1 T_1$ and $R(\lambda_2, \mu_2) = \lambda_2 S_2 + \mu_2 T_2$ in $\mathbb{P}^n(\mathbb{R})$. Suppose that $\det R(\lambda_1, \mu_1)$ and $\det R(\lambda_2, \mu_2)$ are not identically zero and let $(\lambda_{1,i}, \mu_{1,i})$ and $(\lambda_{2,i}, \mu_{2,i})$ be their respective roots. Then the two pencils are projectively equivalent if and only if they have the same Segre symbol and there is an automorphism of $\mathbb{P}^1(\mathbb{C})$ taking $(\lambda_{1,i}, \mu_{1,i})$ to $(\lambda_{2,i}, \mu_{2,i})$.*

With the above definition, we see that (λ_i, μ_i) is a root of all subdeterminants of $R(\lambda, \mu)$ of order $n - t_i + k, k > 0$, but not of any subdeterminant of order $n - t_i$. In other words, the rank r_i of $R(\lambda_i, \mu_i)$ is $n - t_i$. In addition, since $m_i = e_i^1 + \cdots + e_i^{t_i}$, we have that $n - 1 \geqslant r_i \geqslant n - m_i$. Enumerating all possible cases for the e_i^j subject to the constraints induced by its definition gives rise to all possible types of (complex) intersection and accompanying Segre symbols. Tables 1, 2, and 3 list the possible cases for pencils in $\mathbb{P}^3(\mathbb{R})$. Incidentally, we can see that the pair (m_i, r_i) is sufficient to characterize the pencil except in the case $(m_i, r_i) = (4, 2)$.

When the determinantal equation $\mathcal{D}(\lambda, \mu)$ vanishes identically, i.e., all the quadrics are singular (see Tables 2 and 3), the above theory does not apply directly. There are two cases, according to whether the quadrics of the pencil have singular points in common or not:

- When they do not, the pencil can be characterized by a different set of invariants the existence of which was originally proved by Kronecker. We do not detail here how this set is computed (but see [1, p. 55-60]). Suffice it to say that the cases $n = 4$ and $n = 3$ are characterized each by a single set of such invariants, designated by the strings $[1\{3\}]$ and $[\{3\}]$ respectively. In Section 4.1, we carry out the analysis of this situation when $n = 4$ without resorting to these special invariants.

- When the quadrics do have (at least one) singular point in common, say **p**, we may suppose, after a change of frame, that **p** has coordinates $(0, \ldots, 0, 1)$. In the new frame, the matrices

Segre characteristic σ_4	roots of $\mathcal{D}(\lambda,\mu)$ and rank of associated quadric	complex type of intersection
$[1111]$	four simple roots	smooth quartic
$[112]$	one double root, rank 3	nodal quartic
$[11(11)]$	one double root, rank 2	two secant conics
$[13]$	triple root, rank 3	cuspidal quartic
$[1(21)]$	triple root, rank 2	two tangent conics
$[1(111)]$	triple root, rank 1	double conic
$[4]$	quadruple root, rank 3	cubic and tangent line
$[(31)]$	quadruple root, rank 2	conic and two lines crossing on the conic
$[(22)]$	quadruple root, rank 2	two lines and a double line
$[(211)]$	quadruple root, rank 1	two double lines
$[(1111)]$	quadruple root, rank 0	any non-trivial quadric of the pencil
$[22]$	two double roots, both rank 3	cubic and secant line
$[2(11)]$	two double roots, ranks 3 and 2	conic and two lines not crossing on the conic
$[(11)(11)]$	two double roots, both rank 2	four skew lines

Table 1: Classification of pencils by Segre symbol in the case where $\mathcal{D}(\lambda,\mu)$ does not identically vanish. When the determinantal equation has multiple roots, the additional simple roots are not indicated: they correspond to rank 3 quadrics.

Segre characteristic σ_3	roots of $\mathcal{D}_3(\lambda,\mu)$ and rank of associated conic	complex type of intersection
$[1\{3\}]$	no common singular point	conic and double line
$[111]$	three simple roots	four concurrent lines
$[12]$	double root, rank 2	two lines and a double line
$[1(11)]$	double root, rank 1	two double lines
$[3]$	triple root, rank 2	line and triple line
$[(21)]$	triple root, rank 1	quadruple line
$[(111)]$	triple root, rank 0	any non-trivial quadric of the pencil
$[\{3\}]$	$\mathcal{D}_3(\lambda,\mu) \equiv 0$	line and plane

Table 2: Classification of pencils by Segre symbol in the case where $\mathcal{D}(\lambda,\mu) \equiv 0$ and the quadrics of the pencil have zero (top part) or one (bottom part) singular point $\mathbf{p}$ in common. $\mathcal{D}_3(\lambda,\mu)$ is the determinant of the 3×3 upper-left matrix of $R(\lambda,\mu)$ after a congruence transformation sending $\mathbf{p}$ to $(0,0,0,1)$. The conic associated with a root of $\mathcal{D}_3(\lambda,\mu)$ corresponds to the 3×3 upper-left matrix or $R(\lambda,\mu)$.

Segre characteristic σ_2	roots of $\mathcal{D}_2(\lambda,\mu)$ and rank of associated matrix	complex type of intersection
$[11]$	two simple roots	quadruple line
$[2]$	double root, rank 1	plane
$[(11)]$	double root, rank 0	any non-trivial quadric of the pencil
	$\mathcal{D}_2(\lambda,\mu) \equiv 0$	double plane

Table 3: Classification of pencils by Segre symbol in the case where the quadrics of the pencil have (at least) two singular points **p** and **q** in common (i.e., $\mathcal{D}_3(\lambda,\mu) \equiv 0$). $\mathcal{D}_2(\lambda,\mu)$ is the determinant of the 2×2 upper-left matrix of $R(\lambda,\mu)$ after a congruence transformation sending **p** and **q** to $(0,0,0,1)$ and $(0,0,1,0)$. The matrix associated with a root of $\mathcal{D}_2(\lambda,\mu)$ corresponds to the 2×2 upper-left matrix or $R(\lambda,\mu)$.

have their last row and column filled with zeros. To sort out the different types of intersection, we may identify the quadrics with their upper left $(n-1)\times(n-1)$ matrices and classify the restricted pencils by looking at the Segre symbol σ_{n-1} of their degree $n-1$ determinantal equation. This is what we have done in Table 2 for the case of quadrics in $\mathbb{P}^3(\mathbb{R})$.

The above process can be repeated by recursing on dimension.

2.2.2 From the complexes to the reals

Theorem 2.2 can be used to find a canonical form for a pencil of quadrics when $\mathcal{D}(\lambda,\mu)$ is not identically zero (see [5]). Consider the pencil $R(\lambda) = \lambda S - T$ and its determinantal equation $\mathcal{D}(\lambda)$, with roots λ_i of multiplicity m_i. Let

$$[(e_1^1,\ldots,e_1^{t_1}),(e_2^1,\ldots,e_2^{t_2}),\ldots,(e_p^1,\ldots,e_p^{t_p})]$$

be the Segre symbol of the pencil. Then there exists a change of coordinates in $\mathbb{P}^n(\mathbb{C})$ such that, in the new frame, the pencil writes down as $R'(\lambda) = \lambda S' - T'$, where

$$S' = \operatorname{diag}(E_1^1,\ldots,E_1^{t_1},\ldots,E_p^1,\ldots,E_p^{t_p}), \quad T' = \operatorname{diag}(E_1^1J_1^1,\ldots,E_1^{t_1}J_1^{t_1},\ldots,E_p^1J_p^1,\ldots,E_p^{t_p}J_p^{t_p})$$

are block diagonal matrices with blocks:

$$E_i^k = \begin{pmatrix} 0 & & 1 \\ & \iddots & \\ 1 & & 0 \end{pmatrix} \quad \text{and} \quad J_i^k = \begin{pmatrix} \lambda_i & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda_i \end{pmatrix}$$

of size e_i^k. The parentheses in the Segre symbol correspond one-to-one to the singular quadrics in the pencil. The root of $\mathcal{D}$ corresponding to a singular quadric of symbol $(e_i^1,\ldots,e_i^{t_i})$ has multiplicity $m_i = \sum_{k=1}^{t_i} e_i^k$.

Segre string	roots of $\mathcal{D}(\lambda,\mu)$	rank or inertia of $R(\lambda_1,\mu_1)$	rank or inertia of $R(\lambda_2,\mu_2)$	type of (λ_2,μ_2)	s	real type of intersection
[1111]	4 simple roots					smooth quartic or $\emptyset$; see Part I
[112]	1 double root	$(3,0)$		real		point
[112]	1 double root	$(2,1)$		real	$-$	nodal quartic; isolated node
[112]	1 double root	$(2,1)$		real	$+$	nodal quartic; convex sing.
[112]	1 double root	rank 3		complex		nodal quartic; concave sing.
[11(11)]	1 double root	$(2,0)$		real	$+$	$\emptyset$
[11(11)]	1 double root	$(2,0)$		real	$-$	two points
[11(11)]	1 double root	$(1,1)$	$(2,1)$	real	$-$	two non-secant conics
[11(11)]	1 double root	$(1,1)$	$(3,0)$	real	$-$	$\emptyset$
[11(11)]	1 double root	$(1,1)$		real	$+$	two secant conics; convex sing.
[11(11)]	1 double root	rank 2		complex	$-$	conic
[11(11)]	1 double root	rank 2		complex	$+$	two secant conics; concave sing.
[13]	triple root	rank 3				cuspidal quartic
[1(21)]	triple root	$(2,0)$				double point
[1(21)]	triple root	$(1,1)$				two tangent conics
[1(111)]	triple root	rank 1	$(2,1)$			double conic
[1(111)]	triple root	rank 1	$(3,0)$			$\emptyset$
[4]	quadruple root	rank 3				cubic and tangent line
[(31)]	quadruple root	$(1,1)$			$-$	conic
[(31)]	quadruple root	$(1,1)$			$+$	conic and two lines crossing on the conic
[(22)]	quadruple root	$(2,0)$				double line
[(22)]	quadruple root	$(1,1)$			$+$	two single lines & a double line
[(211)]	quadruple root	rank 1			$-$	point
[(211)]	quadruple root	rank 1			$+$	two secant double lines
[(1111)]	quadruple root	rank 0				any smooth quadric of the pencil
[22]	2 double roots	rank 3	rank 3	real		cubic and secant line
[22]	2 double roots	rank 3	rank 3	complex		cubic and non-secant line
[2(11)]	2 double roots	$(3,0)$	rank 2	real		point
[2(11)]	2 double roots	$(2,1)$	rank 2	real	$+$	conic and two intersecting lines
[2(11)]	2 double roots	$(2,1)$	rank 2	real	$-$	conic and point
[(11)(11)]	2 double roots	$(2,0)$	$(2,0)$	real		$\emptyset$
[(11)(11)]	2 double roots	$(2,0)$	$(1,1)$	real		two points
[(11)(11)]	2 double roots	$(1,1)$	$(2,0)$	real		two points
[(11)(11)]	2 double roots	$(1,1)$	$(1,1)$	real		four skew lines
[(11)(11)]	2 double roots	rank 2	rank 2	complex		two secant lines

Table 4: Classification of pencils in the case where $\mathcal{D}(\lambda,\mu)$ does not identically vanish. (λ_1,μ_1) denotes a multiple root of $\mathcal{D}(\lambda,\mu)$ (if any) and (λ_2,μ_2) another root (not necessarily simple). If (λ_1,μ_1) is a double root then s denotes the sign of $\frac{\det(\lambda S+\mu T)}{(\mu_1\lambda-\lambda_1\mu)^2}$ at $(\lambda,\mu)=(\lambda_1,\mu_1)$; if (λ_1,μ_1) is a quadruple root then s denotes the sign of $\det(\lambda S+\mu T)$ for any $(\lambda,\mu)\neq(\lambda_1,\mu_1)$. When the determinantal equation has multiple roots, the additional simple roots are not indicated.

Segre string	roots of $\mathcal{D}_3(\lambda,\mu)$	rank or inertia of $R(\lambda_1,\mu_1)$	inertia of $R(\lambda_2,\mu_2)$	type of (λ_2,μ_2)	real type of intersection
$[1\{3\}]$	no common singular point				conic and double line
$[111]$	3 simple roots	$(1,1)$	$(1,1)$	real	four concurrent lines meeting at **p**
$[111]$	3 simple roots	$(2,0)$		real	point **p**
$[111]$	3 simple roots		$(2,0)$	real	point **p**
$[111]$	3 simple roots			complex	two lines intersecting at **p**
$[12]$	double root	$(1,1)$			2 lines and a double line meeting at **p**
$[12]$	double root	$(2,0)$			double line
$[1(11)]$	double root	rank 1	$(1,1)$		two double lines meeting at **p**
$[1(11)]$	double root	rank 1	$(2,0)$		point **p**
$[3]$	triple root	rank 2			a line and a triple line meeting at **p**
$[(21)]$	triple root	rank 1			a quadruple line
$[(111)]$	triple root	rank 0			any non-trivial quadric of the pencil
$[\{3\}]$	$\mathcal{D}_3(\lambda,\mu)\equiv 0$				the complex intersection is real; see Tables 2 and 3

Table 5: Classification of pencils in the case where $\mathcal{D}(\lambda,\mu)$ identically vanishes. In the bottom part, the quadrics of the pencil have a singular point **p** in common. $\mathcal{D}_3(\lambda,\mu)$ is the determinant of the 3×3 upper-left matrix of $R(\lambda,\mu)$ after a congruence transformation sending **p** to $(0,0,0,1)$. The conic associated with a root of $\mathcal{D}_3(\lambda,\mu)$ corresponds to the 3×3 upper-left matrix or $R(\lambda,\mu)$. (λ_1,μ_1) denotes the multiple root of $\mathcal{D}_3(\lambda,\mu)$ (if any) and (λ_2,μ_2) another root. When $\mathcal{D}_3(\lambda,\mu)$ has a multiple root, the additional simple roots are not indicated.

The parallel between the Canonical Pair Form Theorem introduced in Section I.5.1[1] and the decomposition by Segre symbol should now jump to mind: the first is in a sense a real version of the second, i.e. it gives a canonical form that is projectively equivalent by a *real* congruence transformation to the original pencil. In the real form, complex roots of the determinantal equation are somehow combined in complex Jordan blocks so that quadric pencils are equivalent by a real projective transformation.

When λ_i is real, the J_i^j are the real Jordan blocks associated with λ_i. The sum of the sizes of the blocks corresponding to λ_i is $\sum_{k=1}^{t_i} e_i^k = m_i$ and the number of those blocks is $t_i = n - \operatorname{rank} R(\lambda_i)$, as in Theorem I.5.3.

When λ_i is complex, let λ_j be its conjugate. It is intuitively clear that $t_i = t_j$ in the complex decomposition and that the associated Jordan blocks J_i^k and J_j^k have the same sizes, i.e. $e_i^k = e_j^k$. When the complex roots and their blocks are combined, they give rise to complex Jordan blocks of

[1]When reference is made to a section or result in another part of the paper, it is prefixed by the part number.

size $2e_i^k$. In the real canonical form, the number of these blocks is again t_i but the sum of their sizes is $2m_i$.

The Segre symbol can thus serve as a starting point for the study of real pencils using the Canonical Pair Form Theorem. We illustrate this with two examples concerning pencils in $\mathbb{P}^3(\mathbb{R})$. Consider first the Segre symbol $[(211)]$. The associated pencil has a quadruple root, which is necessarily real (otherwise its conjugate would also be a root of the determinantal equation of the pencil). In view of the above, the real decomposition of the pencil has three Jordan blocks, one of size 2 and two of size 1. Now consider the Segre symbol [22]. The associated pencil has two double roots, which can be either both real or both complex. If they are real, then each of the roots has one Jordan block of size 2. If they are complex, then the two roots appear in the same Jordan block of size 4.

3 Classification of regular pencils of $\mathbb{P}^3(\mathbb{R})$ over the reals

We now turn to the classification of pencils of quadrics of $\mathbb{P}^3(\mathbb{R})$ over the reals. In what follows, we make heavy use of the Canonical Pair Form Theorem for pairs of real symmetric matrices (Part I and [8, 9]). For each possible Segre characteristic, we examine the different cases according to whether the roots of the determinantal equation are real or not and then examine the conditions leading to different types of intersection over the reals.

In each case, we start by computing the canonical form of the pair (S,T) for a given Segre characteristic and type (real or complex) of multiple root(s) of the determinantal equation. We then deduce from this canonical form a *normal form* of the pencil over the reals by rescaling and translating the roots to particularly simple values. Recall that the congruence transformation in the Canonical Pair Form Theorem preserves the roots (values and multiplicities) of the determinantal equation of the pencil. This normal form is in a sense the “simplest pair” of quadrics having a given real intersection type. The normal pencil is equivalent by a real projective transformation to any pencil of quadrics with the same real and complex intersection type.

A word of caution: the projective transformations involved in the classification of real pencils, if they preserve the real type of the intersection, may well involve irrational numbers. This fact should be kept to mind when interpreting the results.

We treat the first case (nodal quartic) in some detail so that the reader gets accustomed to the techniques we use. For the other cases, we move directly to the normal form without first expliciting the canonical form.

Note that the case where the Segre characteristic is $[1111]$, which corresponds to a smooth quartic, has been extensively treated in Part I. Also, the $[(1111)]$ case does not necessitate any further treatment: save for the quadric corresponding to the quadruple root (which is $\mathbb{P}^3(\mathbb{R})$), all the quadrics of the pencil are equal and the intersection is thus any of those non-trivial quadrics. The case $\det R(\lambda,\mu) \equiv 0$ is treated separately in Section 4.

Here and in the ensuing sections, a singularity of the intersection will be called *convex* if the branches of the curve are on the same side of the common tangent plane to the branches at the singularity, *concave* otherwise.

An additional benefit of the classification of pencils over the reals is the ability to draw pictures of all possible situations. Such a gallery of intersection cases is given in Figure 1. The pictures were made with the *surf* visualization tool [7].

3.1 Nodal quartic in $\mathbb{P}^3(\mathbb{C})$, $\sigma_4 = [112]$

The determinantal equation has a double root λ_1, which is necessarily real (otherwise its conjugate would also be a double root of $\det R(\lambda)$). Let λ_2 and λ_3 be the other roots. The Segre characteristic implies that the three quadrics $R(\lambda_i)$ have rank 3 (equal to $n - t_i$; see Section 2.2.1). The Canonical Pair Form Theorem thus implies that to λ_1 corresponds one real Jordan block of size 2.

There are two cases.

λ_2 and λ_3 are real. $R(\lambda_2)$ and $R(\lambda_3)$ are projective cones. The Canonical Pair Form Theorem gives that S and T are simultaneously congruent to the quadrics of equations

$$\begin{cases} 2\varepsilon_1 xy + \varepsilon_2 z^2 + \varepsilon_3 w^2 = 0, \\ 2\varepsilon_1 \lambda_1 xy + \varepsilon_1 y^2 + \varepsilon_2 \lambda_2 z^2 + \varepsilon_3 \lambda_3 w^2 = 0, \end{cases} \qquad \varepsilon_i = \pm 1, i = 1,2,3.$$

$\lambda_1 S - T$ and $\lambda_2 S - T$ are thus simultaneously congruent to the quadrics of equations

$$\begin{cases} -\varepsilon_1 y^2 + \varepsilon_2(\lambda_1 - \lambda_2) z^2 + \varepsilon_3(\lambda_1 - \lambda_3) w^2 = 0, \\ -\varepsilon_1 y^2 + 2\varepsilon_1(\lambda_2 - \lambda_1) xy + \varepsilon_3(\lambda_2 - \lambda_3) w^2 = 0. \end{cases}$$

Let $\varepsilon = \operatorname{sign} \frac{\lambda_2 - \lambda_3}{\lambda_1 - \lambda_3}$ (recall that $\lambda_1 \neq \lambda_3$ and $\lambda_2 \neq \lambda_3$). By multiplying the above two equations by $-\varepsilon_1 \left| \frac{\lambda_2 - \lambda_3}{\lambda_1 - \lambda_3} \right|$ and $-\varepsilon_1$, respectively, we can rewrite them as

$$\begin{cases} \left| \frac{\lambda_2 - \lambda_3}{\lambda_1 - \lambda_3} \right| y^2 - \varepsilon\varepsilon_1\varepsilon_2 \frac{(\lambda_1 - \lambda_2)(\lambda_2 - \lambda_3)}{\lambda_1 - \lambda_3} z^2 - \varepsilon\varepsilon_1\varepsilon_3(\lambda_2 - \lambda_3) w^2 = 0, \\ \sqrt{\left| \frac{\lambda_2 - \lambda_3}{\lambda_1 - \lambda_3} \right|}\, y \left(\sqrt{\left| \frac{\lambda_1 - \lambda_3}{\lambda_2 - \lambda_3} \right|}\, y - 2(\lambda_2 - \lambda_1) \sqrt{\left| \frac{\lambda_1 - \lambda_3}{\lambda_2 - \lambda_3} \right|}\, x \right) - \varepsilon_1\varepsilon_3(\lambda_2 - \lambda_3) w^2 = 0. \end{cases}$$

Now, we apply the following projective transformation:

$$\sqrt{\left| \frac{\lambda_1 - \lambda_3}{\lambda_2 - \lambda_3} \right|}\, y - 2(\lambda_2 - \lambda_1) \sqrt{\left| \frac{\lambda_1 - \lambda_3}{\lambda_2 - \lambda_3} \right|}\, x \mapsto x, \quad \sqrt{\left| \frac{\lambda_2 - \lambda_3}{\lambda_1 - \lambda_3} \right|}\, y \mapsto y,$$
$$\sqrt{\left| \frac{(\lambda_1 - \lambda_2)(\lambda_2 - \lambda_3)}{\lambda_1 - \lambda_3} \right|}\, z \mapsto z, \quad \sqrt{|\lambda_2 - \lambda_3|}\, w \mapsto w.$$

We obtain that $R(\lambda_1) = \lambda_1 S - T$ and $R(\lambda_2) = \lambda_2 S - T$ are simultaneously congruent, by a real projective transformation P, to the quadrics of equations

$$\begin{cases} P^T R(\lambda_1) P : \; y^2 + az^2 + bw^2 = 0, \\ P^T R(\lambda_2) P : \; xy + cw^2 = 0, \end{cases} \tag{2}$$

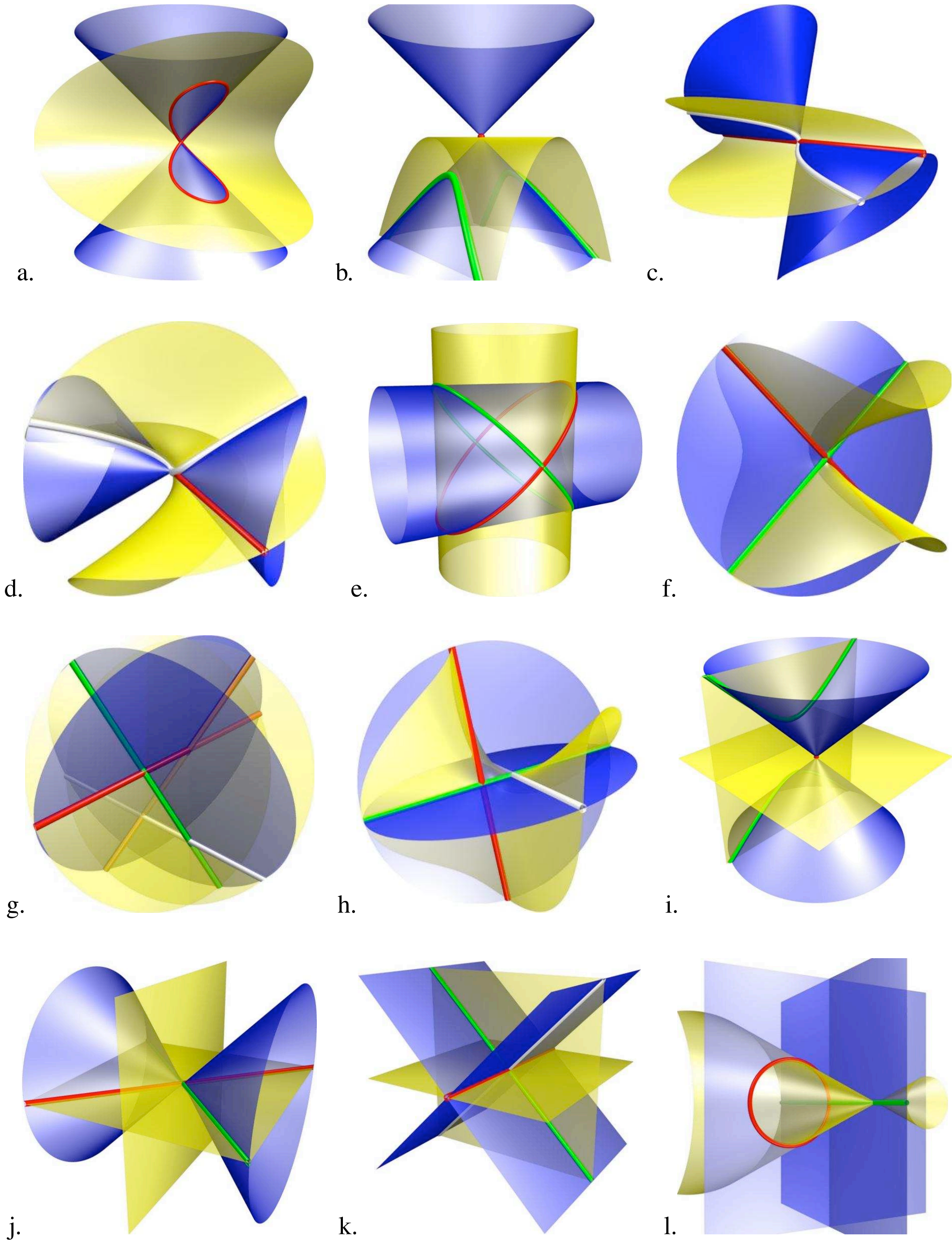

Figure 1: A gallery of intersections. a. Nodal quartic. b. Nodal quartic with isolated singular point. c. Cubic and secant line. d. Cubic and tangent line. e. Two secant conics. f. Two double lines. g. Four skew lines. h. Two lines and a double line. i. Conic and two lines not crossing on the conic, the two lines being imaginary. j. Four concurrent lines, only two of which are real. k. Two lines and a double line, the three being concurrent. l. Conic and double line.

with $a,b,c \in \{-1,1\}$. One can further assume that $c=1$ by changing x by $-x$.

From now on we forget about the transformation P and identify $R(\lambda_i)$ with $P^T R(\lambda_i) P$, but it should be kept to mind that things happen in the local frame induced by P.

If a or b is -1, the cone $R(\lambda_1)$ has inertia $(2,1)$ and thus is real. Otherwise ($a=b=1$), the cone $R(\lambda_1)$ is imaginary but for its real apex $\mathbf{p}=(1,0,0,0)$. The other cone $R(\lambda_2)$ is always real and contains the apex $\mathbf{p}$ of $R(\lambda_1)$. We distinguish the three following cases.

- $a=b=1$. The real part of the nodal quartic is reduced to its node, the apex $\mathbf{p}$ of $R(\lambda_1)$.
- Only one of a and b is 1. Assume for instance that $a=1, b=-1$ (the other case is obtained by exchanging z and w). By substituting the parameterization of the cone $y^2+z^2-w^2=0$ (see Table I.3)
$$\left(s, uv, \frac{u^2-v^2}{2}, \frac{u^2+v^2}{2}\right), \quad (u,v,s) \in \mathbb{P}^{\star 2},$$
into the other cone $xy+w^2=0$, and solving in s, we get the parameterization of the nodal quartic
$$\mathbf{X}(u,v) = \left((u^2+v^2)^2, -4u^2v^2, 2uv(u^2-v^2), 2uv(u^2+v^2)\right)^T, \quad (u,v) \in \mathbb{P}^1.$$
The nodal quartic is thus real and its node, corresponding to the parameters $(1,0)$ and $(0,1)$, is at $\mathbf{p}$. The plane tangent to the quadric $Q_{R(\lambda_2)}$ at the quartic's node $\mathbf{p}$ is $y=0$. In a neighborhood of this node, $x=(u^2+v^2)^2>0$ and $y=-4u^2v^2 \leqslant 0$ (recall that $\mathbf{X}(u,v)$ is projective, so its coordinates are defined up to a non-zero scalar). We conclude that the two branches lie on the same side of the tangent plane and that the singularity is convex.
- $a=-1, b=-1$. Parameterizing the nodal quartic as above, we get the parameterization
$$\mathbf{X}(u,v) = \left(-4u^2v^2, (u^2+v^2)^2, (u^2+v^2)(u^2-v^2), 2uv(u^2+v^2)\right)^T, \quad (u,v) \in \mathbb{P}^1.$$
It can be checked that the point $\mathbf{p}=(1,0,0,0)$ which is on the intersection is not attained by any real value of the parameter (u,v) (it is only attained with the complex parameters $(1,i)$ and $(i,1)$). The nodal quartic is thus real with an isolated singular point.

We now argue that we can easily distinguish between these three cases. For this, we first prove the following lemma.

Lemma 3.1. *Given any pencil of quadrics generated by S and T whose determinantal equation* $\det(\lambda S+\mu T)=0$ *has a double root (λ_1,μ_1), the sign of $\frac{\det(\lambda S+\mu T))}{(\mu_1\lambda-\lambda_1\mu)^2}$ at (λ_1,μ_1) is invariant by a real projective transformation of the pencil and does not depend on the choice of S and T in the pencil.*

Proof. We suppose that $\mathcal{D}(\lambda,\mu)=\det(\lambda S+\mu T)$ has a double root (λ_1,μ_1). The lemma claims that for any real projective transformation P and any $a_1,\ldots,a_4 \in \mathbb{R}$ such that $a_1a_4-a_2a_3 \neq 0$,
$$\mathcal{D}'(\lambda',\mu') = \det(\lambda' P^T(a_1S+a_2T)P + \mu' P^T(a_3S+a_4T)P)$$

has a double root (λ_1', μ_1') such that $\frac{\mathcal{D}(\lambda,\mu)}{(\mu_1\lambda-\lambda_1\mu)^2}$ at (λ_1,μ_1) has same sign as $\frac{\mathcal{D}'(\lambda',\mu')}{(\mu_1'\lambda'-\lambda_1'\mu')^2}$ at (λ_1',μ_1'). We have

$$\mathcal{D}'(\lambda',\mu') = (\det P)^2 \mathcal{D}(a_1\lambda' + a_3\mu', a_2\lambda' + a_4\mu').$$

Thus $\mathcal{D}'(\lambda',\mu') = 0$ has a double root (λ_1',μ_1') defined by

$$\begin{cases} a_1\lambda_1' + a_3\mu_1' = \lambda_1 \\ a_2\lambda_1' + a_4\mu_1' = \mu_1 \end{cases} \Leftrightarrow \begin{cases} \lambda_1' = \frac{a_4\lambda_1 - a_3\mu_1}{a_1a_4 - a_2a_3} \\ \mu_1' = \frac{-a_2\lambda_1 + a_1\mu_1}{a_1a_4 - a_2a_3}. \end{cases}$$

It follows that

$$\frac{\mathcal{D}'(\lambda',\mu')}{(\mu_1'\lambda' - \lambda_1'\mu')^2} = (\det P)^2 \frac{\mathcal{D}(a_1\lambda' + a_3\mu', a_2\lambda' + a_4\mu')}{(\mu_1(a_1\lambda' + a_3\mu') - \lambda_1(a_2\lambda' + a_4\mu'))^2}(a_1a_4 - a_2a_3)^2.$$

Hence $\frac{\mathcal{D}'(\lambda',\mu')}{(\mu_1'\lambda'-\lambda_1'\mu')^2}$ at (λ_1',μ_1') has same sign as $\frac{\mathcal{D}(\lambda,\mu)}{(\mu_1\lambda-\lambda_1\mu)^2}$ at (λ_1,μ_1). □

Proposition 3.2. *If the determinantal equation* $\det(\lambda S + \mu T) = 0$ *has two simple real roots and one double root* (λ_1,μ_1) *whose associated matrix* $\lambda_1 S + \mu_1 T$ *has rank three, then the intersection of* S *and* T *in* $\mathbb{C}^3$ *is a nodal quartic whose node is the apex of* $\lambda_1 S + \mu_1 T$.

Moreover, if the inertia of $\lambda_1 S + \mu_1 T$ *is* $(3,0)$ *then the real part of the nodal quartic is reduced to its node. Otherwise the nodal quartic is real; furthermore, if* $\frac{\det(\lambda S+\mu T)}{(\mu_1\lambda-\lambda_1\mu)^2}$ *is negative for* $(\lambda,\mu) = (\lambda_1,\mu_1)$*, the node is isolated and, otherwise, the singularity is convex.*

Proof. The first part of the proposition follows directly from the Segre characteristic (see Section 2.2.1 and Table 1).

If the inertia of $\lambda_1 S + \mu_1 T$ is $(3,0)$, then $a = b = 1$ in (2) and the result follows as discussed above. Otherwise, considering $S' = P^T(\lambda_1 S - T)P$ and $T' = P^T(\lambda_2 S - T)P$, (2) gives that $\det(\lambda S' + \mu T') = -a\lambda(b\lambda + \mu)\mu^2/4$. Evaluating $\frac{\det(\lambda S'+\mu T')}{\mu^2}$ at $(\lambda,\mu) = (1,0)$, gives by Lemma 3.1 that $-ab$ has same sign as $\frac{\det(\lambda S+\mu T)}{(\mu_1\lambda-\lambda_1\mu)^2}$ at (λ_1,μ_1). The result then follows from the discussion above depending on whether $a = b = -1$ or $ab = -1$. □

λ_2 and λ_3 are complex conjugate. The reduction to normal pencil form is slightly more involved in this case. Let $\lambda_2 = \alpha + i\beta, \lambda_3 = \overline{\lambda_2}, \beta \neq 0$. The Canonical Pair Form Theorem gives that S and T are simultaneously congruent to the quadrics of equations

$$\begin{cases} 2\varepsilon xy + 2zw = 0, \\ 2\varepsilon\lambda_1 xy + \varepsilon y^2 + 2\alpha zw + \beta z^2 - \beta w^2 = 0, \end{cases} \qquad \varepsilon = \pm 1$$

Through this congruence, $S' = \lambda_1 S - T$ has equation

$$\begin{aligned} 0 &= -\varepsilon y^2 + \beta(w^2 - z^2) + 2(\lambda_1 - \alpha)zw, \\ &= -\varepsilon y^2 + \beta(w + \xi z)\left(w - \frac{1}{\xi}z\right), \\ &= -\varepsilon y^2 + \beta z'w', \end{aligned}$$

where ξ is real and positive. Through the congruence and with the above transformation $(z,w)\mapsto(z',w')$, S has equation

$$\begin{aligned}0 &= 2\varepsilon xy + 2zw\\ &= 2\varepsilon xy + \frac{2}{(\xi+\frac{1}{\xi})^2}\left(\frac{1}{\xi}z'^2 - \xi w'^2 + \left(\xi - \frac{1}{\xi}\right)z'w'\right).\end{aligned}$$

Through the above congruence transformations, the quadric of the pencil $T' = \beta S - 2\frac{\xi-\frac{1}{\xi}}{(\xi+\frac{1}{\xi})^2}(\lambda_1 S - T)$ has equation

$$2\varepsilon y\left(\beta x + \frac{\xi-\frac{1}{\xi}}{(\xi+\frac{1}{\xi})^2}y\right) + \frac{2\beta}{(\xi+\frac{1}{\xi})^2}\left(\frac{1}{\xi}z'^2 - \xi w'^2\right) = 0.$$

Finally, by making a shift on x, rescaling on the four axes, and changing the signs of x and z, we get that the two quadrics of the pencil S' and T' are simultaneously congruent to the quadrics of equations

$$\begin{cases} y^2 + zw = 0,\\ xy + z^2 - w^2 = 0.\end{cases} \tag{3}$$

As before, we now drop reference to the accumulated congruence transformation and work in the local frame. By substituting the parameterization of the cone $y^2+zw=0$ (see Table I.3)

$$\left(s, uv, u^2, -v^2\right), \quad (u,v,s)\in\mathbb{P}^{\star 2},$$

into the other quadric $xy+z^2-w^2=0$, and solving in s, we get the parameterization of the nodal quartic

$$\mathbf{X}(u,v) = \left(v^4-u^4, u^2v^2, u^3v, -uv^3\right)^T, \quad (u,v)\in\mathbb{P}^1.$$

The nodal quartic is thus real and its node, corresponding to the parameters $(1,0)$ and $(0,1)$, is at $\mathbf{p}=(1,0,0,0)$, the apex of S'. The plane tangent to the quadric $xy+z^2-w^2=0$ at the quartic's node $\mathbf{p}$ is $y=0$. In a neighborhood of the quartic's node on the branch corresponding to the parameter $(0,1)$, $x=v^4-u^4>0$ and $y=u^2v^2\geqslant 0$. On the other branch corresponding to the parameter $(1,0)$, $x=v^4-u^4<0$ and $y=u^2v^2\geqslant 0$. Hence, the two branches of the quartic are on opposite sides of the tangent plane $y=0$ in a neighborhood of the node, i.e., the singularity is concave.

We thus have the following result.

Proposition 3.3. *If the determinantal equation* $\det(\lambda S+\mu T)=0$ *has two simple complex conjugate roots and one double root* (λ_1,μ_1) *whose associated matrix* $\lambda_1 S+\mu_1 T$ *has rank three, then the intersection of S and T is a real nodal quartic with a concave singularity at its node, the apex of* $\lambda_1 S+\mu_1 T$.

3.2 Two secant conics in $\mathbb{P}^3(\mathbb{C})$, $\sigma_4 = [11(11)]$

The determinantal equation has a double root λ_1 and the rank of $R(\lambda_1)$ is 2. λ_1 is necessarily real and there are two Jordan blocks of size 1 associated with it in the canonical form. Let λ_2 and λ_3 be the other (simple) roots, associated with quadrics of rank 3. We have two cases.

λ_2 and λ_3 are real. λ_2 and λ_3 appear in real Jordan blocks of size 1. The normal form of $R(\lambda_1)$ and $R(\lambda_2)$ is:

$$\begin{cases} z^2 + aw^2 = 0, \\ x^2 + by^2 + cw^2 = 0, \end{cases}$$

with $a, b, c \in \{-1, 1\}$.

The two planes of $R(\lambda_1)$ are real if the matrix has inertia $(1,1)$, i.e. if $a = -1$. The cone $R(\lambda_2)$ is real if its inertia is $(2,1)$, i.e. if $b = -1$ or $c = -1$. The two conics of the intersection are secant over the reals if the singular line $z = w = 0$ of the pair of planes meets the cone in real points, i.e. if $b = -1$. We have the following cases:

- $a = \pm 1, b = 1, c = 1$: The planes are real or imaginary and the cone is imaginary. The apex of the cone is not on the planes, so intersection is empty.

- $a = 1, b = 1, c = -1$: The planes are imaginary and the cone is real. Their real intersection is the intersection of the singular line $z = w = 0$ of the pair of planes with the cone. The real intersection is thus empty.

- $a = 1, b = -1, c = \pm 1$: The planes are imaginary and the cone is real. The line $z = w = 0$ intersects the cone in two points of coordinates $(1,1,0,0)$ and $(-1,1,0,0)$. The intersection is reduced to these two points.

- $a = -1, b = 1, c = -1$: The planes and the cone are real. The line $z = w = 0$ does not intersect the cone, so intersection consists of two non-secant conics.

- $a = -1, b = -1, c = \pm 1$: The planes and the cone are real. The line $z = w = 0$ intersects the conics. Intersection consists of two conics intersecting in two points $\mathbf{p}^{\pm}$ of coordinates $(\pm 1, 1, 0, 0)$. All the quadrics of the pencil have the same tangent plane $P^{\pm} : x \mp y = 0$ at $\mathbf{p}^{\pm}$. The two conics of the intersection are on the same side of $P^{\pm}$.

Computing the inertia of $R(\lambda_1)$ gives a. Also, in normal form, the determinantal equation $\det(\lambda R(\lambda_1) + \mu R(\lambda_2))$ is equal to $b\mu^2\lambda(a\lambda + c\mu)$. Thus, by Lemma 3.1, ab is equal to the sign of $\frac{\det(\lambda S + \mu T)}{(\mu_1\lambda - \lambda_1\mu)^2}$ at (λ_1, μ_1). Hence we can easily compute a and b. Finally, we need to compute c but only in the case where $a = -1$ and $b = 1$. Then $c = 1$ if the inertia of $R(\lambda_2)$ (or $R(\lambda_3)$) is (3,0); otherwise $c = -1$ and the inertia of $R(\lambda_2)$ (or $R(\lambda_3)$) is $(2,1)$.

λ_2 and λ_3 are complex conjugate. There are two complex Jordan blocks of size 2 associated with the two roots. The pencil normal form is obtained as in Section 3.1. The end result is:

$$\begin{cases} zw = 0, \\ x^2 + ay^2 + z^2 - w^2 = 0, \end{cases}$$

with $a \in \{-1, 1\}$.

The pair of planes $R(\lambda_1)$ is always real. The intersection consists of the two conics $z = x^2 + ay^2 - w^2 = 0$ and $w = x^2 + ay^2 + z^2 = 0$. We have two cases:

- $a = 1$: One conic is real, the other is imaginary.
- $a = -1$: The two conics are real. They intersect at the points $\mathbf{p}^{\pm}$ of coordinates $(1, \pm 1, 0, 0)$. All the quadrics of the pencil have the same tangent plane $P^{\pm} : x \mp y = 0$ at $\mathbf{p}^{\pm}$. The two conics of the intersection are on opposite sides of $P^{\pm}$.

Note finally that, in normal form, the determinantal equation $\det(\lambda R(\lambda_1) + \mu R(\lambda_2))$ is equal to $-a\mu^2(\mu^2 + \lambda^2/4)$. Hence a is opposite to the sign of $\frac{\det(\lambda S + \mu T)}{(\mu_1 \lambda - \lambda_1 \mu)^2}$ at (λ_1, μ_1) (by Lemma 3.1).

3.3 Cuspidal quartic in $\mathbb{P}^3(\mathbb{C})$, $\sigma_4 = [13]$

The determinantal equation has a triple root λ_1, which is necessarily real. To it corresponds a real Jordan block of size 3. $R(\lambda_1)$ has rank 3. Let λ_2 be the other root, necessarily real, and $R(\lambda_2)$ the associated cone. The normal form of $R(\lambda_1)$ and $R(\lambda_2)$ is:

$$\begin{cases} w^2 + yz = 0, \\ y^2 + xz = 0. \end{cases}$$

The intersection consists of a cuspidal quartic which can be parameterized (in the local frame of the normal form) by

$$\mathbf{X}(u, v) = (v^4, u^2v^2, -u^4, u^3v), \quad (u, v) \in \mathbb{P}^1.$$

The quartic has a cusp at $\mathbf{p} = (1, 0, 0, 0)$ (the vertex of the first cone), which corresponds to $(u, v) = (0, 1)$. The intersection of $R(\lambda_1)$ with the plane tangent to $R(\lambda_2)$ at $\mathbf{p}$ gives the (double) line tangent to the quartic at $\mathbf{p}$, i.e. $z = w^2 = 0$.

3.4 Two tangent conics in $\mathbb{P}^3(\mathbb{C})$, $\sigma_4 = [1(21)]$

The determinantal equation has a triple root λ_1 and the rank of $R(\lambda_1)$ is 2. λ_1 is necessarily real. Attached to λ_1 are two real Jordan blocks, one of size 2, the other of size 1. Let λ_2 be the other simple real root, with $R(\lambda_2)$ of rank 3. The normal forms of $R(\lambda_1)$ and $R(\lambda_2)$ are:

$$\begin{cases} x^2 + aw^2 = 0, \\ xy + z^2 = 0, \end{cases}$$

where $a \in \{-1, 1\}$.

The pair of planes $R(\lambda_1)$ is real when the matrix has inertia $(1, 1)$, i.e. when $a = -1$. The cone $R(\lambda_2)$ is real since its inertia is $(2, 1)$. So we have two cases:

- $a = 1$: The pair of planes is imaginary. Its real part is restricted to the line $x = w = 0$, which intersects the cone in the real double point $(0, 1, 0, 0)$. The intersection is reduced to that point.
- $a = -1$: The planes are real. The intersection consists of two conics intersecting in the double point $(0, 1, 0, 0)$ and sharing a common tangent at that point.

3.5 Double conic in $\mathbb{P}^3(\mathbb{C})$, $\sigma_4 = [1(111)]$

The determinantal equation has a real triple root λ_1 and the rank of $R(\lambda_1)$ is 1. The Jordan normal form of $S^{-1}T$ contains three blocks of size 1 for λ_1. Let λ_2 be the other real root, with $R(\lambda_2)$ of rank 3. The normal forms of $R(\lambda_1)$ and $R(\lambda_2)$ are:

$$\begin{cases} w^2 = 0, \\ x^2 + ay^2 + z^2 = 0, \end{cases}$$

where $a \in \{-1, 1\}$.

The cone $R(\lambda_2)$ is real if its inertia is $(2,1)$, i.e. if $a = -1$. We have two cases:

- $a = -1$: The cone is real. The intersection consists of a double conic lying in the plane $w = 0$.
- $a = 1$: The cone is imaginary. Its real apex does not lie on the plane $w = 0$, so the intersection is empty.

3.6 Cubic and tangent line in $\mathbb{P}^3(\mathbb{C})$, $\sigma_4 = [4]$

The determinantal equation has a quadruple root λ_1 and the rank of $R(\lambda_1)$ is 3. λ_1 is necessarily real. Associated with it is a unique real Jordan block of size 4. The normal form of $R(\lambda_1)$ and S is:

$$\begin{cases} z^2 + yw = 0, \\ xw + yz = 0. \end{cases}$$

The intersection contains the line $z = w = 0$. The cubic is parameterized by

$$\mathbf{X}(u, v) = \left(u^3, -u^2 v, uv^2, v^3\right), \quad (u, v) \in \mathbb{P}^1.$$

The cubic intersects the line in the point of coordinate $(1,0,0,0)$, corresponding to the parameter $(1,0)$. The cubic and the line are tangent at that point.

3.7 Conic and two lines crossing on the conic in $\mathbb{P}^3(\mathbb{C})$, $\sigma_4 = [(31)]$

The determinantal equation has a quadruple root λ_1, with $R(\lambda_1)$ of rank 2. λ_1 is necessarily real. To it correspond two real Jordan blocks of size 3 and 1. The normal forms of $R(\lambda_1)$ and S are:

$$\begin{cases} yz = 0, \\ y^2 + xz + aw^2, \end{cases}$$

with $a \in \{-1, 1\}$. $z = 0$ gives two real or imaginary lines. $y = 0$ gives a real conic. The lines cross on the conic at the point $\mathbf{p} = (1,0,0,0)$.

Both the pair of planes and the nonsingular quadric are real. We have two cases:

- $a = 1$: The lines are imaginary. The intersection is reduced to the conic.

- $a=-1$: The lines are real. The intersection consists of a conic and two lines crossing on the conic at $\mathbf{p}$.

The determinantal equation in normal form $\det(\lambda R(\lambda_1)+\mu S) = -a\mu^4/4$ has a quadruple root and thus is always non-negative or non-positive. In this case, it is straightforward to show, similarly as in the proof of Lemma 3.1, that the sign $\geqslant 0$ or $\leqslant 0$ of $\det(\lambda S+\mu T)$ is invariant by real projective transformation and independent of the choice of S and T in the pencil. Hence a is opposite to the sign of $\det(\lambda S+\mu T)$ for any (λ,μ) that is not the quadruple root.

3.8 Two lines and a double line in $\mathbb{P}^3(\mathbb{C})$, $\sigma_4 = [(22)]$

The determinantal equation has a quadruple root λ_1, with $R(\lambda_1)$ of rank 2. λ_1 is necessarily real and there are two real Jordan blocks associated with it, both of size 2. The normal forms of $R(\lambda_1)$ and S are:

$$\begin{cases} y^2+aw^2=0, \\ xy+azw=0, \end{cases}$$

with $a\in\{-1,1\}$. The intersection consists of the double line $y=w=0$ and two single lines $y\pm\sqrt{-a}\,w = x\pm\sqrt{-a}\,z=0$ cutting the double line in the points $(\mp\sqrt{-a},0,1,0)$.

The pair of planes is real if its inertia is $(1,1)$, i.e. if $a=-1$. We have two cases:

- $a=1$: The two single lines are imaginary. The intersection is reduced to the double line $y=w=0$.

- $a=-1$: The intersection consists of the two single lines $y\pm w = x\pm z=0$ and the double line $y=w=0$.

Note that the determinantal equation $\det(\lambda R(\lambda_1)+\mu S)$ is equal in normal form to $\frac{a^2\mu^4}{16}$. Thus $\mathcal{D}(\lambda,\mu)$ is positive for any (λ,μ) distinct from the quadruple root.

3.9 Two double lines in $\mathbb{P}^3(\mathbb{C})$, $\sigma_4 = [(211)]$

The determinantal equation has a quadruple root λ_1, with $R(\lambda_1)$ of rank 1. λ_1 is real and there are three real Jordan blocks associated with it, two having size 1 and the last size 2. The normal forms of $R(\lambda_1)$ and S are:

$$\begin{cases} w^2=0, \\ x^2+ay^2+zw=0, \end{cases}$$

with $a\in\{-1,1\}$. The intersection consists of two double lines $w^2=x^2+ay^2=0$.

There are two cases:

- $a=1$: The two double lines are imaginary. The intersection is reduced to their real intersection point, i.e. $(0,0,1,0)$.

- $a=-1$: The two double lines $w^2=x-y=0$ and $w^2=x+y=0$ are real so the intersection consists of these two lines, meeting at $(0,0,1,0)$.

The determinantal equation (in normal form) is equal to $\det(\lambda R(\lambda_1)+\mu S) = -a\lambda^4/4$ thus a is opposite to the sign of $\det(\lambda S+\mu T)$ for any (λ,μ) that is not a root.

3.10 Cubic and secant line in $\mathbb{P}^3(\mathbb{C})$, $\sigma_4 = [22]$

The determinantal equation has two double roots λ_1 and λ_2. The associated quadrics both have rank 3. λ_1 and λ_2 are either both real or complex conjugate.

λ_1 and λ_2 are real. There is a real Jordan block of size 2 associated with each root. The normal form of $R(\lambda_1)$ and $R(\lambda_2)$ is:

$$\begin{cases} y^2+zw=0, \\ xy+w^2=0. \end{cases}$$

The intersection consists of the line $y=w=0$ and a cubic. The cubic is parameterized by

$$\mathbf{X}(u,v) = \left(u^3, -uv^2, -v^3, u^2v\right), \quad (u,v)\in\mathbb{P}^1.$$

The line intersects the cubic in the two points of coordinates $(1,0,0,0)$ and $(0,0,1,0)$, corresponding to the parameters $(1,0)$ and $(0,1)$.

λ_1 and λ_2 are complex conjugate. Let $\lambda_1 = \alpha+i\beta, \lambda_2 = \overline{\lambda_1}, \beta\neq 0$. There is complex Jordan block of size 4 associated with the two roots. The normal form of S and $R(\alpha)$ is:

$$\begin{cases} xw+yz=0, \\ xz-yw+zw=0. \end{cases}$$

The intersection contains the line $z=w=0$. The cubic is parameterized by

$$\mathbf{X}(u,v) = \left(-u^2v, uv^2, u^3+uv^2, u^2v+v^3\right), \quad (u,v)\in\mathbb{P}^1.$$

The cubic intersects the line in the points of coordinates $(1,\mathrm{i},0,0)$ and $(1,-\mathrm{i},0,0)$. Thus, over the reals, the cubic and the line do not intersect.

3.11 Conic and two lines not crossing on the conic, $\sigma_4 = [2(11)]$

The determinantal equation has two double roots λ_1 and λ_2, with associated ranks 3 (a projective cone) and 2 (a pair of planes) respectively. The two roots are necessarily real, otherwise the ranks of the quadrics $R(\lambda_1)$ and $R(\lambda_2)$ would be the same. Associated with λ_1 and λ_2 are respectively a unique real Jordan block of size 2 and two real Jordan blocks of size 1. The pencil normal form is:

$$\begin{cases} y^2+az^2+bw^2=0, \\ xy=0, \end{cases}$$

where $a,b\in\{-1,1\}$. The plane $x=0$ contains a conic which is real when $a=-1$ or $b=-1$ and imaginary otherwise. The plane $y=0$ contains two lines which are real if $ab<0$ and imaginary otherwise. The lines cross at the point $(1,0,0,0)$, the apex of $R(\lambda_1)$, which is not on the conic.

The pair of planes $R(\lambda_2)$ is always real. The cone $R(\lambda_1)$ is real when its inertia is $(2,1)$, i.e. when $a=-1$ or $b=-1$. We have three cases:

- $a=1, b=1$: The lines and the conic are imaginary. The intersection is reduced to the real point of intersection of the two lines, i.e. $(1,0,0,0)$.
- $a=-b$: The lines and the conic are real. The intersection consists of a conic and two intersecting lines, each cutting the conic in a point (at $(0,0,1,1)$ and $(0,0,-1,1)$).
- $a=-1, b=-1$: The lines are imaginary, the conic is real. The intersection consists of a conic and the point $(1,0,0,0)$, intersection of the two lines.

To determine in which of the three situations we are, first compute the inertia of $R(\lambda_1)$. If the inertia is $(3,0)$, this implies that $a=b=1$. Otherwise, we consider as before the determinantal equation in normal form $\det(\lambda R(\lambda_1)+\mu R(\lambda_2)) = -ab\lambda^2\mu^2/4$. By Lemma 3.1, $-ab$ is equal to the sign of $\frac{\det(\lambda S+\mu T)}{(\mu_1\lambda-\lambda_1\mu)^2}$ at (λ_1,μ_1). If $ab>0$ then $a=b=-1$, otherwise $a=-b$.

3.12 Four skew lines in $\mathbb{P}^3(\mathbb{C})$, $\sigma_4 = [(11)(11)]$

The determinantal equation has two double roots λ_1 and λ_2, with associated quadrics of rank 2. λ_1 and λ_2 can be either both real or both complex conjugate.

λ_1 and λ_2 are real. Each root appears in two real Jordan blocks of size 1. The normal forms of $R(\lambda_1)$ and $R(\lambda_2)$ are:

$$\begin{cases} z^2+aw^2, \\ x^2+by^2, \end{cases}$$

where $a,b \in \{-1,1\}$.

The first pair of planes is imaginary if $a=1$. The second pair of planes is imaginary if $b=1$. There are three cases:

- $a=1, b=1$: The four lines are imaginary and the intersection is empty.
- $a=-b$: One pair of planes is real, the other is imaginary. If $a=1$, the intersection consists of the points of intersection of the line $z=w=0$ with the pair of planes $x^2-y^2=0$, i.e. the points $(1,1,0,0)$ and $(-1,1,0,0)$. Similarly, if $b=1$ the intersection is reduced to the two points $(0,0,1,1)$ and $(0,0,-1,1)$.
- $a=-1, b=-1$: The four lines are real. The intersection consists of four skew lines.

The values of a and b follow from the inertia of $R(\lambda_1)$ and $R(\lambda_2)$. Note also that b directly follows from a because, the determinantal equation (in normal form) $\det(\lambda R(\lambda_1)+\mu R(\lambda_2)) = ab\lambda^2\mu^2$ and it is straightforward to show that ab is equal to the sign of $\det(\lambda S+\mu T)$ for any (λ,μ) that is not a root.

λ_1 and λ_2 are complex conjugate. Let $\lambda_1 = a + \mathrm{i}b, \lambda_2 = \overline{\lambda_1}, b \neq 0$. The roots appear in two complex Jordan blocks of size 2. The normal forms of S and $aS - T$ are:

$$\begin{cases} xy + zw = 0, \\ x^2 - y^2 + z^2 - w^2 = 0. \end{cases}$$

The intersection consists of two real lines of equations $x \pm w = y \mp z = 0$ and two imaginary lines of equations $x \pm \mathrm{i}z = y \mp \mathrm{i}w = 0$.

4 Classification of singular pencils of $\mathbb{P}^3(\mathbb{R})$ over the reals

We now examine the singular pencils of $\mathbb{P}^3(\mathbb{R})$, i.e. those whose determinantal equation vanishes identically.

There are two cases according to whether two arbitrary quadrics of the pencil have a singular point in common or not.

4.1 Q_S and Q_T have no singular point in common, $\sigma_4 = [1\{3\}]$

We first prove the following lemma.

Lemma 4.1. *If* $\det R(\lambda,\mu) \equiv 0$ *and* Q_S *and* Q_T *have no singular point in common, then every quadric of the pencil has a singular point such that all the other quadrics of the pencil contains this point and share a common tangent plane at this point.*

Proof. Let Q_R be any quadric of the pencil. First note that R has rank at most 3, otherwise the determinantal equation would not identically vanish.

If R has rank 1, it is a double plane in $\mathbb{P}^3(\mathbb{C})$ containing only singular points. Since there is no quadric of inertia $(4,0)$ in the pencil, the intersection of the double plane with every other quadric of the pencil is not empty in $\mathbb{P}^3(\mathbb{R})$ (by Theorem I.4.3). Hence Q_R contains a singular point that belongs to all the quadrics of the pencil.

If R has rank 2, it is a pair of planes in $\mathbb{P}^3(\mathbb{C})$ with a real singular line. By the Segre classification (see Table 2) we know that the intersection in $\mathbb{P}^3(\mathbb{C})$ contains a conic and a double line. Furthermore, the line is necessarily real because otherwise its conjugate would also be in the intersection. This line lies in one of the two planes of Q_R and thus cuts any other line in that plane and in particular the singular line of the pair of planes. Hence Q_R contains a singular point that belongs to all the quadrics of the pencil.

If R has rank 3, we apply a congruence transformation so that Q_R has the diagonal form $ax^2 + by^2 + cz^2 = 0$, with $abc \neq 0$. We may also change the generators of the pencil, replacing S by R. After these transformations, the determinant $\mathcal{D}(\lambda,\mu)$ becomes the sum of $\delta abc\lambda^3\mu$ and of other terms of degree at least 2 in μ, where δ is the coefficient of w^2 in the equation of Q_T. The hypothesis that $\mathcal{D}(\lambda,\mu) \equiv 0$ thus implies that $\delta = 0$. Hence the singular point $(0,0,0,1)$ of Q_R belongs to Q_T and thus to all the quadrics of the pencil.

Thus, in all cases, every quadric of the pencil has a singular point that belongs to all the quadrics of the pencil. Any such point $\mathbf{p}$ lies on the intersection of Q_S and Q_T and is a singular point of the intersection: since **p** is a singular point of a quadric of the pencil, the rank of the Jacobian matrix (1) is less than two. We conclude on the common tangent plane by applying Proposition 2.1 under the assumption that **p** is not a singular point of both Q_S and Q_T. $\square$

By Lemma 4.1, there exist a singular point **s** of Q_S and a singular point **t** of Q_T that belong to all the quadrics $Q_{\lambda S+\mu T}$ of the pencil. Quadrics Q_S, Q_T, and Q_{S+T} have rank at most 3 since the determinantal equation identically vanishes, and they are not of inertia $(3,0)$ (see Table I.1) since they contain **s** and **t** that are distinct by assumption. Hence Q_S, Q_T, and Q_{S+T} are cones or pairs of (possibly complex) planes. Thus, since **s** and **t** are singular points of Q_S and Q_T, respectively, the line **st** is entirely contained in Q_S and Q_T, and thus is also contained in Q_{S+T}. Moreover, **s** and **t** are not singular points of Q_{S+T} because otherwise all the quadrics of the pencil would be singular at these points, contradicting the hypothesis. It now follows from the fact that Q_{S+T} is a cone or a pair of planes that its tangent planes at **s** and **t** coincide. Therefore, by Lemma 4.1, the tangent plane of Q_S at **t** is the same as the tangent plane of Q_T at **s**.

Now we change of frames in such a way that **s** and **t** have coordinates $(0,0,0,1)$ and $(0,0,1,0)$ and that the common tangent plane has equation $x=0$. Then the equations of Q_S and Q_T become $xz+q_1(x,y)=0$ and $xw+q_2(x,y)=0$, where q_1 and q_2 are binary quadratic forms. The two equations can thus be expressed in the form $ay^2+x(by+cx+z)=0$ and $a'y^2+x(b'y+c'x+w)=0$. By a new change of frame, we get equations of the form $ay^2+xz=0$ and $a'y^2+xw=0$. Replacing the second quadric by a linear combination of the two and applying the change of coordinates $a'z-aw \to w$ and a scaling on y, gives as normal form for the pencil:

$$\begin{cases} xw=0, \\ xz+ay^2=0, \end{cases}$$

with $a \in \{-1,1\}$. Furthermore, we can set $a=1$ by changing z in $-z$.

Therefore, the intersection consists of the double line $x=y^2=0$ and the conic $w=xz-y^2=0$. The line and the conic meet at $(0,0,1,0)$ in the local frame of the normal form.

4.2 Q_S and Q_T have a singular point in common

Let **p** be the common singular point. We proceed as already outlined in Section 2.2. After a rational change of frame, we may suppose that **p** has coordinates $(0,0,0,1)$. In the new frame, the equations of the quadrics are homogeneous polynomials of degree 2 in three variables. To classify the different types of intersection, we may identify the quadrics with their upper left 3×3 matrices and look at the multiple roots of the cubic determinantal equation, which we refer to as the *restricted determinantal equation*, and the ranks of the associated matrices. We thus apply the Canonical Pair Theorem to pairs of conics.

The case $[(111)]$ is trivial and left aside: in that situation, the cubic determinantal equation has a (real) triple root, the associated quadric has rank 0 and all the other quadrics of the pencil are cones. The intersection consists of any cone of the pencil, that is any quadric of the pencil distinct from $\mathbb{P}^3(\mathbb{R})$.

4.2.1 Four concurrent lines in $\mathbb{P}^3(\mathbb{C})$, $\sigma_3 = [111]$

The restricted determinantal equation has three simple roots. At least one is real: call it λ_1. Let λ_2 be another root. To these roots correspond quadrics of rank 2.

If λ_2 is real, then the three roots are real. The normal form of $R(\lambda_1)$ and $R(\lambda_2)$ is:

$$\begin{cases} ay^2 + z^2 = 0, \\ bx^2 + z^2 = 0, \end{cases}$$

with $a, b \in \{-1, 1\}$. Note that the equation of the third pair of planes of the pencil is obtained by subtracting the two equations, giving $ay^2 - bx^2 = 0$. We have two cases:

- $a = b = -1$: The intersection consists of four concurrent lines $y - \varepsilon_1 z = x - \varepsilon_2 z = 0$, with $\varepsilon_1, \varepsilon_2 \in \{-1, 1\}$, meeting at $\mathbf{p}$.
- $a = b = 1$ or $a = -b$: When $a = b = 1$, both $R(\lambda_1)$ and $R(\lambda_2)$ represent imaginary pairs of planes. When $a = -b$, the third pair of planes is imaginary, as well as one of the first two. In both cases, the intersection is reduced to the point $\mathbf{p}$.

Both a and b are equal to -1 if and only if $R(\lambda_1)$ and $R(\lambda_2)$ have inertia $(1, 1)$.

If $\lambda_2 = \alpha + i\beta$ is complex, $\beta \neq 0$, we obtain the following normal form (proceeding as in Section 3.1):

$$\begin{cases} x^2 + y^2 - z^2 = 0, \\ yz = 0. \end{cases}$$

The intersection consists of two real lines $y = x^2 - z^2 = 0$, intersecting at $\mathbf{p}$, and two complex lines $z = x^2 + y^2 = 0$.

4.2.2 Two lines and a double line in $\mathbb{P}^3(\mathbb{C})$, $\sigma_3 = [12]$

The restricted determinantal equation has a double root λ_1, which is real. The Jordan normal form of $S^{-1}T$ contains one real Jordan block of size 2. Let λ_2 be the other root, also real. The normal forms of $R(\lambda_1)$ and $R(\lambda_2)$ are:

$$\begin{cases} y^2 + az^2 = 0, \\ xy = 0, \end{cases}$$

where $a \in \{-1, 1\}$. There are two cases:

- $a = -1$: The intersection consists of the double line $y = z^2 = 0$ and the two single lines $x = y - z = 0$ and $x = y + z = 0$. The three lines are concurrent at $\mathbf{p}$.
- $a = 1$: The two single lines are imaginary. Their common point is on the double line, so the intersection consists of this double line $y = z^2 = 0$.

Note that the value of a follows from the inertia of $R(\lambda_1)$.

4.2.3 Two double lines in $\mathbb{P}^3(\mathbb{C})$, $\sigma_3 = [1(11)]$

The restricted determinantal equation has a double root λ_1, which is real. The canonical pair form has two real Jordan blocks of size 1 associated with λ_1. Let λ_2 be the other root, also real. The normal forms of $R(\lambda_1)$ and $R(\lambda_2)$ are:

$$\begin{cases} z^2 = 0, \\ x^2 + ay^2 = 0, \end{cases}$$

where $a \in \{-1, 1\}$. The pair of planes $R(\lambda_2)$ is real when its inertia is $(1,1)$, i.e. when $a = -1$. We have two cases:

- $a = 1$: The intersection is reduced to the point **p**.
- $a = -1$: The intersection consists of the two double lines $x - y = z^2 = 0$ and $x + y = z^2 = 0$, meeting at **p**.

Note that the value of a follows from the inertia of $R(\lambda_2)$.

4.2.4 Line and triple line in $\mathbb{P}^3(\mathbb{C})$, $\sigma_3 = [3]$

The restricted determinantal equation has a triple root λ_1, which is real. The Jordan normal form of $S^{-1}T$ contains one real Jordan block of size 3. The normal forms of S and $R(\lambda_1)$ are:

$$\begin{cases} xz + y^2 = 0, \\ yz = 0. \end{cases}$$

The intersection consists of the triple line $z = y^3 = 0$ and the simple line $x = y = 0$. The two lines cut at **p**, the singular point of all the quadrics of the pencil.

4.2.5 Quadruple line in $\mathbb{P}^3(\mathbb{C})$, $\sigma_3 = [(21)]$

The restricted determinantal equation has a real triple root λ_1. The canonical pair form has two real Jordan blocks of size 2 and 1. The normal form of $R(\lambda_1)$ and S is:

$$\begin{cases} y^2 = 0, \\ z^2 + xy = 0. \end{cases}$$

The intersection consists of the quadruple line $y^2 = z^2 = 0$.

4.2.6 $\sigma_3 = [\{3\}]$ and remaining cases

In this case, the restricted determinantal equation identically vanishes. One can easily prove that if the two conics S and T have no singular point in common, the pencil can be put in the normal form $\lambda xy + \mu xz$. The intersection consists of the plane $x = 0$ and the line $y = z = 0$, which meets the plane at **p**.

If the two conics have a singular point in common (call it **q**), we can go from 3×3 matrices to 2×2 matrices pretty much as above by sending **q** to $(0,0,1,0)$. Consider the new determinantal equation, a quadratic equation. The cases are:

- Two simple real roots: The pencil can be put in the normal form $\lambda x^2 + \mu y^2$. The intersection consists of the quadruple line $x^2 = y^2 = 0$ which goes through **p** and **q**.
- Two simple complex roots: A normal form for the pencil is $\lambda xy + \mu(x^2 - y^2)$, giving the quadruple line $x^2 = y^2 = 0$ for the intersection.
- A double real root, with a real Jordan block of size 2: The normal form is $\lambda xy + \mu y^2$. The intersection consists of the plane $y = 0$.
- Vanishing quadratic equation: The intersection consists of a double plane.

5 Classifying degenerate intersections

Our near-optimal algorithm for parameterizing intersections of quadrics works in two stages: first it determines the real type of the intersection and, second, it computes a parameterization of this intersection. The purpose of this section is to detail the first stage, called the *type-detection* phase. The second stage, which consists of case-by-case algorithms for computing (near-)optimal parameterizations of the real part of the intersection, will be presented in Part III.

The splitting in two stages reflects a key design choice of our parameterization algorithm, which sums up as: the sooner you know what is the type of the intersection, the less prone you are of making errors in later stages. Information obtained in the type-detection phase is used to drive the algorithm and make it efficiently compute precisely and uniquely what is needed.

Note however that presenting the type detection distinctly from the parameterization is quite an oversimplification. In the actual implementation, there is no clear cut separation between the two stages, which are largely intertwined. Sometimes detecting the type of the intersection is doing a very small step towards parameterization. And sometimes almost everything takes place in the type-detection phase.

We now turn to a high-level description of the detection phase, which relies heavily on the results of Sections 3 and 4 on real pencils of quadrics. We start by presenting some global tools, and then outline the type-detection algorithm for each of the possible root patterns (vanishing determinantal equation, one double root, one triple root, one quadruple root, two double roots).

5.1 Preliminaries

In what follows, we assume that the two input quadrics Q_S and Q_T have rational coefficients. We now briefly describe the basic operations needed for detecting the real type of the intersection $Q_S \cap Q_T$. They essentially fall in two categories: linear algebra routines and elementary algebraic manipulations. Most computations involve rational numbers. We give special attention to the limited number of situations where this is not the case.

Let $R(\lambda,\mu) = \lambda S + \mu T$ be the pencil generated by S and T. Computing the coefficients of the determinantal equation $\mathcal{D}(\lambda,\mu) = \det R(\lambda,\mu)$ involves nothing but computing determinants of rational matrices, so there is nothing difficult here. Next, we need to compute the inertia and rank of a matrix $R_0 = R(\lambda_0,\mu_0)$ of the pencil, where (λ_0,μ_0) is a root of $\mathcal{D}$ or a real rational projective number.

Assume first that R_0 has real and rational coefficients. Let $p(\omega) = \det(R_0 - \omega I)$, where I is the identity matrix. Since R_0 is symmetric, all its eigenvalues are real. We can thus compute the number e^+ of positive eigenvalues and the number e^- of negative eigenvalues of R_0 by applying Descartes' Sign Rule to $p(\omega)$ and $p(-\omega)$ respectively. Then the inertia I_0 of R_0 is the pair (e^+, e^-) and its rank r_0 is $e^+ + e^-$.

When the coefficients of R_0 are not rational or real, the worst situation that we have to deal with is when $\mathcal{D}$ has two real or complex conjugate double roots. In both cases, the coefficients of R_0 involve one square root. When the roots are real conjugate, we could use Descartes' Sign Rule again, except that we have to evaluate the signs of coefficients that now involve a square root. In these cases we however propose a more efficient approach based on the rank of R_0 (see Algorithm 6) which only deals with rational numbers. We show below how the rank of R_0 can be computed using only standard linear algebra operations on rational numbers.

Let $(\lambda_0,\mu_0) = (a, b \pm \sqrt{c})$, where $(a,b,c) \in \mathbb{P}^2(\mathbb{Q})$, c is not a square and c is either > 0 (real conjugate roots) or < 0 (complex conjugate roots). Form the 8×8 rational matrix

$$M_0 = \begin{pmatrix} aS + bT & cT \\ T & aS + bT \end{pmatrix}.$$

If the vector $\mathbf{k}_1 + \sqrt{c}\mathbf{k}_2$ is in the kernel of R_0, then the column vector $(\mathbf{k}_1^T, \mathbf{k}_2^T)^T$ is in the kernel of M_0. But the vector $(c\mathbf{k}_2^T, \mathbf{k}_1^T)$ also is in the kernel of M_0. It is not too difficult to realize that the kernel of M_0 has twice the number of elements of the kernel of R_0, i.e. $\dim\ker R_0 = \frac{1}{2}(\dim\ker M_0 - 1)$. We can thus conclude that

$$r_0 = 3 - \dim\ker R_0 = \frac{1}{2}(7 - \dim\ker M_0).$$

Computing the singular space of a quadric with rational coefficients is another operation we need. This only amounts to computing the kernel of the associated matrix. Also, intersecting the singular spaces of two quadrics is like computing the intersection between two linear spaces: it is a standard linear algebra operation.

In terms of algebraic computations, we need to be able to compute the gcd of polynomials of degree at most 3 and to isolate the roots of a cubic polynomial (in the four concurrent lines case). This last task can be done using Uspensky's algorithm as in Part I or using special methods for comparing the roots of low-degree polynomials (see [4]).

The top level type-detection loop is given in Algorithm 1. It does not necessitate much comment save for the fact that when the gcd $\mathcal{H}$ of the two partial derivatives of $\mathcal{D}$ has degree 2, then either its discriminant vanishes, in which case $\mathcal{D}$ has a triple root, or it does not vanish and $\mathcal{D}$ has two double roots.

Algorithm 1 Main loop for degenerate intersection classification.

Require: a pencil of quadrics $R(\lambda,\mu) = \lambda S + \mu T$
compute $\mathcal{D}(\lambda,\mu) := \det R(\lambda,\mu)$
if $\mathcal{D} \equiv 0$ **then** *// vanishing determinantal equation*
  execute Algorithm 2
else
  compute $\mathcal{H} := \gcd(\partial\mathcal{D}/\partial\lambda, \partial\mathcal{D}/\partial\mu)$ and let $d := \mathrm{degree}(\mathcal{H}, \{\lambda,\mu\})$
  if $d = 0$ **then** *// no multiple root*
    output: `smooth quartic` $(\mathbb{C})$ – see Part I
  else if $d = 1$ **then** *// double real root*
    execute Algorithm 3
  else if $d = 2$ **then**
    if discriminant$(\mathcal{H}) = 0$ **then** *// triple real root*
      execute Algorithm 4
    else *// two double roots*
      execute Algorithm 6
    end if
  else *// $d = 3$: quadruple real root*
    execute Algorithm 5
  end if
end if

5.2 $\mathcal{D}$ vanishes identically

The type-detection algorithm when $\mathcal{D}$ is identically zero, outlined in Algorithm 2, is little more than a reprise of the information contained in Section 4. The general idea is: determine if two arbitrary quadrics of the pencil have a singular point **p** in common. If they do, send **p** to infinity and work on the pencil of conics living in the plane $w = 0$. To actually compute the restricted pencil $R_3(\lambda, mu)$, build the matrix of a real projective transformation P obtained by putting **p** as the last column and completing P so that its columns form a basis of $\mathbb{P}^3(\mathbb{R})$. $R_3(\lambda,\mu)$ is then the principal submatrix of the matrix $P^T(\lambda S + \mu T)P$.

Two comments are in order. First, a multiple root of a cubic form – the determinantal equation of the pencil of conics – is necessarily real (otherwise its complex conjugate would also be a multiple root) and rational (otherwise its real conjugate would also be a multiple root).

So the only place where we might end up working with non-rational numbers is in the four concurrent lines case. Indeed, in this situation the restricted determinantal equation $\mathcal{D}_3$ is a cubic form with three generically non-rational simple roots. Computing the sign of the discriminant of $\mathcal{D}_3$ can help us distinguish between the cases when only two lines are real and when the four lines are either all real or all imaginary. But this is not enough to give a complete characterization over the reals. We have thus decided to apply Finsler's Theorem (Theorem I.4.3), as in the smooth quartic case, after isolating the roots of the cubic. If the restricted pencil contains a conic of inertia $(3,0)$, then the intersection of conics is empty in the plane $w = 0$ and the intersection of the two initial quadrics is reduced to **p**.

Algorithm 2 Classifying the intersection when the determinantal equation vanishes.

Require: $R(\lambda,\mu)$ (from Algorithm 1)

let $\gamma := \text{singular}(Q_S) \cap \text{singular}(Q_T), \kappa := \dim\gamma$
if $\kappa = -1$ **then** *// conic and double line* $(\mathbb{C})$
  output: `conic and double line`
else *//* $\kappa \geqslant 0$*: at least one common singular point* $\gamma(1)$
  send $\gamma(1)$ to the point $[0\ \ 0\ \ 0\ \ 1]$
  compute the restricted pencil $R_3(\lambda,\mu)$ of upper left 3×3 matrices and $\mathcal{D}_3(\lambda,\mu) := \det R_3(\lambda,\mu)$
  if $\mathcal{D}_3 \equiv 0$ **then** *// vanishing restricted determinantal equation*
    either $\sigma_3 = [\{3\}]$ or repeat restriction
  else *//* $\mathcal{D}_3 \not\equiv 0$
    compute $\mathcal{H}_3 := \gcd(\partial\mathcal{D}_3/\partial\lambda, \partial\mathcal{D}_3/\partial\mu)$ and let $d_3 := \text{degree}(\mathcal{H}_3, \{\lambda,\mu\})$
    if $d_3 = 0$ **then** *// no multiple root: four concurrent lines* $(\mathbb{C})$
      if $\mathcal{D}_3$ has only one real root **then**
        output: `two concurrent lines`
      else if $R_3(\lambda,\mu)$ contains a conic of inertia $(3,0)$ **then**
        output: `point`
      else
        output: `four concurrent lines`
      end if
    else *// one multiple root*
      let (λ_0,μ_0) be the multiple root of $\mathcal{D}_3$, I_0 and r_0 the inertia and rank (resp.) of $R(\lambda_0,\mu_0)$
      if $d_3 = 1$ **then** *//* $\mathcal{D}_3$ *has one double root*
        if $r_0 = 2$ **then** *// two concurrent lines and a double line* $(\mathbb{C})$
          if $I_0 = (1,1)$ **then** *// pair of planes* $R_3(\lambda_0,\mu_0)$ *is real*
            output: `two concurrent lines and a double line`
          else *// pair of planes* $R_3(\lambda_0,\mu_0)$ *is imaginary*
            output: `double line`
          end if
        else *//* $r_0 = 1$*: two double lines* $(\mathbb{C})$
          let (λ_1,μ_1) be the other root of $\mathcal{D}_3$, I_1 the inertia of $R(\lambda_1,\mu_1)$
          if $I_1 = (1,1)$ **then** *// pair of planes* $R_3(\lambda_1,\mu_1)$ *is real*
            output: `two double lines`
          else *// pair of planes* $R_3(\lambda_1,\mu_1)$ *is imaginary*
            output: `point`
          end if
        end if
      else *//* $d_3 = 2$*:* $\mathcal{D}_3$ *has one triple root*
        if $r_0 = 2$ **then** *// line and triple line* $(\mathbb{C})$
          output: `line and triple line`
        else if $r_0 = 1$ **then** *// quadruple line* $(\mathbb{C})$
          output: `quadruple line`
        else *//* $r_0 = 0$*: projective cone* $(\mathbb{C})$
          if S or T (in restricted form) has inertia $(2,1)$ **then**
            output: `cone`
          else
            output: `point`
          end if
        end if
      end if
    end if
  end if
end if

5.3 $\mathcal{D}$ has a single multiple root

The type-detection algorithms when $\mathcal{D}$ has a unique multiple root are given in Algorithms 3 (double root), 4 (triple root) and 5 (quadruple root).

First note that when $\mathcal{D}$ has a single multiple root, it is necessarily real and rational, for the same reasons as above. So the singular quadrics that we deal with all have rational coefficients and their singular set can be parameterized rationally.

In the double real root case, we use the result of Lemma 3.1 and classify the intersections according (among others) the sign

$$s := \operatorname{sign} \mathcal{E}(\lambda_0, \mu_0), \quad \text{with} \quad \mathcal{E}(\lambda, \mu) := \mathcal{D}(\lambda, \mu)/(\mu_0\lambda - \lambda_0\mu)^2, (\lambda_0, \mu_0) \text{ double root of } \mathcal{D}.$$

The other slight difficulty occurs in the two secant conics case, when the pair of planes $R(\lambda_0, \mu_0)$ is real and $s = -1$. To separate the two subcases (two non-secant conics or empty set), we can compute the inertia of $R(\lambda_1, \mu_1)$, where (λ_1, μ_1) is a simple root of $\mathcal{D}$, when this root is rational. But in the general case, we use again Finsler's theorem, looking for a quadric of inertia $(4,0)$ between and outside the two simple roots of $\mathcal{D}$. If such a quadric is found, the intersection is empty.

The triple real root case does not necessitate further comment except for noticing that the additional simple root (λ_1, μ_1) of $\mathcal{D}$ is necessarily real and rational, so the associated quadric $R(\lambda_1, \mu_1)$ has rational coefficients.

The type detection in the quadruple real root case is pretty straightforward. The only subtlety is that the case of a quadruple real root (λ_0, μ_0) with associated quadric of rank 2 corresponds to two different Jordan decompositions, with Segre symbols [(22)] and [(31)], as already mentioned in Section 2.2. To distinguish between the two, we simply note that in the first situation, the singular line of the pair of planes $R(\lambda_0, \mu_0)$ is entirely contained in all the quadrics of the pencil.

5.4 $\mathcal{D}$ has two double roots

When $\mathcal{D}$ has two double roots, we have distinguished between two situations: the roots are either real and rational, or they are not. In the first case, computing the inertia of singular quadrics is easy since computations take place over the rationals. Also, note that if one of the singular quadrics has rank 2 and the other has rank 3, then the associated roots of the determinantal equation are necessarily rational.

So assume that the roots are not real or not rational. The ranks of the non-rational singular quadrics are necessarily the same, so we need only compute one of them, in the way indicated above. When this rank is 2 and the roots are real conjugate, we distinguish between the remaining subcases (four real skew lines or empty set) by testing whether any quadric of the pencil between the two roots has inertia $(4,0)$ or not.

6 Examples

We now give several examples for which the type of the real part of the intersection is determined using the type-detection algorithms of the previous section.

Algorithm 3 Classifying the intersection: the double real root case, $d = 1$.

Require: $R(\lambda, \mu)$, $\mathcal{D}$ (from Algorithm 1)
Require: double real root (λ_0, μ_0), inertia I_0 and rank r_0 of $R(\lambda_0, \mu_0)$
let $s := \operatorname{sign} \mathcal{E}(\lambda_0, \mu_0)$ and $\delta := \operatorname{sign}(\operatorname{discriminant}(\mathcal{E}))$, where $\mathcal{E}(\lambda, \mu) := \mathcal{D}(\lambda, \mu)/(\mu_0\lambda - \lambda_0\mu)^2$
if $r_0 = 3$ **then** *// nodal quartic* $(\mathbb{C})$
  if $\delta = -1$ **then** *// other roots are complex*
    output: `nodal quartic, concave singularity`
  else *// other roots are real*
    if $s = +1$ **then**
      output: `nodal quartic, convex singularity`
    else if $I_0 = (2, 1)$ **then** *// cone $R(\lambda_0, \mu_0)$ is real*
      output: `nodal quartic with isolated singular point`
    else *// cone $R(\lambda_0, \mu_0)$ is imaginary*
      output: `point`
    end if
  end if
else *// $r_0 = 2$: two secant conics* $(\mathbb{C})$
  if $\delta = -1$ **then** *// other roots are complex*
    if $s = +1$ **then**
      output: `two secant conics, concave singularities`
    else
      output: `one conic`
    end if
  else *// other roots are real*
    if $I_0 = (1, 1)$ **then** *// pair of planes $R(\lambda_0, \mu_0)$ is real*
      if $s = +1$ **then**
        output: `two secant conics, convex singularities`
      else
        if $R(\lambda, \mu)$ contains a quadric of inertia $(4, 0)$ **then**
          output: $\emptyset$
        else
          output: `two non-secant conics`
        end if
      end if
    else *// pair of planes $R(\lambda_0, \mu_0)$ is imaginary*
      if $s = +1$ **then**
        output: $\emptyset$
      else
        output: `two points`
      end if
    end if
  end if
end if

Algorithm 4 Classifying the intersection: the triple real root case, $d = 2$ and discriminant$(\mathcal{H}) = 0$

Require: $R(\lambda,\mu)$, $\mathcal{D}$ (from Algorithm 1)
Require: triple real root (λ_0,μ_0), inertia I_0 and rank r_0 of $R(\lambda_0,\mu_0)$

```
if r_0 = 3 then    // cuspidal quartic (C)
  output: cuspidal quartic
else if r_0 = 2 then    // two tangent conics (C)
  if I_0 = (1,1) then    // pair of planes R(λ_0, μ_0) is real
    output: two tangent conics
  else    // pair of planes R(λ_0, μ_0) is imaginary
    output: point
  end if
else    // r_0 = 1: double conic (C)
  let I_1 be the inertia of R(λ_1, μ_1), (λ_1, μ_1) the second root of D
  if I_1 = (2,1) then    // cone R(λ_1, μ_1) is real
    output: double conic
  else    // cone R(λ_1, μ_1) is imaginary
    output: ∅
  end if
end if
```

6.1 Example 1

Consider the following pair of quadrics:

$$\begin{cases} Q_S : -x^2 - 4xy + 4xz - 6y^2 + 2yz - 4yw + 2zw - 2w^2 = 0, \\ Q_T : -x^2 - 6xy + 4xz - 2xw - 6y^2 - 8yw - 6w^2 = 0. \end{cases}$$

The determinantal equation is

$$\mathcal{D}(\lambda,\mu) = \det(\lambda S + \mu T) = -16(2\lambda^4 - 10\lambda^3\mu - 19\lambda^2\mu^2 - 16\lambda\mu^3 - 5\mu^4).$$

The gcd of the partial derivatives is equal to $32(\lambda+\mu)$. So, by Algorithm 1, $\mathcal{D}$ has a double real root at $(\lambda_0,\mu_0) = (1,-1)$.

We then follow Algorithm 3. Let $R_0 = \lambda_0 S + \mu_0 T$. We have:

$$\det(R_0 - xI) = x^4 - 4x^3 - 8x^2.$$

Descartes' Sign Rule gives that the inertia of R_0 is $I_0 = (1,1)$ and the rank is $r_0 = 2$. The intersection thus consists of two secant conics over the complexes. We compute

$$\mathcal{E}(\lambda,\mu) = \frac{\mathcal{D}(\lambda,\mu)}{(\mu_0\lambda - \lambda_0\mu)^2} = -16(2\lambda^2 + 6\lambda\mu + 5\mu^2).$$

So $\delta = \mathrm{sign}(\mathrm{discriminant}(\mathcal{E})) = -1$ and $s = \mathrm{sign}\,\mathcal{E}(\lambda_0,\mu_0) = -1$. We conclude that the intersection consists, over the reals, of a single conic.

This conic can be parameterized by (see Part III)

$$\mathbf{X}(u,v) = (2u^2 - 12uv + 18v^2, -u^2 + 2uv + 3v^2, 8v^2, u^2 - 2uv - 3v^2), \quad (u,v) \in \mathbb{P}^1(\mathbb{R}).$$

Algorithm 5 Classifying the intersection: the quadruple real root case, $d = 3$.

Require: $R(\lambda,\mu), \mathcal{D}$ (from Algorithm 1)
Require: quadruple real root (λ_0,μ_0), inertia I_0 and rank r_0 of $R(\lambda_0,\mu_0)$
if $r_0 = 3$ **then** *// cubic and tangent line* $(\mathbb{C})$
  output: `cubic and tangent line`
else if $r_0 = 2$ **then** *// conic and two lines crossing or two skew lines and a double line* $(\mathbb{C})$
  if $I_0 = (2,0)$ **then** *// pair of planes $R(\lambda_0,\mu_0)$ is imaginary*
    output: `double line`
  else *// pair of planes $R(\lambda_0,\mu_0)$ is real*
    if $s = +1$ **then**
      let l_0 be the singular line of $R(\lambda_0,\mu_0)$
      if l_0 is contained in Q_S and Q_T **then**
        output: `two skew lines and a double line`
      else
        output: `conic and two lines crossing on conic`
      end if
    else *// $s = -1$*
      output: `conic`
    end if
  end if
else if $r_0 = 1$ **then** *// two double lines* $(\mathbb{C})$
  if $s = +1$ **then**
    output: `two secant double lines`
  else *// $s = -1$*
    output: `point`
  end if
else *// $r_0 = 0$: smooth quadric* $(\mathbb{C})$
  if S or T has inertia $(2,2)$ **then**
    output: `smooth quadric`
  else
    output: $\emptyset$
  end if
end if

6.2 Example 2

Consider the following pair of quadrics:

$$\begin{cases} Q_S : -5x^2 - 2xy - 4y^2 - 12yz - 6yw - 8z^2 - 4zw + w^2 = 0, \\ Q_T : -2x^2 - 2xy + 3y^2 + 6yz + 4z^2 + 2zw + w^2 = 0. \end{cases}$$

The determinantal equation is

$$\mathcal{D}(\lambda,\mu) = \det(\lambda S + \mu T) = -3(16\lambda^4 - 8\lambda^2\mu^2 + \mu^4).$$

Algorithm 6 Classifying the intersection: the two double roots case, $d = 2$ and discriminant $(\mathcal{H}) \neq 0$.

Require: $R(\lambda,\mu), \mathcal{D}, \mathcal{H}$ (from Algorithm 1)
Require: double roots (λ_0,μ_0) and (λ_1,μ_1)

let $\delta :=$ discriminant $(\mathcal{H})$ and s be the sign of $\mathcal{D}$ outside the roots
if $\delta > 0$ and δ is a square **then** *// double roots are real and rational*
  let r_0 and r_1 $(r_0 \leqslant r_1)$ be the ranks of $R(\lambda_0,\mu_0)$ and $R(\lambda_1,\mu_1)$, I_1 the inertia of the second
  if $r_0 = 3$ and $r_1 = 3$ **then** *// cubic and secant line* $(\mathbb{C})$
    output: `cubic and secant line`
  else *//* $r_0 = 2$
    if $r_1 = 3$ **then** *// conic and two lines not crossing on conic* $(\mathbb{C})$
      if $I_1 = (3,0)$ **then** *// cone* $R(\lambda_1,\mu_1)$ *is imaginary*
        output: `point`
      else *// cone* $R(\lambda_1,\mu_1)$ *is real*
        if $s = +1$ **then**
          output: `conic and two lines`
        else
          output: `conic and point`
        end if
      end if
    else *//* $r_1 = 2$*: four skew lines* $(\mathbb{C})$
      if $s = -1$ **then**
        output: `two points`
      else if $R(\lambda,\mu)$ contains a quadric of inertia $(4,0)$ **then**
        output: $\emptyset$
      else
        output: `four skew lines`
      end if
    end if
  end if
else *// double roots are complex or real non-rational*
  let $r_0 := 3 - \dim(\text{singular}(Q_{R(\lambda_0,\mu_0)}))$
  if $r_0 = 2$ **then** *//* $R(\lambda_0,\mu_0)$ *and* $R(\lambda_1,\mu_1)$ *are pairs of planes*
    if $\delta < 0$ **then** *// roots are complex conjugate*
      output: `two skew lines`
    else *//* $\delta > 0$*: roots are real conjugate*
      if $s = -1$ **then**
        output: `two points`
      else if $R(\lambda,\mu)$ contains a quadric of inertia $(4,0)$ **then**
        output: $\emptyset$
      else
        output: `four skew lines`
      end if
    end if
  else *//* $r_0 = 3$*:* $R(\lambda_0,\mu_0)$ *and* $R(\lambda_1,\mu_1)$ *are cones*
    if $\delta < 0$ **then** *// roots are complex conjugate*
      output: `cubic and non-secant line`
    else
      output: `cubic and secant line`
    end if
  end if
end if

The gcd of the partial derivatives is equal to $\mathcal{H} = 12(\mu^2 - 4\lambda^2)$. Since the discriminant δ of $\mathcal{H}$ is not zero, $\mathcal{D}$ has two double roots, according to Algorithm 1. Further, δ is positive and is a square, so $\mathcal{D}$ has two real rational double roots. These roots are $(\lambda_0, \mu_0) = (-1, -2)$ and $(\lambda_1, \mu_1) = (-1, 2)$.

We now follow Algorithm 6. Let $R_0 = \lambda_0 S + \mu_0 T$ and $R_1 = \lambda_1 S + \mu_1 T$. Applying Descartes' Sign Rule, we find that $r_0 = r_1 = 2$. So the intersection, over the complexes, consists of four skew lines. Since $\mathcal{D}(1,0) < 0$, the determinantal equation is negative outside the roots, and $s = -1$. So the intersection, over the reals, consists of two points.

The two points can be computed with the algorithms of Part III:

$$(-3, -3, 3 + \sqrt{3}, -3 - 4\sqrt{3}) \quad \text{and} \quad (-3, -3, 3 - \sqrt{3}, -3 + 4\sqrt{3}).$$

6.3 Example 3

Consider the following pair of quadrics:

$$\begin{cases} Q_S : -2xy + 2xw - y^2 - z^2 + w^2, \\ Q_T : 4xy - 4xw + 2y^2 + z^2 - 2w^2. \end{cases}$$

The determinantal equation vanishes identically. We then follow Algorithm 2. Q_S has rank 3, and its singular point $\mathbf{p}$ has coordinates $(-1, 1, 0, 1)$. Q_T has rank 3, and its singular point is again $\mathbf{p}$. So the dimension κ of the intersection of the singular sets of Q_S and Q_T is 0. Let P be the transformation matrix sending $\mathbf{p}$ to $(0,0,0,1)$, completed as follows:

$$P = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Let $S' = P^T S P$ and $T' = P^T T P$, and remove the last line and column of the two matrices. This gives:

$$\begin{cases} Q_{S'} : -2xy - y^2 - z^2, \\ Q_{T'} : 4xy + 2y^2 + z^2. \end{cases}$$

The restricted determinantal equation is then

$$\mathcal{D}_3(\lambda, \mu) = \lambda^3 - 5\lambda^2\mu + 8\lambda\mu^2 - 4\mu^3.$$

It has a double root at $(\lambda_0, \mu_0) = (2, 1)$. The associated conic $R'_0 = \lambda_0 S' + \mu_0 T'$ has rank 1. So the intersection consists, over the complexes, of two double lines. $\mathcal{D}_3$ has a second root at $(\lambda_1, \mu_1) = (1, 1)$. The associated conic has inertia $(1, 1)$, from which we conclude that the intersection consists, over the reals, of two double lines.

The two lines can easily be parameterized as follows:

$$\mathbf{X}_1(u, v) = (v, -u - v, 0, u - v) \quad \text{and} \quad \mathbf{X}_2(u, v) = (u, v, 0, v), \quad (u, v) \in \mathbb{P}^1(\mathbb{R}).$$

The two lines meet at point $\mathbf{p}$.

7 Conclusion

In this second part of our paper, we have shown how the real type of the intersection of two quadrics can be determined by extracting simple information from the pencil of the two quadrics, and in particular its determinantal equation. Our type-detection algorithm relies on a classification of real pencils of quadrics of $\mathbb{P}^3(\mathbb{R})$, itself derived from the Canonical Pair Form Theorem for pairs of real symmetric matrices ([8, 9]).

In Part III [3], we will use the structural information gathered in the type-detection phase to drive the parameterization process. In each case, we will show that the parameterization computed is near-optimal.

References

[1] T. Bromwich. *Quadratic Forms and Their Classification by Means of Invariant Factors*. Cambridge Tracts in Mathematics and Mathematical Physics, 1906.

[2] L. Dupont, D. Lazard, S. Lazard, and S. Petitjean. Near-optimal parameterization of the intersection of quadrics: I. The generic algorithm. Research Report n° 5667, INRIA, September 2005.

[3] L. Dupont, D. Lazard, S. Lazard, and S. Petitjean. Near-optimal parameterization of the intersection of quadrics: III. Parameterizing singular intersections. Research Report n° 5669, INRIA, September 2005.

[4] I. Emiris and E. Tsigaridas. Comparing real algebraic numbers of small degree. In *Proc. of European Symposium on Algorithms*, pages 652–663, 2004.

[5] W. Hodge and D. Pedoe. *Methods of Algebraic Geometry*. Cambridge University Press, 1953. Volumes I and II.

[6] C. Segre. Studio sulle quadriche in uno spazio lineare ad un numero qualunque di dimensioni. *Mem. della R. Acc. delle Scienze di Torino*, 36(2):3–86, 1883.

[7] Surf: A Tool To Visualize Algebraic Curves and Surfaces. Stephan Endrass, Hans Huelf, Ruediger Oertel, Ralf Schmitt, Kai Schneider and Johannes Beigel. `http://surf.sourceforge.net`.

[8] F. Uhlig. Simultaneous block diagonalization of two real symmetric matrices. *Linear Algebra and Its Applications*, 7:281–289, 1973.

[9] F. Uhlig. A canonical form for a pair of real symmetric matrices that generate a nonsingular pencil. *Linear Algebra and Its Applications*, 14:189–209, 1976.

[10] K. Weierstrass. Zur Theorie der bilinearen und quadratischen Formen. *Monatshefte Akademie der Wissenschaften, Berlin*, pages 310–338, 1868.

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Near-Optimal Parameterization of the Intersection of Quadrics: III. Parameterizing Singular Intersections

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Thème SYM —Systèmes symboliques
Projets Vegas et Salsa

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Abstract: In Part II [3] of this paper, we have shown, using a classification of pencils of quadrics over the reals, how to determine quickly and efficiently the real type of the intersection of two given quadrics.

For each real type of intersection, we design, in this third part, an algorithm for computing a near-optimal parameterization. We also give here examples covering all the possible situations, in terms of both the real type of intersection and the number and depth of square roots appearing in the coefficients.

Key-words: Intersection of surfaces, quadrics, pencils of quadrics, curve parameterization, singular intersections.

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Paramétrisation quasi-optimale de l'intersection de quadriques : III. Paramétrisation des courbes d'intersection singulières

Résumé : Dans la partie II [3] de cet article, nous avons montré, en se basant sur une classification des faisceaux de quadriques de $\mathbb{P}^3(\mathbb{R})$ sur les réels, comment déterminer rapidement et efficacement le type réel de l'intersection de deux quadriques données.

Dans cette troisième partie, nous utilisons l'information collectée pendant la phase de détection du type pour diriger le paramétrage de l'intersection. Dans chaque cas possible, nous donnons la trame d'un algorithme quasi-optimal pour paramétrer la partie réelle de l'intersection et donnons des exemples couvrant toutes les situations possibles en terme de nombre et de profondeur de radicaux impliqués.

Mots-clés : Intersection de surfaces, quadriques, faisceaux de quadriques, paramétrisation, intersections singulières.

1 Introduction

Building on the classification of pencils of quadrics of $\mathbb{P}^3(\mathbb{R})$ over the reals achieved in Part II [3] and the type-detection algorithm that we deduced from this classification, we now are ready to present near-optimal parameterization algorithms for all the possible types of real intersection.

Since the smooth quartic case has already been thoroughly studied in Part I [2], we focus here on the singular cases. For each case, we prove the near-optimality of the parameterization and, when there is possibly an extra square root, we describe the test needed to assert the full optimality which always boils down to finding a rational point on a (possibly non-rational) conic.

In what follows, Q_S, Q_T refer to the initial quadrics and Q_R (assumed to be distinct from Q_S) to the intermediate quadric used to parameterize the intersection C of Q_S and Q_T. As in Section I.7.1, denote by Ω the equation in the parameters:

$$\Omega : \mathbf{X}^T S \mathbf{X} = 0,$$

where $\mathbf{X}$ is the parameterization of Q_R. Denote also by C_Ω the curve zero-set of Ω. Recall that the parameterization of Q_R defines an isomorphism between C and the plane curve C_Ω. When C is singular, its genus is 0 so it can be parameterized by rational functions (i.e. $\sqrt{\Delta}$ can be avoided).

Our general philosophy is to use for Q_R the rational quadric of the pencil of smallest rank. This will lead us to use repeatedly the results of Section I.6 on the optimality of parameterizations of projective quadrics and to parameterize cones without a rational point, cones with a rational point, pairs of planes, etc. As will be seen, this philosophy has the double advantage of (i) avoiding $\sqrt{\Delta}$ in all singular cases, and (ii) minimizing the number of radicals. As an additional benefit, it helps keep the size of the numbers involved in intermediate computations and in the final parameterizations to a minimum (see Part IV [4]).

For every type of real intersection, we give a set of worst-case examples where the maximum number of square roots is reached, both in the optimal and near-optimal situations (the best-case examples are those given by the canonical forms of Section II.3). Examples covering all possible situations are gathered in Appendix C.

A summary of the results of this part is given in Table 1.

The rest of this third part is as follows. Section 2 gives near-optimal parameterization algorithms for all types of real intersection when the pencil is regular. Section 3 does the same for singular pencils (i.e., when the determinantal equation vanishes identically). Several examples are detailed in Section 4 and it is shown how our implementation fares on these examples. Finally, we conclude in Section 5 and give a few perspectives.

2 Parameterizing degenerate intersections: regular pencils

In this section, we outline parameterization algorithms for all cases of regular pencils, i.e. when the determinantal equation does not identically vanish. Information gathered in the type-detection phase (Part II) is used as input; see in particular Table II.4 but also the details of the classification of pencils over the reals. In each case, we study optimality issues and give worst-case examples.

Segre string	real type of intersection	worst case format of parameterization	worst-case optimality of parameterization
[1111]	nonsingular quartic (see part I)	$\mathbb{Q}(\sqrt{\delta})[\xi, \sqrt{\Delta}]$, $\Delta \in \mathbb{Q}(\sqrt{\delta})[\xi]$	rational point on degree-8 surface
[112]	point	$\mathbb{Q}$	optimal
	nodal quartic	$\mathbb{Q}(\sqrt{\delta})[\xi]$	rational point on conic
[13]	cuspidal quartic	$\mathbb{Q}[\xi]$	optimal
[22]	cubic and non-tangent line	$\mathbb{Q}[\xi]$	optimal
[4]	cubic and tangent line	$\mathbb{Q}[\xi]$	optimal
[11(11)]	two points	$\mathbb{Q}(\sqrt{\delta})$	optimal
	conic	$\mathbb{Q}(\sqrt{\delta}, \sqrt{\mu})[\xi]$, $\mu \in \mathbb{Q}(\sqrt{\delta})$	optimal if $\sqrt{\delta} \notin \mathbb{Q}$ rational point on conic if $\sqrt{\delta} \in \mathbb{Q}$
	two non-tangent conics	$\mathbb{Q}(\sqrt{\delta}, \sqrt{\delta'})[\xi]$	$\mathbb{Q}(\sqrt{\delta'})$-rational point on $\mathbb{Q}(\sqrt{\delta'})$-conic
[1(21)]	point	$\mathbb{Q}$	optimal
	two tangent conics	$\mathbb{Q}(\sqrt{\delta})[\xi]$	optimal
[1(111)]	double conic	$\mathbb{Q}(\sqrt{\delta})[\xi]$	rational point on conic
[2(11)]	point	$\mathbb{Q}$	optimal
	conic and point	$\mathbb{Q}(\sqrt{\delta})[\xi]$	rational point on conic
	conic and two lines not crossing on the conic	$\mathbb{Q}(\sqrt{\delta})[\xi]$	rational point on conic
[(31)]	conic	$\mathbb{Q}[\xi]$	optimal
	conic and two lines crossing on the conic	$\mathbb{Q}(\sqrt{\delta})[\xi]$	optimal
[(11)(11)]	two points	$\mathbb{K}[\xi]$, degree $(\mathbb{K}) = 4$	optimal
	two skew lines	$\mathbb{K}[\xi]$, degree $(\mathbb{K}) = 4$	optimal
	four skew lines	$\mathbb{K}[\xi]$, degree $(\mathbb{K}) = 4$	optimal
[(22)]	double line	$\mathbb{Q}[\xi]$	optimal
	two simple skew lines cutting a double line	$\mathbb{Q}(\sqrt{\delta})[\xi]$	optimal
[(211)]	point	$\mathbb{Q}$	optimal
	two double concurrent lines	$\mathbb{Q}(\sqrt{\delta})[\xi]$	optimal
[1{3}]	conic and double line	$\mathbb{Q}[\xi]$	optimal
[111]	point	$\mathbb{Q}$	optimal
	two concurrent lines	$\mathbb{K}[\xi]$, degree $(\mathbb{K}) = 4$	optimal
	four concurrent lines	$\mathbb{K}[\xi]$, degree $(\mathbb{K}) = 4$	optimal
[12]	double line	$\mathbb{Q}[\xi]$	optimal
	two simple and a double concurrent lines	$\mathbb{Q}(\sqrt{\delta})[\xi]$	optimal
[3]	concurrent simple and triple lines	$\mathbb{Q}[\xi]$	optimal
[1(11)]	point	$\mathbb{Q}$	optimal
	two concurrent double lines	$\mathbb{Q}(\sqrt{\delta})[\xi]$	optimal
[(21)]	quadruple line	$\mathbb{Q}[\xi]$	optimal
[11]	quadruple line	$\mathbb{Q}[\xi]$	optimal

Table 1: Ring of definition of the projective coordinates of the parameterization of each component of the intersection and optimality, in all cases where the real part of the intersection is 0- or 1-dimensional. $\delta, \delta' \in \mathbb{Q}$.

In the following, we often need to compute the parameterization of the intermediate quadric Q_R and this is achieved using the normal form of Q_R. Recall that a rational congruence sending a quadric with rational coefficients into normal form can be computed using Gauss reduction of quadratic forms into sums of squares (see Part I).

Recall also that the discriminant of a quadric is the determinant of the associated matrix. In the following, we also call *discriminant of a pair of planes* Q_R the product ab where $ax^2 - by^2 = 0$ is the canonical equation of a pair of planes obtained from Q_R by a real rational congruence transformation; the discriminant is defined up to a rational square factor.

2.1 Nodal quartic in $\mathbb{P}^3(\mathbb{C})$, $\sigma_4 = [112]$

If we parameterize C using the generic algorithm (see Part I), we will not be able to avoid the appearance of $\sqrt{\Delta}$ because C_Ω (as C) is irreducible. However, since the intersection curve is singular, we we know that $\sqrt{\Delta}$ is avoidable by Proposition I.7.1. We thus proceed differently.

2.1.1 Algorithms

Let λ_1 be the real and rational double root of the determinantal equation. Let Q_R be the rational cone associated with λ_1. As we have found in Section II.3, there are essentially two cases depending on the real type of the intersection.

Point. Q_R is an imaginary cone. The intersection is reduced to a point, which is the apex of Q_R. Since λ_1 is rational, this apex is rational (otherwise its algebraic conjugate would also be a singular point of the cone). Thus the intersection in this case is defined in $\mathbb{Q}$.

Real nodal quartic (with or without isolated singularity). Let P be a real rational congruence transformation sending the apex of Q_R to $(0,0,0,1)$. The parameterization $\mathbf{X}(u,v,s) = P\,(x_1(u,v), x_2(u,v), x_3(u,v), s)^T, (u,v,s) \in \mathbb{P}^{\star 2}$ of the cone (see Table I.3) introduces a square root $\sqrt{\delta}$. Equation Ω in the parameters is

$$as^2 + b(u,v)s + c(u,v) = 0,$$

with a and the coefficients of b, c defined in $\mathbb{Q}(\sqrt{\delta})$. The nodal quartic passes through the vertex of Q_R and this point corresponds to the value $u = v = 0$ of the parameters. At this point $s \neq 0$ because $(u,v,s) \in \mathbb{P}^{\star 2}$. Thus $a = 0$, Ω is linear in s and it can be solved rationally for s. This leads to the parameterization of the quartic

$$\mathbf{X}(u,v) = P\,(b(u,v)x_1(u,v), b(u,v)x_2(u,v), b(u,v)x_3(u,v), -c(u,v))^T.$$

The coefficients of $\mathbf{X}(u,v)$ clearly live in $\mathbb{Q}(\sqrt{\delta})[\xi]$, where $\xi = (u,v)$.

When the node of the quartic is not isolated, the singularity is now reached by two different values of $(u,v) \in \mathbb{P}^1(\mathbb{R})$ which are precisely those values such that $b(u,v) = 0$. When the node of the quartic is isolated, the singularity is not reached by $(u,v) \in \mathbb{P}^1(\mathbb{R})$, i.e. $b(u,v) = 0$ has no real solution. In that situation, the node has to be added to the output. Since this point is the vertex of the cone Q_R, it is rational.

2.1.2 Optimality

By Proposition I.6.3, if the cone Q_R contains a rational point other than its vertex, it can be parameterized with rational coefficients and thus the parameterization of the nodal quartic is defined over $\mathbb{Q}[\xi]$. Otherwise, if Q_R contains no rational point other than its vertex, then the nodal quartic also contains no rational point other than its singular point. Hence the nodal quartic admits no parameterization over $\mathbb{Q}[\xi]$. Therefore, testing whether $\sqrt{\delta}$ can be avoided in the parameterization of real nodal quartics is akin to deciding whether Q_R has a rational point outside its singular locus; furthermore, finding a parameterization in $\mathbb{Q}[\xi]$ amounts to finding a rational point on Q_R outside its singular locus.

There are cases where $\sqrt{\delta}$ cannot be avoided. Example of these are

$$\begin{cases} x^2+y^2-3z^2=0, \\ xw+z^2=0 \end{cases}$$

when the singularity is not isolated and

$$\begin{cases} x^2+y^2-3z^2=0, \\ zw+x^2=0 \end{cases}$$

when the singularity is isolated. In both cases, the projective cone corresponding to the double root of the determinantal equation is the first equation. By Proposition I.6.3, this cone has no rational point except its singular point and $\sqrt{\delta}$ cannot be avoided in the parameterization of the intersection.

2.2 Cuspidal quartic in $\mathbb{P}^3(\mathbb{C})$, $\sigma_4 = [13]$

The intersection in this case is always a real cuspidal quartic. As above, using the generic algorithm is not good idea: it would introduce an unnecessary and unwanted $\sqrt{\Delta}$.

We consider instead the cone Q_R associated with the real and rational triple root of the determinantal equation. The singular point of the quartic is the vertex $\mathbf{p}$ of Q_R. The intersection of Q_R with the tangent plane of Q_S at $\mathbf{p}$ consists of the double line tangent to C at the cusp. Since it is double, this line is necessarily rational. So we have a rational cone containing a rational line. By Theorem I.6.1, this cone admits a rational parameterization.

So we are left with an equation $\Omega : as^2+b(u,v)s+c(u,v)=0$ whose coefficients are defined on $\mathbb{Q}$. As above, the singularity is reached at $(u,v)=(0,0)$ and at this point $s \neq 0$, so $a=0$. Thus Ω can be solved rationally for s and the intersection is in $\mathbb{Q}[\xi], \xi=(u,v)$. This is optimal.

2.3 Cubic and secant line in $\mathbb{P}^3(\mathbb{C})$, $\sigma_4 = [22]$

The real intersection consists of a cubic and a line. The cubic and the line are either secant or skew. Note that the line of the intersection is necessarily rational, otherwise its algebraic conjugate would also belong to the intersection.

When the double roots of the determinantal equation are real and rational, the pencil contains two rational cones Q_{R_1} and Q_{R_2}. The line of C is the rational line joining the vertices of Q_{R_1} and Q_{R_2}.

Also, the vertex of Q_{R_2} is a rational point on Q_{R_1}, and vice versa, so the two cones can be rationally parameterized (see Theorem I.6.1). Setting up Ω, we have again that it is linear in s, because the line and the cubic intersect at the vertex of the cone, corresponding to $(u,v) = (0,0)$. But here the content of Ω in s is linear in (u,v) and it corresponds to the line of C. The cubic is found after dividing by this content and rationally solving for s. The parameterization of the cubic is defined in $\mathbb{Q}[\xi]$.

When the double roots of the determinantal equation are either complex conjugate (the cubic and the line are not secant) or real algebraic conjugate (the cubic and the line are secant), there exists quadrics of inertia $(2,2)$ in the pencil (by Theorems I.4.1 and I.4.3). We use the generic algorithm of Part I: first find a quadric Q_R of inertia $(2,2)$ of the pencil through a rational point. Since C contains a rational line, the discriminant of this quadric is a square by Lemma I.7.4 and Q_R can be rationally parameterized by Theorem I.6.1. Now compute the bidegree $(2,2)$ equation Ω. The line of C corresponds to a fixed value of one of the parameters and the contents provide factors of bidegree $(1,0)$ and $(1,2)$ (or $(0,1)$ and $(2,1)$), which are linear in one of the parameters and thus easy to solve rationally for getting a parameterization of the intersection. The parameterization of C is defined in $\mathbb{Q}[\xi]$.

2.4 Cubic and tangent line in $\mathbb{P}^3(\mathbb{C})$, $\sigma_4 = [4]$

The real intersection consists of a cubic and a tangent line. The line is necessarily rational, by the same argument as above. The determinantal equation has a real and rational quadruple root. To it corresponds a real rational projective cone. Since this cone contains a rational line, it can be rationally parameterized (by Theorem I.6.1). The rest is as in the cubic and secant line case when the two roots are rational. The parameterization of the cubic is defined in $\mathbb{Q}[\xi]$.

2.5 Two secant conics in $\mathbb{P}^3(\mathbb{C})$, $\sigma_4 = [11(11)]$

In this case, the determinantal equation has a double root corresponding to a rational pair of planes Q_R. There are several cases depending on the real type of the intersection.

2.5.1 Two points

The pair of planes Q_R is imaginary. Its rational singular line intersects any other quadric of the pencil in two points. So parameterize the line and intersect it with any quadric of the pencil having rational coefficients. A square root is needed to parameterize the two points if and only if the equation in the parameters of the line has irrational roots.

This situation can happen as the following example shows:

$$\begin{cases} z^2 + w^2 = 0, \\ x^2 - 2y^2 + w^2 = 0. \end{cases}$$

Clearly, the two points are defined by $z = w = 0$ and $x^2 - 2y^2 = 0$ so they live in $\mathbb{Q}[\sqrt{2}]$.

2.5.2 One conic

In this case, the pair of planes is real, the pencil has no quadric of inertia $(2,2)$ and only one of the planes of Q_R intersects the other quadrics of the pencil.

The algorithm is as follows. First parameterize the pair of planes and separate the two individual planes. Plugging the parameterization of each plane into the equation of Q_S gives two equations of conics in parameter space, with coefficients in $\mathbb{Q}(\sqrt{\delta})$ where δ is the discriminant of the pair of planes. The conics in parameter space correspond to the components of the intersection, thus one of these conics is real and the other is imaginary. Determine the real conic, that is the one with inertia $(2,1)$, and parameterize it. Substituting this parameterization into the parameterization of the corresponding plane gives a parameterization of the conic of intersection. The parameterization is in $\mathbb{Q}(\sqrt{\delta},\sqrt{\mu})$, where δ is the discriminant of the pair of planes Q_R and $\sqrt{\mu}$ is the square root needed to parameterize the conic in parameter space, $\mu \in \mathbb{Q}(\sqrt{\delta})$.

If δ is not a square, the parameterization is optimal. Indeed, if the intersection had a real $\mathbb{Q}(\sqrt{\delta})$-rational point, the conjugate of that point would be on the conjugate conic which is not real. So such a point does not exist and the parameterization is optimal. If δ is a square, the parameterization is defined in $\mathbb{Q}(\sqrt{\mu})[\xi]$ with $\mu \in \mathbb{Q}$. By Proposition I.6.3, the parameterization is optimal if and only if the (rational) conic contains no rational point; moreover, testing if the parameterization is non-optimal and, if so, finding an optimal parameterization is equivalent to finding a rational point on this rational conic.

The situation where δ is a square but the conic has no rational point (the field of the coefficients is of degree two) can be attained for instance with the following pair of quadrics:

$$\begin{cases} (x-w)(x-3w)=0, \\ x^2+y^2+z^2-4w^2=0. \end{cases}$$

The two planes of the first quadric are rational. The plane $x-w=0$ cuts the second quadric in the conic $x-w=y^2+z^2-3w^2=0$. By Proposition I.6.3, this conic has no rational point, so $\sqrt{\delta}$ cannot be avoided and the parameterization of the conic is in $\mathbb{Q}(\sqrt{3})$.

A field extension of degree 4 is obtained with the following quadrics:

$$\begin{cases} x^2-4xw-3w^2=0, \\ x^2+y^2+z^2-w^2=0. \end{cases}$$

The pair of planes is defined on $\mathbb{Q}(\sqrt{7})$, so, by the above argument, a field extension of degree 4 is unavoidable.

2.5.3 Two (secant or non-secant) conics

By contrast to the one conic case, the pencil now contains quadrics of inertia $(2,2)$. But going through the generic algorithm and factoring C_Ω directly in two curves of bidegree $(1,1)$ can induce nested radicals. So we proceed as follows. First, find a rational quadric Q_R of inertia $(2,2)$ through a rational point. This introduces one square root, say $\sqrt{\delta}$. Independently, factor the pair of planes, which introduces another square root $\sqrt{\delta'}$. Now plug the parameterization of Q_R in each of the planes. This gives linear equations in the parameters of Q_R which can be solved without introducing nested radicals. The two conics have a parameterization defined in $\mathbb{Q}(\sqrt{\delta}, \sqrt{\delta'})$.

Note that when the two simple roots of the determinantal equation are rational, an alternate approach is to parameterize one of the two rational cones of the pencil instead of a quadric of inertia $(2,2)$, and then proceed as above.

In terms of optimality, $\sqrt{\delta'}$ cannot be avoided if the planes are irrational. As for the other square root, it can be avoided if and only if the conics contain a point that is rational in $\mathbb{Q}(\sqrt{\delta'})$ (by Proposition I.6.3 in which the field $\mathbb{Q}$ can be replaced by $\mathbb{Q}(\sqrt{\delta'})$); moreover, testing if this square root can be avoided and, if so, finding a parameterization avoiding it is equivalent to finding a $\mathbb{Q}(\sqrt{\delta'})$-rational point on this conic whose coefficients are in $\mathbb{Q}(\sqrt{\delta'})$.

All cases can happen. We illustrate this in the non-secant case. An extension of $\mathbb{Q}$ of degree 4 is needed to parameterize the intersection of the following pair:

$$\begin{cases} x^2 - 33w^2 = 0, \\ y^2 + z^2 - 3w^2 = 0. \end{cases}$$

Indeed, $\sqrt{\delta'} = \sqrt{33}$ cannot be avoided. In addition, by Proposition I.6.6, $y^2+z^2-3w^2-11x^2=0$ has no rational point on $\mathbb{Q}(\sqrt{33})$, thus its intersection with the plane $x=0$, the conic $y^2+z^2-3w^2=0$, also has no rational point on $\mathbb{Q}(\sqrt{33})$; hence the cone $y^2+z^2-3w^2=0$ has no rational point on $\mathbb{Q}(\sqrt{33})$ except for its singular locus.

An extension field of degree 2 can be obtained by having conics without rational point, but living in rational planes, as in this example:

$$\begin{cases} x^2 - w^2 = 0, \\ y^2 + z^2 - 3w^2 = 0. \end{cases}$$

It can also be attained by having conics living in non-rational planes but having rational points in the extension of $\mathbb{Q}$ defined by the planes:

$$\begin{cases} x^2 - 3w^2 = 0, \\ y^2 + z^2 - 3w^2 = 0. \end{cases}$$

As can be seen, the points of coordinates $(\sqrt{3}, 0, \pm\sqrt{3}, 1)$ belong to the intersection. So the conic has a parameterization in $\mathbb{Q}(\sqrt{3})[\xi]$.

2.6 Two tangent conics in $\mathbb{P}^3(\mathbb{C})$, $\sigma_4 = [1(21)]$

Here, the determinantal equation has a real and rational triple root, corresponding to a pair of planes Q_R. The other (real and rational) root corresponds to a real projective cone. There are two types of intersection over the reals.

Point. The pair of planes is imaginary and its rational singular line intersects the cone in a double point, which is the only component of the intersection. This point is necessarily rational, otherwise its conjugate would also be in the intersection. One way to compute it is to parameterize the singular line, plug the parameterization in the rational equation of the cone and solve the resulting equation in the parameters.

Two real tangent conics. The pair of planes is real and each of the planes intersects the cone. The singular line of Q_R is tangent to the cone. As above, the point of tangency of the two conics is rational. So, by Proposition I.6.3, the conics have a rational parameterization in the extension of $\mathbb{Q}$ defined by the planes. In other words, the conics have a parameterization defined in $\mathbb{Q}(\sqrt{\delta})[\xi]$, where δ is the discriminant of the pair of planes Q_R, if and only if δ is not a square.

One situation where $\sqrt{\delta}$ cannot be avoided is the following:

$$\begin{cases} x^2 - 2w^2 = 0, \\ xy + z^2 = 0. \end{cases}$$

2.7 Double conic in $\mathbb{P}^3(\mathbb{C})$, $\sigma_4 = [1(111)]$

The determinantal equation has a real rational triple root, corresponding to a double plane. The other root gives a rational cone. Assume this cone is real (otherwise the intersection is empty).

To obtain the parameterization of the double conic, first parameterize the double plane. Then plug this parameterization in the equation of the cone. This gives the rational equation of the conic (in the parameters of the plane). If the conic has a rational point, then it can be rationally parameterized. Otherwise, one square root is needed.

One worst-case situation where a square root is always needed is the following:

$$\begin{cases} x^2 = 0, \\ y^2 + z^2 - 3w^2 = 0. \end{cases}$$

By Proposition I.6.3, the second quadric (a cone) has no rational point outside its vertex. Thus the conic cannot be parameterized rationally.

2.8 Conic and two lines not crossing on the conic in $\mathbb{P}^3(\mathbb{C})$, $\sigma_4 = [2(11)]$

The determinantal equation has two double roots, corresponding to a cone and a pair of planes which is always real. The two roots are necessarily real and rational, otherwise the quadrics associated with them in the pencil would have the same rank. So both the cone and the pair of planes are rational. Also, the vertex of the cone falls on the pair of planes outside its singular line. Thus, by Proposition I.6.2, the discriminant of the pair of planes is a square and each individual plane has a rational parameterization.

Over the reals, there are three cases.

Point. The projective cone is imaginary. The intersection is limited to its real vertex. Since the cone is rational, its vertex is rational.

Point and conic. The cone is now real. One of the planes cuts the cone in a conic living in a rational plane, the other plane cuts the cone in its vertex. The point of the intersection is this vertex and it is rational. To parameterize the conic of the intersection, plug the parameterization of the plane that does not go through the vertex of the cone. This gives a rational conic in the parameters of the plane. One square root is possibly needed to parameterize this conic. It can be avoided if and only if the conic has a rational point.

One example where the square root cannot be avoided is the following:

$$\begin{cases} xw = 0, \\ y^2 + z^2 - 3w^2 = 0. \end{cases}$$

By Proposition I.6.3, the projective cone has no rational point other than its vertex $(1,0,0,0)$. So the conic $x = y^2 + z^2 - 3w^2 = 0$ has no rational point.

Two lines and conic. Again, the cone is real and one plane cuts it in a rational nonsingular conic. But now the second plane, going through the vertex of the cone, further cuts the cone in two lines. The parameterization of the conic goes as above. To represent the lines, we plug the second plane in the equation of the cone and parameterize.

Note that if the lines are rational, then the cone contains a rational line and can be rationally parameterized. Since the conic is the intersection of this cone with a rational plane, it has a rational parameterization. So in that case all three components have parameterizations in $\mathbb{Q}[\xi]$. If the lines are irrational, it can still happen that the conic has a rational point and thus a rational parameterization.

We give examples for the three situations we just outlined. First, the pair

$$\begin{cases} xy = 0, \\ y^2 + z^2 - w^2 = 0 \end{cases}$$

gives birth to the rational lines $y = z \pm w = 0$ and the rational conic $x = y^2 + z^2 - w^2 = 0$ which contains the rational point $(0,0,1,1)$ and can be rationally parameterized. Second, the pair of quadrics

$$\begin{cases} xy = 0, \\ 2y^2 + z^2 - 3w^2 = 0 \end{cases}$$

has as intersection the two irrational lines $y = z \pm \sqrt{3}w = 0$ and the conic $x = 2y^2 + z^2 - 3w^2 = 0$ which contains the rational point $(0,1,1,1)$ so can be rationally parameterized. Finally, the lines and the conic making the intersection of the quadrics

$$\begin{cases} xy = 0, \\ y^2 + z^2 - 3w^2 = 0 \end{cases}$$

cannot be rationally parameterized. Indeed, by Proposition I.6.3, the cone has no rational point outside the vertex $(1,0,0,0)$, so the conic $x = y^2 + z^2 - 3w^2 = 0$ has no rational point.

2.9 Conic and two lines crossing on the conic in $\mathbb{P}^3(\mathbb{C})$, $\sigma_4 = [(31)]$

The determinantal equation has a real pair of planes Q_R corresponding to a real and rational quadruple root. The asymmetry in the sizes of the Jordan blocks associated with this root (the two blocks have size 1 and 3) implies that the individual planes of this pair are rational. The conic of the intersection is always real and the two lines (real or imaginary) cross on the conic.

There are two types of intersection over the reals.

Conic. The point at which the two lines cross is the double point that is the intersection of the singular line of Q_R with any other quadric of the pencil. This point is necessarily rational. So the conic can be rationally parameterized by Proposition I.6.3.

Conic and two lines. To parameterize the intersection, first compute the parameterization of the two planes of Q_R. Plugging these parameterizations in the equation of any other quadric of the pencil yields a conic on one side and a pair of lines on the other side. As above, the conic can be rationally parameterized. As for the two lines, they have a rational parameterization if and only if the discriminant of the pair of lines is a square.

One situation where this discriminant is not a square is as follows:

$$\begin{cases} yz = 0, \\ y^2 + xz - 2w^2 = 0. \end{cases}$$

The conic is given by $y = xz - 2w^2 = 0$ which contains the rational point $(1,0,0,0)$ and can be rationally parameterized. The lines are defined by $z = y^2 - 2w^2 = 0$. But the pair of planes $y^2 - 2w^2 = 0$ has no rational point outside its singular locus so the lines are defined in $\mathbb{Q}(\sqrt{2})$.

2.10 Two skew lines and a double line in $\mathbb{P}^3(\mathbb{C})$, $\sigma_4 = [(22)]$

The determinantal equation has a real and rational quadruple root, which corresponds to a pair of planes. The singular line of the pair of planes is contained in all the quadrics of the pencil. There are two cases.

Double line. The pair of planes is imaginary. The intersection is reduced to the rational singular line of the pair of planes. So C is defined in $\mathbb{Q}[\xi]$.

Two simple lines and a double line. The pair of planes is real. We can factor it into simple planes, parameterize these planes and plug them in any other quadric of the pencil. The two resulting equations in the parameters of the planes are pairs of lines, each pair containing the double line of the intersection and one of the simple lines. The simple lines are rational if and only if the discriminant of the pair of planes is a square.

A situation where the two simple lines are irrational is the following:

$$\begin{cases} y^2 - 2w^2 = 0, \\ xy - zw = 0. \end{cases}$$

2.11 Two double lines in $\mathbb{P}^3(\mathbb{C})$, $\sigma_4 = [(211)]$

The determinantal equation has a real rational quadruple root, which corresponds to a double plane. The double plane cuts any other quadric of the pencil in two double lines in $\mathbb{P}^3(\mathbb{C})$. There are two cases.

Point. Except for the double plane, the pencil consists of quadrics of inertia $(3,1)$. The two lines are imaginary. The intersection is reduced to their rational intersection point, i.e. the point at which the double plane is tangent to the other quadrics of the pencil.

Two real double lines. Except for the double plane, the pencil consists of quadrics of inertia $(2,2)$. The two lines are real. To parameterize them, first compute a parameterization of the double plane and then plug it in any quadric of inertia $(2,2)$ of the pencil. The resulting pair of lines can easily be parameterized. The intersection is thus parameterized with one square root if and only if the lines are irrational.

One case where the square root cannot be avoided is as follows:

$$\begin{cases} w^2 = 0, \\ x^2 - 2y^2 + zw = 0. \end{cases}$$

The lines $w = x^2 - 2y^2 = 0$ have no rational point except for their singular point $(0,0,1,0)$ so their parameterization is in $\mathbb{Q}(\sqrt{2})[\xi]$.

2.12 Four skew lines in $\mathbb{P}^3(\mathbb{C})$, $\sigma_4 = [(11)(11)]$

We start by describing the algorithms we use in this case. We then prove the optimality of the parameterizations and conclude the section by giving examples of pairs of rational quadrics for all possible types of real intersections and extension fields.

2.12.1 Algorithms

In this case the determinantal equation has two double roots that correspond to (possibly imaginary) pairs of planes. It can be written in the form

$$\mathcal{D}(\lambda, \mu) = \gamma \left(a\lambda^2 + b\lambda\mu + c\mu^2\right)^2 = 0, \tag{1}$$

with γ, a, b, and c in $\mathbb{Q}$.

In order to minimize the number and depth of square roots in the coefficients of the parameterization of the intersection, we proceed differently depending on the type of the real intersection and the values of γ and $\delta = b^2 - 4ac$.

Note that the roots of the determinantal equation are defined in $\mathbb{Q}(\sqrt{\delta})$ and thus the coefficients of the pairs of planes in the pencil also live in $\mathbb{Q}(\sqrt{\delta})$. Let $d^+, d^- \in \mathbb{Q}(\sqrt{\delta})$ be the discriminants of the two pairs of planes, with $d^+ > d^-$. When $d^+ > 0$ (resp. $d^- > 0$), the corresponding pair of planes is real and can be factored into two planes that are defined over $\mathbb{Q}(\sqrt{d^+})$ (resp. $\mathbb{Q}(\sqrt{d^-})$). The algorithms in the different cases are as follows.

Two points. In this case one pair of planes of the pencil is real (the one with discriminant d^+) and the other is imaginary. We factor the two real planes and substitute in each a parameterization of the (real) singular line of the imaginary pair of planes. The singular line is defined in $\mathbb{Q}(\sqrt{\delta})$

and each of the real planes are defined in $\mathbb{Q}(\sqrt{d^+})$. We thus obtain the two points of intersection with coordinates in $\mathbb{Q}(\sqrt{\delta}, \sqrt{d^+})$. The two points are thus defined over $\mathbb{Q}(\sqrt{d^+})$, $d^+ \in \mathbb{Q}(\sqrt{\delta})$, an extension field of degree 4 (in the worst case) with one nested square root.

Two or four lines. Since the intersection is contained in every quadric of the pencil, there are no quadric of inertia $(3,1)$ in the pencil in this case (such quadrics contain no line) and thus $\gamma > 0$. Furthermore all the non-singular quadrics of the pencil have inertia $(2,2)$ (by Theorem I.4.3) and their discriminant is equal to γ, up to a square factor (by Eq. (1)). Hence we can parameterize a quadric Q_R of inertia $(2,2)$ in the pencil using the parameterization of Table I.3 with coefficients in $\mathbb{Q}(\sqrt{\gamma})$ (see Part I).

There are three subcases.

$\sqrt{\delta} \in \mathbb{Q}$. The roots of the determinantal equation are real (since $\delta > 0$), thus the intersection consists of four real lines and the two pairs of planes of the pencil are real (see Table II.4). We factor the two pairs of planes into four planes with coefficients in $\mathbb{Q}(\sqrt{d^\pm})$ and intersect them pairwise. We thus obtain a parameterization of the four lines over $\mathbb{Q}(\sqrt{d^+}, \sqrt{d^-})$ with $d^\pm \in \mathbb{Q}$ (since δ is a square), an extension field of degree 4 (in the worst case) with no nested square root.

$\sqrt{\delta} \notin \mathbb{Q}$ **and** $\sqrt{\gamma\delta} \in \mathbb{Q}$. Here again $\delta > 0$ thus the intersection consists of four real lines and the two pairs of planes of the pencil are real. We factor one of these pairs of planes (say the one with discriminant d^+) in two planes with coefficients in $\mathbb{Q}(\sqrt{d^+})$; if the discriminant of one of the pair of planes is a square, we choose this pair of planes for the factorization. We then substitute the parameterization of the quadric Q_R into each plane. This leads to an equation of bidegree $(1,1)$ in the parameters with coefficients in $\mathbb{Q}(\sqrt{d^+}, \sqrt{\gamma})$. This field is equal to $\mathbb{Q}(\sqrt{d^+})$ because $d^+ \in \mathbb{Q}(\sqrt{\delta})$ and $\gamma\delta$ is a square. We finally obtain each line by factoring the equation in the parameters into to terms of bidegree $(1,0)$ and $(0,1)$ and by substituting the solutions of these factors into the parameterization of Q_R. We thus obtain a parameterization of the four lines defined over $\mathbb{Q}(\sqrt{d^+})$, $d^+ \in \mathbb{Q}(\sqrt{\delta})$, an extension field of degree 4 (in the worst case) with one nested square root.

$\sqrt{\delta} \notin \mathbb{Q}$ **and** $\sqrt{\gamma\delta} \notin \mathbb{Q}$. In this case we apply the generic algorithm of Part I: we substitute the parameterization of Q_R into the equation of another quadric of the pencil (with rational coefficients). The resulting equation in the parameters of bidegree $(2,2)$ has coefficients in $\mathbb{Q}(\sqrt{\gamma})$. We factor it into two terms of bidegree $(2,0)$ and $(0,2)$, whose coefficients also live in $\mathbb{Q}(\sqrt{\gamma})$. We solve each term separately and each real solution leads to a real line. At least one of the two factors has two real solutions, which are defined in an extension field of the form $\mathbb{Q}(\sqrt{\alpha_1 + \alpha_2\sqrt{\gamma}})$, $\alpha_i \in \mathbb{Q}$. If the other factor has real solutions, they are defined in $\mathbb{Q}(\sqrt{\alpha_1 - \alpha_2\sqrt{\gamma}})$. Thus in the case where the intersection consists of two real lines, we obtain parameterization defined over an extension field $\mathbb{Q}(\sqrt{\alpha_1 + \alpha_2\sqrt{\gamma}})$ of degree 4 (in the worst case), with one nested square root. In the case where the intersection consists of four real lines, the parameterization of the four lines altogether is defined over an extension field of degree 8 (in the worst case) but each of the lines is parameterized over an extension $\mathbb{Q}(\sqrt{\alpha_1 + \alpha_2\sqrt{\gamma}})$ or $\mathbb{Q}(\sqrt{\alpha_1 - \alpha_2\sqrt{\gamma}})$ of degree 4 (in the worst case), with one nested square root.

2.12.2 Optimality

We prove that the algorithms described above output parameterizations that are always optimal in the number and depth of square roots appearing in their coefficients. This proof needs some considerations of Galois theory that can be found in Appendix A.

The two input quadrics intersect here in four lines in $\mathbb{P}^3(\mathbb{C})$. The pencil contains two (possibly complex) pair of planes and these lines are the intersections between two planes taken in two different pairs of planes. Let $\mathbf{p}_1, \ldots, \mathbf{p}_4$ be their pairwise intersection points of the four lines. These points are the singular points of the intersection. These points are also the intersections of the singular line of a pair of planes with the other pair of planes, and vice versa. Let the points be numbered such that $\mathbf{p}_1$ and $\mathbf{p}_3$ are on the singular line of one pair of planes of the pencil; $\mathbf{p}_2$ and $\mathbf{p}_4$ are then on the singular line of the other pair of planes of the pencil. The four lines of intersection are thus $\mathbf{p}_1\mathbf{p}_2$, $\mathbf{p}_2\mathbf{p}_3$, $\mathbf{p}_3\mathbf{p}_4$, and $\mathbf{p}_4\mathbf{p}_1$.

Let $\mathbb{K}$ be the field of smallest degree on which the four points $\mathbf{p}_i$ are rational. The above algorithms show that $\mathbb{K}$ has degree 1, 2, 4 or 8 (since two rational lines in $\mathbb{K}$ intersect in a rational point in $\mathbb{K}$). Let G be its Galois group, which acts by permutations on the points $\mathbf{p}_i$. It follows that G is a subgroup of the dihedral group D_4 of order 8 of the symmetries of the square. This group D_4 acts on the four points $\mathbf{p}_i$ and on the lines joining them the way the 8 isometries of a square act on its vertices and edges. We show that the optimal number of square roots needed for parameterizing the four lines and the way this optimal number is reached only depend on G and on its action on the $\mathbf{p}_i$.

The eight elements of D_4 are the identity, the transpositions τ_{13} and τ_{24} which exchange $\mathbf{p}_1$ and $\mathbf{p}_3$ or $\mathbf{p}_2$ and $\mathbf{p}_4$ (symmetries with respect to the diagonal), the permutation $\tau_{12,34}$ (resp. $\tau_{14,23}$) of order 2 which exchange $\mathbf{p}_1$ with $\mathbf{p}_2$ and $\mathbf{p}_3$ with $\mathbf{p}_4$ (resp. $\mathbf{p}_1$ with $\mathbf{p}_4$ and $\mathbf{p}_2$ with $\mathbf{p}_3$), the circular permutations ρ and ρ^{-1} of order 4, and the permutation $\rho^2 = \tau_{13}\tau_{24} = \tau_{12,34}\tau_{14,23}$ of order 2.

If G is included in the group $G_{\mathcal{L}}$ of order 4 generated by τ_{13} and τ_{24} (symmetries of the lozenge), its action leaves fixed the pairs $\{\mathbf{p}_1, \mathbf{p}_3\}$ and $\{\mathbf{p}_2, \mathbf{p}_4\}$ and thus also the lines $\mathbf{p}_1\mathbf{p}_3$ and $\mathbf{p}_2\mathbf{p}_4$ and the two singular quadrics of the pencil (the two pairs of planes). It follows that the roots of the determinantal equation $\mathcal{D}$ are rational. Conversely, if these roots are rational, the singular quadrics and their singular lines are invariant under the action of G, as well as the pairs $\{\mathbf{p}_1, \mathbf{p}_3\}$ and $\{\mathbf{p}_2, \mathbf{p}_4\}$, which implies that G is included in $G_{\mathcal{L}}$. A similar argument shows that G is the identity (resp. is generated by τ_{13} (or τ_{24}), or contains $\tau_{13}\tau_{24}$), if and only if 0 (resp. 1 or 2) of the singular quadrics consist of irrational planes. Moreover, in the case where G contains $\tau_{13}\tau_{24}$, the group is different from $G_{\mathcal{L}}$ if and only if any element which exchanges $\mathbf{p}_1$ and $\mathbf{p}_3$ also exchanges $\mathbf{p}_2$ and $\mathbf{p}_4$, i.e. if and only if the conjugations exchanging the planes in each singular quadric is the same (implying that the square roots needed for factoring them are one and the same). As the degree of $\mathbb{K}$ is the order of G, this shows that the number of square roots needed in our algorithm is always optimal if the roots of $\mathcal{D}$ are rational (i.e. δ is a square).

When the roots of $\mathcal{D}$ are not rational, we consider, in the algorithm, a rational quadric Q_R passing through a rational point $\mathbf{p}$. Let D be the line of Q_R passing through $\mathbf{p}$ and intersecting the lines $\mathbf{p}_1\mathbf{p}_2$ and $\mathbf{p}_3\mathbf{p}_4$ in two points $\mathbf{q}_1$ and $\mathbf{q}_2$. If the discriminant of Q_R is a square (and its parameterization is rational), then D is rational and is fixed by any Galois automorphism. It follows that the lines $\mathbf{p}_1\mathbf{p}_2$ and $\mathbf{p}_3\mathbf{p}_4$ are either fixed or exchanged, which implies that G is included in the group $G_{\mathcal{R}}$ of order 4 generated by $\tau_{12,34}$ and $\tau_{14,23}$ (symmetries of the rectangle). Conversely, if $G \subset G_{\mathcal{R}}$, the lines $\mathbf{p}_1\mathbf{p}_2$ and $\mathbf{p}_3\mathbf{p}_4$ are fixed or exchanged by any Galois automorphism; the image of D by such an automorphism is D itself or the other line of Q_R passing through $\mathbf{p}$; as this image contains the images of $\mathbf{q}_1$ and $\mathbf{q}_2$ which are on $\mathbf{p}_1\mathbf{p}_2$ or $\mathbf{p}_3\mathbf{p}_4$, we may conclude that D is fixed by any Galois automorphism, and is rational; this shows that the discriminant of Q_R is a square by Lemma I.7.4. Pushing these arguments a little more, it is easy to see that G is generated by $\tau_{12,34}$ or $\tau_{14,23}$ (or is the identity) if and only if the roots of either or both of the factors of bidegree $(2,0)$ and $(0,2)$ of the equation Ω in the parameters are rational.

By similar arguments of invariance, we may also conclude that the group G is generated by the circular permutation ρ if and only if any Galois automorphism which exchanges the lines $\mathbf{p}_1\mathbf{p}_3$ and $\mathbf{p}_2\mathbf{p}_4$ exchanges also the lines of Q_R passing through $\mathbf{p}$ (and if there is such an automorphism). It follows that this case occurs when the square root of the discriminant of Q_R and the roots of $\mathcal{D}$ generate the same field.

Finally, G is of order 8 if none of the preceding cases occur.

Optimality in all cases is proved by checking that, for each possible group, the algorithm involves exactly 0, 1 or 2 square roots for parameterizing the lines if the size of the orbit of a line is 1, 2 or 4 respectively.

2.12.3 Examples

We now give examples in all the possible cases outlined in Section 2.12.1. These examples are obtained using the following proposition which gives a rational canonical form for pencils having two double roots corresponding to quadrics of rank 2. Its proof is postponed to Appendix B and needs again elements of Galois theory that can be found in Appendix A.

Proposition 2.1. *Let $R(\lambda,\mu)$ be a rational pencil of quadrics whose determinantal equation has two double roots corresponding to two quadrics of rank 2. Then there is a rational change of frame such that the pencil is generated by quadrics (Q_S, Q_T) of equation*

$$\begin{cases} x^2-\gamma y^2-2wz=0, \\ \alpha x^2+2\gamma xy+\alpha\gamma y^2-z^2-(\alpha^2-\gamma)w^2=0, \end{cases}$$

where $\alpha,\gamma\in\mathbb{Q},\delta=\alpha^2-\gamma\neq 0$. Moreover, a field $\mathbb{K}$ of smallest degree on which the four lines of the intersection are rationally parameterized is generated by the roots of $t^4-2\alpha t+\gamma=0$.

The different cases are as follows:

- $\delta>0,\gamma>0$*, and* $\alpha<0$: *the real intersection is empty.*
- $\delta>0,\gamma>0$*, and* $\alpha>0$: *the real intersection consists of four lines.*
- $\delta>0$ *and* $\gamma<0$: *the real intersection consists of two points.*
- $\delta<0$: *the real intersection consists of two lines.*

Note that the determinantal equation for the reduced pair of quadrics Q_S and Q_T is

$$\mathcal{D}(\lambda,\mu)=\gamma\left(\lambda^2-\delta\mu^2\right)^2=0.$$

Its roots are $(\lambda_0, \mu_0) = (\pm\sqrt{\delta}, 1)$ and, when $\delta > 0$, the associated quadrics of the pencil are the pairs of planes of equations

$$\lambda_0 Q_S + \mu_0 Q_T = (\alpha \pm \sqrt{\delta})\left(x + (\alpha \mp \sqrt{\delta})y\right)^2 - (z \pm \sqrt{\delta} w)^2 = 0$$

and of discriminants $d^{\pm} = \alpha \pm \sqrt{\delta}$. Note also that, when $\gamma > 0$, the quadric Q_S has inertia $(2,2)$ and can be parameterized, using the parameterization of Table I.3, by:

$$\mathbf{X} = \left(ut + vs, \frac{ut - vs}{\sqrt{\gamma}}, vt, 2us \right), \quad (u,v),(s,t) \in \mathbb{P}^1(\mathbb{R}).$$

Plugging $\mathbf{X}$ into the equation of T gives the following biquadratic equation in the parameters:

$$\Omega : \left(2\,(\alpha + \sqrt{\gamma})\,u^2 - v^2\right)\left(2\,(\alpha - \sqrt{\gamma})\,s^2 - t^2\right) = 0.$$

We can now give examples in all the possible cases outlined cases outlined in Section 2.12.1. We start with the four real lines case:

- δ is a square:
  - If $(\alpha, \gamma) = (5,9)$, then $\sqrt{\delta} = 4$, the discriminants of the pairs of planes are $d^{\pm} = 5 \pm 4$, so $\sqrt{d^{\pm}} \in \mathbb{Q}$, and the four lines are defined in $\mathbb{Q}$.
  - If $(\alpha, \gamma) = (3,5)$, then $\sqrt{\delta} = 2$, the discriminants of the pairs of planes are $d^{\pm} = 3 \pm 2$, so $\sqrt{d^+} \notin \mathbb{Q}$ and $\sqrt{d^-} \in \mathbb{Q}$, and the four lines are defined in $\mathbb{Q}(\sqrt{5})$.
  - If $(\alpha, \gamma) = (5,16)$, then $\sqrt{\delta} = 3$, the discriminants of the pair of planes are $d^{\pm} = 5 \pm 3$, so $\sqrt{d^{\pm}} \notin \mathbb{Q}$ but $\sqrt{d^-}$ and $\sqrt{d^+}$ generate the same field $\mathbb{Q}(\sqrt{2})$, and the four lines are defined in $\mathbb{Q}(\sqrt{2})$.
  - If $(\alpha, \gamma) = (6,20)$, then $\sqrt{\delta} = 4$, the discriminants of the pairs of planes are $d^{\pm} = 6 \pm 4$, so $\sqrt{d^{\pm}} \notin \mathbb{Q}$, $\sqrt{d^+}$ and $\sqrt{d^-}$ do not generate the same field and the four lines are defined in $\mathbb{Q}(\sqrt{2}, \sqrt{10})$.
- δ is not a square but $\gamma\delta$ is a square: let $(\alpha, \gamma) = (2,2)$, then $\delta = 2$, $\sqrt{\delta} \notin \mathbb{Q}$ but $\sqrt{\gamma\delta} = 2 \in \mathbb{Q}$. The discriminant of the pairs of planes are $d^{\pm} = 2 \pm \sqrt{2}$, so the four lines are defined in $\mathbb{Q}(\sqrt{2+\sqrt{2}})$.
- Neither δ nor $\gamma\delta$ are squares:
  - If $(\alpha, \gamma) = (3,1)$, then $\delta = \gamma\delta = 8$ is not a square, $\sqrt{\gamma} \in \mathbb{Q}$ so Q_R can be rationally parameterized and the factors of bidegree $(2,0)$ and $(0,2)$ of Ω have rational coefficients. Since $2(\alpha - \sqrt{\gamma}) = 4$ is a square, one of those factors splits in two rational linear factors. Thus two lines have a rational parameterization. Since $2(\alpha + \sqrt{\gamma}) = 8$, the other two lines are defined in $\mathbb{Q}(\sqrt{8}) = \mathbb{Q}(\sqrt{2})$.

  - If $(\alpha,\gamma) = (2,1)$, then $\delta = \gamma\delta = 3$ is not a square, $\sqrt{\gamma} \in \mathbb{Q}$ so Q_R can be rationally parameterized. Since $2(\alpha + \sqrt{\gamma}) = 6$ and $2(\alpha - \sqrt{\gamma}) = 2$, two lines have a rational parameterization in $\mathbb{Q}(\sqrt{2})$ and the other two lines have a rational parameterization in $\mathbb{Q}(\sqrt{6})$.
  - If $(\alpha,\gamma) = (3,3)$, then $\delta = 6$ and $\gamma\delta = 18$ are not squares, $\sqrt{\gamma} \notin \mathbb{Q}$ so Q_R cannot be rationally parameterized. Two lines are defined in $\mathbb{Q}(\sqrt{6+2\sqrt{3}})$ and the other two lines are rational in $\mathbb{Q}(\sqrt{6-2\sqrt{3}})$.

Now we give examples for the two real lines case:

- If $(\alpha,\gamma) = (3,25)$, then $\delta = -16 < 0$, $\sqrt{\gamma} = 5$ and $\sqrt{2(\alpha+\sqrt{\gamma})} = 4$ are rational, so the two lines are rational.
- If $(\alpha,\gamma) = (1,4)$, then $\delta = -3 < 0$, $\sqrt{\gamma} \in \mathbb{Q}$ and $\sqrt{2(\alpha+\sqrt{\gamma})} = \sqrt{6} \notin \mathbb{Q}$, so the two lines are defined in $\mathbb{Q}(\sqrt{6})$.
- If $(\alpha,\gamma) = (1,3)$, then $\delta = -2 < 0$, $\sqrt{\gamma}$ and $\sqrt{2(\alpha+\sqrt{\gamma})}$ are not rational, so the two lines are defined in $\mathbb{Q}(\sqrt{\sqrt{3}-1})$.

Finally, here are examples for the two points case:

- If $(\alpha,\gamma) = (0,-1)$, then $\sqrt{\delta} = 1$ is rational and the discriminant $\alpha + \sqrt{\delta} = 1$ of the real pair of planes is a square, so the two points are in $\mathbb{Q}$.
- If $(\alpha,\gamma) = (1,-3)$, then $\sqrt{\delta} = 2$ is rational but the discriminant $\alpha + \sqrt{\delta} = 3$ of the real pair is not a square, so the two points are in $\mathbb{Q}(\sqrt{3})$.
- If $(\alpha,\gamma) = (1,-2)$, then $\sqrt{\delta}$ is not rational and the two points are in $\mathbb{Q}(\sqrt{1+\sqrt{3}})$.

3 Parameterizing degenerate intersections: singular pencils

We now turn to singular pencils. Except when the intersection consists of four concurrent lines, the parameterization algorithms are straightforward and therefore only briefly sketched.

3.1 Conic and double line in $\mathbb{P}^3(\mathbb{C})$, $\sigma_4 = [1\{3\}]$

As we have seen in Section II.4.1, the pencil contains in this case one pair of planes. Furthermore each of the planes is rational by Proposition I.6.2 because the pair of planes contains a rational point outside its singular locus (by Proposition II.4.1). One plane is tangent to all the cones of the pencil, giving a rational double line. The other plane intersects all the cones transversally, giving a conic. The conic contains a rational point (its intersection with the singular line of the planes), so it can be rationally parameterized.

To actually parameterize the line and the conic, we proceed as follows. If Q_S is a pair of planes, replace Q_S by $Q_S + Q_T$. Now, Q_S is a real projective cone whose vertex is on Q_T. Use this rational vertex to obtain a rational parameterization of Q_T. Plug this parameterization into the equation of Q_S. This equation in the parameters factors in a squared linear factor (corresponding to the double line) and a bilinear factor, corresponding to the conic. It can rationally be solved. The parameterization of C is defined in $\mathbb{Q}[\xi]$.

3.2 Four concurrent lines in $\mathbb{P}^3(\mathbb{C})$, $\sigma_3 = [111]$

In this case, and in the three following cases, the two quadrics Q_S and Q_T have a singular point in common. So first compute this singular point $\mathbf{p}$, which is rational, and compute the rational transformation sending this point to $(0,0,0,1)$. In this new frame, Q_S and Q_T are functions of x,y,z only and we can look at the restricted determinantal equation of the 3×3 upper left matrices to determine the real type of the intersection.

When the restricted determinantal equation has three simple roots (in $\mathbb{C}$), there are three types of intersection over the reals: a point, two concurrent lines, or four concurrent lines (see Table I.5). In the first case, the four lines are imaginary and the real part of the intersection consists of their common rational point, i.e. the point $\mathbf{p}$.

We now look at the two other cases.

3.2.1 Algorithm and optimality

The algorithm for computing the lines is as follows. Determine a plane $x=0, y=0, z=0$, or $w=0$, that does not contain the singular point $\mathbf{p}$. Substitute the equation of that plane (say $x=0$) into the equations of $Q_S(x,y,z,w)$ and $Q_T(x,y,z,w)$. This gives a system of two non-homogeneous degree-two equations in three variables having four distinct complex projective solutions $\mathbf{q}_i$. The real lines of C are then the two or four lines going through $\mathbf{p}$ and one of the $\mathbf{q}_i$ with real coordinates, $i=1,\ldots,4$.

This algorithm outputs an optimal parameterization of C. Indeed, since the common singular point $\mathbf{p}$ of Q_S and Q_T is rational and the plane $(x=0)$ used to cut Q_S and Q_T is rational, the lines are rational if and only if their intersection with the planes (the points $\mathbf{q}_i$) are rational.

3.2.2 Degree of the extension

The following result shows that the roots of any polynomial of degree 4 without multiple root may be needed to parameterize four real concurrent lines. It uses notions of Galois theory that can be found in Appendix A.

Proposition 3.1. *For any rational univariate polynomial of degree 4 without multiple root, there are rational pencils of quadrics whose intersection is four (real or imaginary) concurrent lines, such that each of them is rational on the field generated by one of the roots of the polynomial and is not rational on any smaller field (for the inclusion and the degree).*

Proof. Let us consider a polynomial of degree 4 with rational coefficients and without multiple factors. Let us consider its four real or complex roots $t_1,\ldots,t_4$ and the four points $\mathbf{q}_i$ of coordinates $(1,t_i,t_i^2,0)$. Let us consider also two rational points $\mathbf{r}_j=(a_j,b_j,c_j,0)$, $j=1,2$. Exactly one conic exists in the plane $w = 0$, which passes through the four points $\mathbf{q}_i$ and one of the $\mathbf{r}_j$. Each of these conics has necessarily a rational equation, because, if it were not, the conjugate conics (under the action of the Galois group of the field containing the coefficients) would pass through the same five points. In other words, the equation of the conic is invariant under the Galois group and is thus rational. Now the rational cones containing these conics and having the point $(0,0,0,1)$ as vertex intersect in four (real or imaginary) lines passing through this vertex and the points $\mathbf{q}_i$.

The equations of these conics are easy to compute explicitly. Consider a conic with generic coefficients. Expressing that it passes through 5 points induces five linear equations in the coefficients of the equation of the conic. Solving this linear system expresses these coefficients as symmetric functions of the t_i, and thus as rational functions of the coefficients of the polynomial of degree 4. □

3.2.3 Examples

The proof of Proposition 3.1 gives a way of constructing examples of pencils of quadrics whose intersection is four concurrent lines for any quartic f without multiple root. Table 2 shows an exhaustive list of examples covering the possible degrees of field extension on which the lines of intersection are defined. We here focus on the cases where f has two or four real roots, corresponding, respectively, to the two and four concurrent lines cases.

When f has four real roots, the degree of the extension of $\mathbb{Q}$ needed to parameterize the four lines together is the order of the Galois group of f, in view of Proposition 3.1 and Appendix A. In other words, this degree is either 1, 2, 3, 4, 6, 8, 12 or 24. However, each line is defined individually on an extension of degree at most 4. For instance, when the Galois group is the dihedral group D_4 of order 8, the four lines are collectively defined in an extension of degree 8 but each line is defined in an extension of degree 4.

When f has two real and two complex roots, the degree of the extension on which the four lines are defined is again the order of the Galois group of f, but the degree of the extension on which the two real lines are collectively defined is only half the order of the Galois group. This degree is 1, 2, 3, 4 or 12.

Every extension degree can be attained by picking the right polynomial f. To build examples in all cases, it is sufficient to find the equations of two distinct rational cones containing the four points $\mathbf{q}_i = (1, t_i, t_i^2, 0)$ and having the same vertex. Assume f is given by:

$$f = t^4 + \alpha t^3 + \beta t^2 + \gamma t + \delta.$$

Then the following pair (Q_S, Q_T) satisfies the constraints:

$$\begin{cases} xz - y^2 = 0, \\ \delta x^2 + \gamma xy + \beta y^2 + \alpha yz + z^2 = 0. \end{cases}$$

Any two distinct quadrics of the pencil generated by Q_S and Q_T intersect in four (real or imaginary) concurrent lines defined collectively on an extension of $\mathbb{Q}$ of degree equal to the order of the Galois group of f.

By picking the right polynomial, we can generate pairs of quadrics intersecting in four concurrent lines for all types of Galois groups of a quartic. For instance, taking $f = t^4 + t + 1$, we build the two quadrics

$$\begin{cases} xz - y^2 = 0, \\ x^2 + xy + z^2 = 0. \end{cases}$$

The four real concurrent lines of the intersection are defined on an extension of $\mathbb{Q}$ of degree 24, since the Galois group of f is the group S_4 of permutations of four elements (of order 24). Each line is defined in an extension of degree 4.

3.3 Two concurrent lines and a double line in $\mathbb{P}^3(\mathbb{C})$, $\sigma_3 = [12]$

In this case, the restricted pencil has a real and rational double root corresponding to a pair of planes Q_R and a rational simple root corresponding to another pair of planes. The second pair is always real. There are two cases.

Double line. The pair of planes Q_R is imaginary. The intersection is reduced to the singular line of this pair, which is clearly rational.

Double line and two simple lines. The double line is rational (otherwise its conjugate would also be in the intersection). The two simple lines are contained in Q_R and go through the common singular point $\mathbf{p}$ of Q_S and Q_T. They are rational if and only if Q_R has a rational point outside its singular line, i.e. if the discriminant of Q_R is a square.

A simple example where this is not the case is as follows:

$$\begin{cases} xy = 0, \\ y^2 - 2z^2 = 0. \end{cases}$$

3.4 Two double lines in $\mathbb{P}^3(\mathbb{C})$, $\sigma_3 = [1(11)]$

The determinantal equation has a real and rational double root corresponding to a double plane Q_R. The other root corresponds to a (real or imaginary) pair of planes. There are two cases.

Point. The pair of planes is imaginary. The intersection is reduced to the intersection of its singular line with Q_R, i.e. the rational point $\mathbf{p}$.

Two real double lines. The two double lines are conjugate. They contain the rational point $\mathbf{p}$ and are rational if they go through another rational point. This happens when the discriminant of the pair of planes is a square.

This situation does not necessarily happen, as the following example shows:

$$\begin{cases} z^2 = 0, \\ x^2 - 2y^2 = 0. \end{cases}$$

3.5 Line and triple line in $\mathbb{P}^3(\mathbb{C})$, $\sigma_3 = [3]$

The determinantal equation has a real and rational triple root corresponding to a real pair of planes. The intersection consists of the (triple) singular line of this pair of planes, which is clearly rational, and of a single line which is also rational, otherwise its conjugate would also be in the intersection. The simple line is found by intersecting one of the planes with any other cone of the pencil.

3.6 Quadruple line in $\mathbb{P}^3(\mathbb{C})$, $\sigma_3 = [(21)]$

Here, the determinantal equation has a real and rational triple root, corresponding to a double plane. The intersection consists of the (quadruple) line of tangency of this double plane with a cone of the pencil. This line is clearly rational.

3.7 Remaining cases

In the remaining cases, that is when the restricted determinantal equation identically vanishes, the description of the possibles cases given in Section II.4.2.6 directly yields algorithms for computing parameterizations of the intersection over $\mathbb{Q}[\xi]$.

4 Examples

The algorithm described in this paper for computing a near-optimal parameterization of the intersection of two arbitrary quadrics with integer coefficients has been fully implemented in C++. The implementation details as well as an analysis of the complexity (i.e., the height) of the integer coefficients appearing in the parameterizations are presented in Part IV [4].

In this section, we illustrate our algorithm with three examples covering different situations. The output given is the actual output of our implementation, with debug information turned on so as to follow what the algorithm is doing.

4.1 Example 1

Our first example is as shown in Output 1. As explained in Section II.5, we first determine the type of the intersection by looking at the multiple roots of the determinantal equation and the ranks of the associated quadrics. Here, we find two real double roots corresponding to quadrics of rank 3: Algorithm II.6 tells us that the real type of the intersection is "cubic and secant line". We have rational roots, so we can parameterize the intersection using cones. One of the two cones of the pencil is

$$Q_R = -2Q_S + Q_T = -25wx - 30wy - 5wz - 20xy - 30w^2 - 5x^2 - 20y^2.$$

This cone has the point $(-2,1,4,0)$ as vertex and contains the vertex of the second cone, i.e. $(-4,1,4,1)$. The line of the intersection is the line joining these two points. Here, we have applied a simple reparameterization to the line by picking two other representative points with smaller "height" than the original two. On the reparameterized line, a very simple point is $(-2,0,0,1)$ which

Output 1 Execution trace for Example 1.

```
>> quadric 1: 3*x^2 + 5*x*y + 2*x*z + 13*x*w + 14*y^2 - 4*y*z + 4*y*w + z^2 + 6*z*w + 14*w^2
>> quadric 2: x^2 - 10*x*y + 4*x*z + x*w + 8*y^2 - 8*y*z - 22*y*w + 2*z^2 + 7*z*w - 2*w^2

>> launching intersection
>> determinantal equation: 4*l^4 + 12*l^3*m + l^2*m^2 - 12*l*m^3 + 4*m^4
>> gcd of derivatives of determinantal equation: 2*l^2 + 3*l*m - 2*m^2
>> ranks of singular quadrics: 3 and 3
>> two real rational double roots: [ 1 2 ] and [ -2 1 ]
>> complex intersection: cubic and secant line
>> real intersection: cubic and secant line
>> reparameterization of line
>> singular point of cone: [ -2 1 4 0 ]
>> rational point on cone: [ -2 0 0 1 ]
>> parameterization of cone with rational point
>> cubic and line intersect at [ -4 1 4 1 ] and [ -2 1 4 0 ]
>> status of intersection param: optimal
>> end of intersection

>> parameterization of cubic:
[4*u^3 + u^2*v + 2*v^3, - u^3 - v^3, - 4*u^3 - u^2*v + u*v^2 - 4*v^3, - u^3]
>> parameterization of line:
[- 2*u, v, 4*v, u - v]

>> time spent: 10 ms
```

we use as rational point for parameterizing Q_R. Plugging the parameterization of Q_R in the second cone and leaving aside the linear factor corresponding to the line gives the cubic.

4.2 Example 2

Our second example is displayed in Output 2. Here, the determinantal equation has a double real root at $(\lambda,\mu) = (0,1)$, the two other simple roots are real (solution of $\mathcal{E}(\lambda,\mu) = 4\mu^2 - \lambda^2 = 0$), the singular quadric $R = R(0,1)$ is a real pair of planes and $\mathcal{E}(0,1) > 0$, so Algorithm II.3 tells us that the intersection is two secant conics, the singularities of the intersection being convex. Here, the two planes of R are rational:

$$Q_R = (w - y + 2z)(w - 4x + 3y - 2z).$$

We can parameterize each of these planes and plug their parameterization in turn in any other quadric of the pencil. This gives the implicit equations of the two conics which we can parameterize. As explained in Section 2.5, we are here in one of the few situations where we cannot guarantee that what we output is optimal: the square root in the parameterization of the conics might well be unnecessary. (It turns out that in this particular example it is necessary.) This explains why the implementation reports that the parameterizations of the two conics are only near-optimal.

4.3 Example 3

Our last example is presented in Output 3. Here, the determinantal equation vanishes identically and all the quadrics of the pencil share a common singular point, with coordinates $(1,3,-1,-2)$. We apply to all the quadrics of the pencil a projective transformation sending this point to the point

Output 2 Execution trace for Example 2.

```
>> quadric 1: 16*x^2 - 12*x*y + 8*x*z + 4*x*w - y^2 - 20*y*z + 2*y*w - 2*z^2 + 3*w^2
>> quadric 2: 4*x*y - 8*x*z - 4*x*w - 3*y^2 + 8*y*z + 2*y*w - 4*z^2 + w^2

>> launching intersection
>> determinantal equation: - l^4 + 4*l^2*m^2
>> gcd of derivatives of determinantal equation: l
>> double real root: [ 0 1 ]
>> inertia: [ 1 1 ]
>> parameterization of pair of planes
>> complex intersection: two secant conics
>> real intersection: two secant conics, convex singularities
>> parameterization of rational conic
>> status of conic 1 param: near-optimal
>> parameterization of rational conic
>> status of conic 2 param: near-optimal
>> end of intersection

>> parameterization of conic 1:
[- 3*u^2 - 9*v^2 - 14*u*v*sqrt(2), - 12*u^2 - 36*v^2 + (- 7*u^2 - 14*u*v + 21*v^2)*sqrt(2),
- 2*u^2 - 6*v^2, - 8*u^2 - 24*v^2 + (- 7*u^2 - 14*u*v + 21*v^2)*sqrt(2)]
>> parameterization of conic 2:
[(u^2 - 28*u*v - 42*v^2)*sqrt(2), - 2*u^2 - 84*v^2 - 28*u*v*sqrt(2), - 8*u^2 - 336*v^2
+ (- 6*u^2 + 252*v^2)*sqrt(2), - 10*u^2 - 420*v^2 + (- 8*u^2 - 28*u*v + 336*v^2)*sqrt(2)]

>> time spent: 10 ms
```

Output 3 Execution trace for Example 3.

```
>> quadric 1: 17*x^2 - 12*x*y + 14*x*z - 8*x*w - 4*y^2 - 18*y*w + 5*z^2 + 2*z*w - 16*w^2
>> quadric 2: - 3*x^2 + 28*x*y + 30*x*z + 24*x*w - 4*y^2 + 8*y*z - 2*y*w + 9*z^2 + 18*z*w

>> launching intersection
>> vanishing 4 x 4 determinantal equation
>> dimension of common singular locus: 0
>> common singular point of quadrics: [ 1 3 -1 -2 ]
>> computing matrix sending singular point to [ 0 0 0 1 ]
>> 3 x 3 determinantal equation: - l^3 + l^2*m + 5*l*m^2 + 3*m^3
>> gcd of derivatives of 3 x 3 determinantal equation: l + m
>> double real root: [ -1 1 ]
>> second root: [ 3 1 ]
>> complex intersection: two concurrent lines and double line
>> real intersection: two concurrent lines and double line
>> reparameterization of line
>> parameterization of pair of lines
>> reparameterization of lines
>> the three lines meet at [ 1 3 -1 -2 ]
>> status of intersection param: optimal
>> end of intersection

>> parameterization of double line:
[v, 3*v, - 2*u - v, u - 2*v]
>> parameterization of line 1:
[- 3*u + v, 3*v, u - v, - 2*u - 2*v]
>> parameterization of line 2:
[u - v, 3*u + 6*v, - u + 5*v, - 2*u]

>> time spent: 10 ms
```

$(0,0,0,1)$. The determinantal equation of the pencil restricted to the upper left 3×3 part has a double real root at $(-1,1)$. Corresponding to this double root is a real pair of planes Q_R and Algorithm II.2 tells us that the real type of the intersection is "two concurrent lines and a double line".

As we have seen in Section 3.3, the double line of the intersection is the singular line of Q_R. To parameterize the other two lines, we first parameterize Q_R and plug the result in the equation of the other pair of planes of the pencil, corresponding to the second root $(3,1)$ of the restricted determinantal equation. The result follows.

5 Conclusion

We have presented in Parts I, II, and III of this paper a new algorithm for computing an exact parametric representation of the intersection of two quadrics in three-dimensional real space given by implicit equations with rational coefficients. We have shown that our algorithm computes projective parameterizations that are optimal in terms of the functions used in the sense that they are polynomials whenever it is possible and contain the square root of some polynomial otherwise. The parameterizations are also near-optimal in the sense that the number of square roots appearing in the coefficients of these functions is minimal except in a small number of cases (characterized by the real type of the intersection) where there may be an extra square root (see Table 1 for a summary). Furthermore, we have shown that in the latter cases, testing whether the extra square root is unnecessary and, if so, finding an optimal parameterization are equivalent to finding a rational point on a curve or a surface. Hence, leaving for a moment that well-known problem aside, our algorithm closes the problem of finding parameterizations of the intersection that are optimal in the senses discussed above. It should be emphasized that our algorithm is not only theoretically powerful but is also practical: a complete, robust and efficient C++ implementation is described in Part IV [4].

For most applications, the near-optimal parameterizations of intersections of quadrics computed by our algorithm are good enough since they are at most one square root away from being optimal. However, there may be situations where one is interested in fully asserting the optimality of a parameterization and, if a given parameterization is not optimal, in obtaining one. As we have seen, this is akin to deciding whether a given curve or surface has a rational point and to computing such a point. The problem of finding integer (or rational) points on an algebraic variety is known to be hard in general, and many instances of the problem are undecidable [6]. When the intersection is a smooth quartic, deciding whether the extra square root can be avoided amounts to finding a rational point on a surface of degree 8 (see Section I.7) and very little is known about this problem, to the best of our knowledge. The situation is, however, better for the other near-optimal cases, which boil down to finding a rational point on a (possibly non-rational) conic. Indeed, when the conic is rational, Cremona and Rusin [1] recently gave an efficient algorithm for solving this problem, which has been implemented in recent releases of the Magma computational algebra system [5]. As an example, this implementation solves the problem for an equation of the form $ax^2+by^2=cz^2$, where a,b and c are 200-digit primes, in less than 2 seconds on a mainstream PC. In the future, we plan to use this algorithm in our intersection software.

Finally, it should be stressed that the classification, presented in Part II, of pairs of quadrics depending on the type of their real intersection is of independent interest. For instance, it could be used in a collision detection context to predict when collisions between two moving quadrics will occur.

Acknowledgments

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A A primer on Galois theory

Galois theory was introduced in the 19th century for deciding when a polynomial is solvable by radicals. We give here a brief introduction to this theory, which is especially geared towards geometric objects.

Let K be a finite field extension of the field $\mathbb{Q}$ of the rational numbers. Its dimension as $\mathbb{Q}$-vector space is called the *degree* of K. In our context, K is usually the smallest field containing the coefficients of the equations or of the parameterization of a geometric object such as a point, a line, a curve or a plane. This field K may always be defined as $\mathbb{Q}(\alpha)$, where α is a root of some polynomial f of degree $n = \text{degree}(K)$.

The *splitting field* K' of K is defined as the smallest field containing all the roots of f. It may be proved that it is independent from the choice of f and α. The *Galois group* G of K and K' is the group of the field automorphisms of K'. It is immediate that the elements of G permute the roots of f, and this allows to identify G to a subgroup of the group S_n of all the permutations of the n roots of f.

The important fact for geometric considerations is that any element g of G acts on any object defined from the elements of K by the four field operations $(+,-,*,/)$ simply by replacing any element of K appearing in the definition of the object by its image by the automorphism g, exactly as the complex conjugation acts on any object defined with complex numbers. The different images of an object under the action of the elements of G are called the *conjugates* of this object.

If H is a subgroup of the Galois group G, one may define K'^H, the field of the elements of K' such that $g(x) = x$ for any $g \in H$. The main result of Galois theory is that the map $H \mapsto K'^H$ is a bijection between the subgroups of G and the subfields of K'^H. Moreover, we have degree $\left(K'^H\right) = \text{order}(G)/\text{order}(H)$. It follows that an element of K which has k conjugates lies in a subfield of K of degree k.

This may be extended to the

> **Galois principle**: *Two conjugate objects are isomorphic and any object which has k conjugates (including itself) may be defined on a field of degree k and may not be defined on a smaller field.*

This principle is behind all our proofs of optimality. We describe briefly some other consequences in the context of intersection of quadrics.

One of its first consequences is that any object which has no conjugate except itself may be rationally defined. For example this is the case for the singular point of a singular quartic appearing as the intersection of two rational quadrics. If the intersection of two quadrics is decomposed in a cubic and a line, both may be rationally defined, because their conjugates have to be components of the intersection and the conjugate of a line is a line. Similarly, if the determinantal equation of a pencil of quadrics has two double roots and if the corresponding singular quadrics are a cone and a pair of planes, both are rational because they are not isomorphic.

A more involved application of the Galois principle occurs when the intersection of two quadrics consists in four non concurrent lines. As a conjugate of a line in the intersection may only be another line of the intersection, each line may be defined on a field of degree 4. Moreover, if a point lies on a line, any conjugate of the point lies on the corresponding conjugate of the line. Thus the arrangement of the four lines and their four intersection points is preserved by the action of the Galois group. It follows that the Galois group is included in a group of order 8 which acts on the lines and their intersections as the 8 isometries of a square act on its edges and its vertices. By looking at the subgroups of this group, the Galois principle says that the field K of definition of any of the lines has a degree which divides 4, and that if its degree is 4 it has a subfield of degree 2. Therefore each line may be parameterized with at most 2 square roots. As the splitting field has degree at most 8, it is possibly generated by another square root. This shows, without any explicit computation, that at most 3 square roots are needed to define all the four intersection points and the four lines.

B Rational canonical form for the four skew lines case

We here give a proof of Proposition 2.1.

Proof. Let $\mathbb{K}$ be a $\mathbb{Q}$-extension field of smallest degree on which the four lines of the intersection are rationally parameterized, and let $\mathbb{L}$ be the field generated by the roots of the determinantal equation. To decompose the singular quadrics of the pencil in two planes, we have to extract the square roots of two elements d_1 and d_2 of $\mathbb{L}$, which are algebraically conjugate on $\mathbb{Q}$ if $\mathbb{L}$ is different from $\mathbb{Q}$. Let $t^2 - 2\alpha t + \gamma$ be the (rational) polynomial having d_1 and d_2 as roots. It is easy to verify that $\mathbb{K}$ is generated by the roots of the biquadratic polynomial $t^4 - 2\alpha t^2 + \gamma$.

Let the points $\mathbf{p}_i$, $i = 1, \ldots, 4$, be the singular points of the intersection in $\mathbb{P}^3(\mathbb{C})$ of any two distinct quadrics of the pencil. Now, let us choose four points $\mathbf{q}_i$ on which the Galois group G of this polynomial acts in the same way as on the $\mathbf{p}_i$. For instance, take for $\mathbf{q}_i$, $i = 1, \ldots, 4$, the points of coordinates $(1, t_i, t_i^2, t_i^3)$, where the t_i are the roots of the biquadratic polynomial, numbered such that $t_1 = -t_3$. It is now easy to compute the equations H_j of the planes containing all the points $\mathbf{q}_i$ except $\mathbf{q}_j$. The quadrics of equations

$$\begin{cases} H_1 H_3 + H_2 H_4 = 0, \\ \sqrt{\alpha^2 - \gamma}\,(H_1 H_3 - H_2 H_4) = 0 \end{cases}$$

are rational and their intersection consists in the four (not necessarily real) lines $\mathbf{q}_1\mathbf{q}_2$, $\mathbf{q}_2\mathbf{q}_3$, $\mathbf{q}_3\mathbf{q}_4$, and $\mathbf{q}_4\mathbf{q}_1$. Now, the change of frame sending the $\mathbf{q}_i$ to the points of coordinates $(t_i, -1/t_i, t_i^2 - \alpha, 1)$ is rational and leads to the equations

$$\begin{cases} x^2 - \gamma y^2 - 2wz = 0, \\ \alpha x^2 + 2\gamma xy + \alpha\gamma y^2 - z^2 - (\alpha^2 - \gamma) w^2 = 0. \end{cases}$$

There is a unique projective transformation sending the points $\mathbf{p}_i$ on the $\mathbf{q}_i$ and leaving fixed some rational point which is not on any of the planes defined by the $\mathbf{p}_i$ or by the $\mathbf{q}_i$. This transformation is invariant under the action of the Galois group G. Thus it is rational, showing the existence of the rational change of frame which is sought.

We now show how the different types of real intersection follow from the signs of $\delta = \alpha^2 - \gamma, \alpha$, and γ. First note that the determinantal equation for the rational canonical pencil above is

$$\mathcal{D}(\lambda, \mu) = \gamma\left(\lambda^2 - \delta\mu^2\right)^2.$$

Furthermore, the double roots of $\mathcal{D}$ are $(\lambda_0, \mu_0) = (\pm\sqrt{\delta}, 1)$ and, when $\delta > 0$, the associated pairs of planes have equations

$$\lambda_0 Q_S + \mu_0 Q_T = (\alpha \pm \sqrt{\delta})\left(x + (\alpha \mp \sqrt{\delta})y\right)^2 - (z \pm \sqrt{\delta}w)^2.$$

The discriminants of the two pairs of planes of the pencil are thus $d^\pm = \alpha \pm \sqrt{\delta}$.

We know from Table I.4 that, when the real intersection consists of two lines, the double roots are complex, hence $\delta < 0$. In the other cases, the double roots are real and distinct thus $\delta > 0$. Note that d^+d^- is then equal to c. When the real intersection is empty, the inertia of the two pairs of planes is $(2,0)$, hence $d^\pm < 0$; thus $d^+ + d^- = 2\alpha < 0$ and $d^+d^- = \gamma > 0$. When the real intersection consists of two points, the discriminants of the two pairs of planes have opposite signs, thus $d^+d^- = \gamma < 0$. Finally, when the intersection consists of four skew lines, the discriminants of the two pairs of planes are both positive, thus $d^+ + d^- = 2\alpha > 0$ and $d^+d^- = \gamma > 0$. This completes the proof since these cases for the signs of δ, α, and γ are disjoint. □

C Examples in all cases

Table 2 gives an exhaustive list of examples covering all possible degrees of extension fields on which the components of the intersection are defined, for all real types of intersection. The next-to-last column gives the optimal ring of definition on which a parameterization of the given example is known to exist (ξ is the parameter of the parameterization). When the parameterization output by our algorithm is optimal, the last column gives the degree of the $\mathbb{Q}$-extension field on which the coefficients of the parameterization of each real component of the intersection is defined. When our algorithm is only near-optimal, the last column gives both the optimal degree and the near-optimal one.

Table 2: Exhaustive list of examples when the intersection is 0- or 1-dimensional over $\mathbb{C}$.

complex type	real type	example	fi eld of defi nition	degree
smooth quartic, $\sigma_4 = [1111]$	$\emptyset$	$\begin{cases} 6xy+5y^2+2z^2+6zw-w^2=0 \\ 3x^2+y^2-z^2+11w^2=0 \end{cases}$		

Table 2: (continued)

complex type	real type	example	field of definition	degree
	smooth quartic, two finite components	$\begin{cases} x^2+y^2-z^2-w^2=0 \\ xy-2zw=0 \end{cases}$	quartic in $\mathbb{Q}[\xi,\sqrt{\Delta}]$, $\Delta\in\mathbb{Q}[\xi]$	**1 / 2**
		$\begin{cases} 6xy+5y^2+2z^2+6zw-w^2=0 \\ 3x^2+y^2-z^2-w^2=0 \end{cases}$	quartic in $\mathbb{Q}(\sqrt{\delta})[\xi,\sqrt{\Delta}]$, $\Delta\in\mathbb{Q}(\sqrt{\delta})[\xi]$	2
	smooth quartic, one finite component	$\begin{cases} x^2+y^2+2zw=0 \\ x^2+z^2+zw-w^2=0 \end{cases}$	quartic in $\mathbb{Q}[\xi,\sqrt{\Delta}]$	**1 / 2**
		$\begin{cases} 2x^2-2xy+2xz-2xw+y^2 \\ \quad +4yz-4yw+2z^2-4zw=0 \\ x^2-2xy+4xz+4xw-y^2 \\ \quad +2yz+4yw+4zw-2w^2=0 \end{cases}$	quartic in $\mathbb{Q}(\sqrt{\delta})[\xi,\sqrt{\Delta}]$, $\Delta\in\mathbb{Q}(\sqrt{\delta})[\xi]$	2
	smooth quartic, two infinite components	$\begin{cases} xy+zw=0 \\ x^2-y^2+z^2+2zw-w^2=0 \end{cases}$	quartic in $\mathbb{Q}[\xi,\sqrt{\Delta}]$, $\Delta\in\mathbb{Q}[\xi]$	**1 / 2**
		$\begin{cases} x^2-2y^2+4zw=0 \\ xy+z^2+2zw-w^2=0 \end{cases}$	quartic in $\mathbb{Q}(\sqrt{\delta})[\xi,\sqrt{\Delta}]$, $\Delta\in\mathbb{Q}(\sqrt{\delta})[\xi]$	2
nodal quartic, $\sigma_4=[112]$	point	$\begin{cases} y^2+z^2+w^2=0 \\ xy+w^2=0 \end{cases}$	point in $\mathbb{Q}$	1
	nodal quartic	$\begin{cases} y^2+z^2-w^2=0 \\ xy+w^2=0 \end{cases}$	quartic in $\mathbb{Q}[\xi]$	**1 / 2**
		$\begin{cases} y^2+z^2-3w^2=0 \\ xy+w^2=0 \end{cases}$	quartic in $\mathbb{Q}(\sqrt{3})[\xi]$	2
	nodal quartic with isolated singularity	$\begin{cases} y^2-z^2-w^2=0 \\ xy+w^2=0 \end{cases}$	quartic in $\mathbb{Q}[\xi]$, point in $\mathbb{Q}$	**1 / 2** 1
		$\begin{cases} y^2+z^2-3w^2=0 \\ xw+y^2=0 \end{cases}$	quartic in $\mathbb{Q}(\sqrt{3})[\xi]$, point in $\mathbb{Q}$	2 1
cuspidal quartic, $\sigma_4=[13]$	cuspidal quartic	$\begin{cases} yz+w^2=0 \\ xz+y^2=0 \end{cases}$	quartic in $\mathbb{Q}[\xi]$	1
cubic and secant line, $\sigma_4=[22]$	cubic and secant line	$\begin{cases} y^2+zw=0 \\ xy+w^2=0 \end{cases}$	cubic in $\mathbb{Q}[\xi]$, line in $\mathbb{Q}[\xi]$	1 1
	cubic and non-secant line	$\begin{cases} xw+yz=0 \\ xz-yw+zw=0 \end{cases}$	cubic in $\mathbb{Q}[\xi]$, line in $\mathbb{Q}[\xi]$	1 1
cubic and tangent line, $\sigma_4=[4]$	cubic and tangent line	$\begin{cases} yw+z^2=0 \\ xw+yz=0 \end{cases}$	cubic in $\mathbb{Q}[\xi]$, line in $\mathbb{Q}[\xi]$	1 1
two secant conics, $\sigma_4=[11(11)]$	$\emptyset$	$\begin{cases} z^2-w^2=0 \\ x^2+y^2+w^2=0 \end{cases}$		

Table 2: (continued)

complex type	real type	example	field of definition	degree
	two points	$\begin{cases} z^2+w^2=0 \\ x^2-y^2+w^2=0 \end{cases}$	two points in $\mathbb{Q}$	1
		$\begin{cases} z^2+w^2=0 \\ x^2-2y^2+w^2=0 \end{cases}$	two points in $\mathbb{Q}(\sqrt{2})$	2
	one conic	$\begin{cases} zw=0 \\ x^2+y^2+z^2-w^2=0 \end{cases}$	one conic in $\mathbb{Q}[\xi]$	**1 / 2**
		$\begin{cases} x^2-4xw+3w^2=0 \\ x^2+y^2+z^2-4w^2=0 \end{cases}$	one conic in $\mathbb{Q}(\sqrt{3})[\xi]$	2
		$\begin{cases} x^2-4xw-3w^2=0 \\ x^2+y^2+z^2-w^2=0 \end{cases}$	one conic in $\mathbb{Q}(\sqrt{4\sqrt{7}-10})$	4
	two non-secant conics	$\begin{cases} x^2-w^2=0 \\ y^2+z^2-w^2=0 \end{cases}$	two conics in $\mathbb{Q}[\xi]$	**1 / 2**
		$\begin{cases} x^2-w^2=0 \\ y^2+z^2-3w^2=0 \end{cases}$	two conics in $\mathbb{Q}(\sqrt{3})[\xi]$	2
		$\begin{cases} x^2-3w^2=0 \\ y^2+z^2-3w^2=0 \end{cases}$	two conics in $\mathbb{Q}(\sqrt{3})[\xi]$	**2 / 4**
		$\begin{cases} x^2-33w^2=0 \\ y^2+z^2-3w^2=0 \end{cases}$	two conics in $\mathbb{Q}(\sqrt{3},\sqrt{11})[\xi]$	4
	two secant conics	$\begin{cases} x^2-y^2=0 \\ y^2+z^2-w^2=0 \end{cases}$	two conics in $\mathbb{Q}[\xi]$	**1 / 2**
		$\begin{cases} x^2-y^2=0 \\ y^2+z^2-3w^2=0 \end{cases}$	two conics in $\mathbb{Q}(\sqrt{3})[\xi]$	2
		$\begin{cases} x^2-3y^2=0 \\ y^2+z^2-3w^2=0 \end{cases}$	two conics in $\mathbb{Q}(\sqrt{3})[\xi]$	**2 / 4**
		$\begin{cases} x^2-33y^2=0 \\ y^2+z^2-3w^2=0 \end{cases}$	two conics in $\mathbb{Q}(\sqrt{3},\sqrt{11})[\xi]$	4
two tangent conics, $\sigma_4=[1(21)]$	point	$\begin{cases} x^2+w^2=0 \\ xy+z^2=0 \end{cases}$	point in $\mathbb{Q}$	1
	two conics	$\begin{cases} x^2-w^2=0 \\ xy+z^2=0 \end{cases}$	two conics in $\mathbb{Q}[\xi]$	1
		$\begin{cases} x^2-2w^2=0 \\ xy+z^2=0 \end{cases}$	two conics in $\mathbb{Q}(\sqrt{2})[\xi]$	2
double conic, $\sigma_4=[1(111)]$	$\emptyset$	$\begin{cases} w^2=0 \\ x^2+y^2+z^2=0 \end{cases}$		
	double conic	$\begin{cases} x^2=0 \\ y^2+z^2-w^2=0 \end{cases}$	conic in $\mathbb{Q}[\xi]$	**1 / 2**
		$\begin{cases} x^2=0 \\ y^2+z^2-3w^2=0 \end{cases}$	conic in $\mathbb{Q}(\sqrt{3})[\xi]$	2

Table 2: (continued)

complex type	real type	example	field of definition	degree
conic and two lines not crossing, $\sigma_4 = [2(11)]$	point	$\begin{cases} xy=0 \\ y^2+z^2+w^2=0 \end{cases}$	point in $\mathbb{Q}$	1
	conic and point	$\begin{cases} xw=0 \\ y^2+z^2-w^2=0 \end{cases}$	point in $\mathbb{Q}$, conic in $\mathbb{Q}[\xi]$	1 **1 / 2**
		$\begin{cases} xw=0 \\ y^2+z^2-3w^2=0 \end{cases}$	point in $\mathbb{Q}$, conic in $\mathbb{Q}(\sqrt{3})[\xi]$	1 2
	conic and two lines	$\begin{cases} xy=0 \\ y^2+z^2-w^2=0 \end{cases}$	two lines in $\mathbb{Q}[\xi]$, conic in $\mathbb{Q}[\xi]$	1 1
		$\begin{cases} xy=0 \\ 2y^2+z^2-3w^2=0 \end{cases}$	two lines in $\mathbb{Q}(\sqrt{3})[\xi]$, conic in $\mathbb{Q}[\xi]$	2 **1 / 2**
		$\begin{cases} xy=0 \\ y^2+z^2-3w^2=0 \end{cases}$	two lines in $\mathbb{Q}(\sqrt{3})[\xi]$, conic in $\mathbb{Q}(\sqrt{3})[\xi]$	2 2
conic and two lines crossing, $\sigma_4 = [(31)]$	conic	$\begin{cases} yz=0 \\ xz+y^2+w^2=0 \end{cases}$	conic in $\mathbb{Q}[\xi]$	1
	conic and two lines	$\begin{cases} yz=0 \\ xz+y^2-w^2=0 \end{cases}$	conic in $\mathbb{Q}[\xi]$, two lines in $\mathbb{Q}[\xi]$	1 1
		$\begin{cases} yz=0 \\ xz+y^2-2w^2=0 \end{cases}$	conic in $\mathbb{Q}[\xi]$, two lines in $\mathbb{Q}(\sqrt{2})[\xi]$	1 2
four skew lines, $\sigma_4 = [(11)(11)]$	$\emptyset$	$\begin{cases} x^2+y^2=0 \\ z^2+w^2=0 \end{cases}$		
	two points	$\begin{cases} x^2+y^2-2zw=0 \\ 2xy+z^2+w^2=0 \end{cases}$	two points in $\mathbb{Q}$	1
		$\begin{cases} x^2+3y^2-2zw=0 \\ x^2-6xy-3y^2-z^2-4w^2=0 \end{cases}$	two points in $\mathbb{Q}(\sqrt{3})$	2
		$\begin{cases} x^2+2y^2-2zw=0 \\ x^2-4xy-2y^2-z^2-3w^2=0 \end{cases}$	two points in $\mathbb{Q}(\sqrt{1+\sqrt{3}})$	4
	two skew lines	$\begin{cases} x^2-25y^2-2zw=0 \\ 3x^2+50xy+75y^2-z^2+16w^2=0 \end{cases}$	two lines in $\mathbb{Q}[\xi]$	1
		$\begin{cases} x^2-4y^2-2zw=0 \\ x^2+8xy+4y^2-z^2+3w^2=0 \end{cases}$	two lines in $\mathbb{Q}(\sqrt{6})[\xi]$	2
		$\begin{cases} x^2-3y^2-2zw=0 \\ x^2+6xy+3y^2-z^2+2w^2=0 \end{cases}$	two lines in $\mathbb{Q}(\sqrt{\sqrt{3}-1})[\xi]$	4
	four skew lines	$\begin{cases} x^2-9y^2-2zw=0 \\ 5x^2+18xy+45y^2-z^2-16w^2=0 \end{cases}$	four lines in $\mathbb{Q}[\xi]$	1
		$\begin{cases} x^2-y^2-2zw=0 \\ 3x^2+2xy+3y^2-z^2-8w^2=0 \end{cases}$	two lines in $\mathbb{Q}[\xi]$, two lines in $\mathbb{Q}(\sqrt{2})[\xi]$	1 2

Table 2: (continued)

complex type	real type	example	field of definition	degree
		$\begin{cases} x^2-16y^2-2zw=0 \\ 5x^2+32xy+80y^2-z^2-9w^2=0 \end{cases}$	four lines in $\mathbb{Q}(\sqrt{2})[\xi]$	2
		$\begin{cases} x^2-y^2-2zw=0 \\ 2x^2+2xy+2y^2-z^2-3w^2=0 \end{cases}$	two lines in $\mathbb{Q}(\sqrt{2})[\xi]$, two lines in $\mathbb{Q}(\sqrt{6})[\xi]$	2 2
		$\begin{cases} x^2-20y^2-2zw=0 \\ 6x^2+40xy+120y^2-z^2-16w^2=0 \end{cases}$	four lines in $\mathbb{Q}(\sqrt{2},\sqrt{10})[\xi]$	4
		$\begin{cases} x^2-2y^2-2zw=0 \\ 2x^2+4xy+4y^2-z^2-2w^2=0 \end{cases}$	four lines in $\mathbb{Q}(\sqrt{2+\sqrt{2}})[\xi]$	4
		$\begin{cases} x^2-3y^2-2zw=0 \\ 3x^2+6xy+9y^2-z^2-6w^2=0 \end{cases}$	two lines in $\mathbb{Q}(\sqrt{3-\sqrt{3}})[\xi]$, two lines in $\mathbb{Q}(\sqrt{3+\sqrt{3}})[\xi]$	4 4
two skew lines and a double line, $\sigma_4=[(22)]$	double line	$\begin{cases} y^2+w^2=0 \\ xy+zw=0 \end{cases}$	double line in $\mathbb{Q}[\xi]$	1
	two skew lines and a double line	$\begin{cases} y^2-w^2=0 \\ xy-zw=0 \end{cases}$	double line in $\mathbb{Q}[\xi]$ two simple lines in $\mathbb{Q}[\xi]$	1 1
		$\begin{cases} y^2-2w^2=0 \\ xy-zw=0 \end{cases}$	double line in $\mathbb{Q}[\xi]$ two simple lines in $\mathbb{Q}(\sqrt{2})[\xi]$	1 2
two concurrent double lines, $\sigma_4=[(211)]$	point	$\begin{cases} w^2=0 \\ x^2+y^2+zw=0 \end{cases}$	point in $\mathbb{Q}$	1
	two double lines	$\begin{cases} w^2=0 \\ x^2-y^2+zw=0 \end{cases}$	two lines in $\mathbb{Q}[\xi]$	1
		$\begin{cases} w^2=0 \\ x^2-2y^2+zw=0 \end{cases}$	two lines in $\mathbb{Q}(\sqrt{2})[\xi]$	2
conic and double line, $\sigma_4=[1\{3\}]$	conic and double line	$\begin{cases} xw=0 \\ xz+y^2=0 \end{cases}$	conic in $\mathbb{Q}[\xi]$, line in $\mathbb{Q}[\xi]$	1 1
four concurrent lines, $\sigma_3=[111]$	point	$\begin{cases} x^2+z^2=0 \\ y^2+z^2=0 \end{cases}$	point in $\mathbb{Q}$	1
	two concurrent lines	$\begin{cases} xz-y^2=0 \\ -x^2+z^2=0 \end{cases}$	two lines in $\mathbb{Q}[\xi]$	1
		$\begin{cases} xz-y^2=0 \\ -2x^2-y^2+z^2=0 \end{cases}$	two lines in $\mathbb{Q}(\sqrt{2})[\xi]$	2
		$\begin{cases} xz-y^2=0 \\ 2xy+z^2=0 \end{cases}$	one line in $\mathbb{Q}[\xi]$, one line in $\mathbb{K}[\xi]$, degree$(\mathbb{K})=3$	1 3
		$\begin{cases} xz-y^2=0 \\ -3x^2+z^2=0 \end{cases}$	two lines in $\mathbb{Q}(\sqrt{3\sqrt{3}})[\xi]$	4
		$\begin{cases} xz-y^2=0 \\ -3x^2-3xy+z^2=0 \end{cases}$	one line in $\mathbb{K}[\xi]$, degree$(\mathbb{K})=4$ one line in $\mathbb{K}'[\xi]$, degree$(\mathbb{K}')=4$	4 4

Table 2: (continued)

complex type	real type	example	field of definition	degree
	four concurrent lines	$\begin{cases} xz-y^2=0 \\ 4x^2-5y^2+z^2=0 \end{cases}$	four lines in $\mathbb{Q}[\xi]$	1
		$\begin{cases} xz-y^2=0 \\ 2x^2-3y^2+z^2=0 \end{cases}$	two lines in $\mathbb{Q}[\xi]$, two lines in $\mathbb{Q}(\sqrt{2})[\xi]$	1 2
		$\begin{cases} xz-y^2=0 \\ -4x^2+8xy-4yz+z^2=0 \end{cases}$	four lines in $\mathbb{Q}(\sqrt{2})[\xi]$	2
		$\begin{cases} xz-y^2=0 \\ xy-3y^2+z^2=0 \end{cases}$	one line in $\mathbb{Q}[\xi]$, three lines l_i in $\mathbb{K}_i[\xi]$, degree$(\mathbb{K}_i)=3$	1 3
		$\begin{cases} xz-y^2=0 \\ 2x^2-4y^2+z^2=0 \end{cases}$	four lines in $\mathbb{Q}(\sqrt{2+\sqrt{2}})[\xi]$	4
		$\begin{cases} xz-y^2=0 \\ 4x^2-10y^2+z^2=0 \end{cases}$	four lines in $\mathbb{Q}(\sqrt{6},\sqrt{14})[\xi]$	4
		$\begin{cases} xz-y^2=0 \\ 2x^2+10xy+15y^2+7yz+z^2=0 \end{cases}$	two lines in $\mathbb{Q}(\sqrt{2})[\xi]$, two lines in $\mathbb{Q}(\sqrt{5})[\xi]$	2 2
		$\begin{cases} xz-y^2=0 \\ xy-4y^2+z^2=0 \end{cases}$	one line in $\mathbb{Q}[\xi]$, three lines l_i in $\mathbb{K}_i[\xi]$, degree$(\mathbb{K})=3$	1 3
		$\begin{cases} xz-y^2=0 \\ 2x^2-5y^2+z^2=0 \end{cases}$	two lines in $\mathbb{Q}(\sqrt{5+\sqrt{17}})[\xi]$, two lines in $\mathbb{Q}(\sqrt{5-\sqrt{17}})[\xi]$	4 4
		$\begin{cases} xz-y^2=0 \\ x^2-3xy-7y^2+z^2=0 \end{cases}$	four lines l_i in $\mathbb{K}_i[\xi]$, degree$(\mathbb{K}_i)=4$	4
		$\begin{cases} xz-y^2=0 \\ x^2+xy+z^2=0 \end{cases}$	four lines l_i in $\mathbb{K}_i[\xi]$, degree$(\mathbb{K}_i)=4$	4
two concurrent lines and a double line, $\sigma_3=[12]$	double line	$\begin{cases} xy=0 \\ y^2+z^2=0 \end{cases}$	double line in $\mathbb{Q}[\xi]$	1
	two concurrent lines and a double line	$\begin{cases} xy=0 \\ y^2-z^2=0 \end{cases}$	double line in $\mathbb{Q}[\xi]$, two simple lines in $\mathbb{Q}[\xi]$	1 1
		$\begin{cases} xy=0 \\ y^2-2z^2=0 \end{cases}$	double line in $\mathbb{Q}[\xi]$, two simple lines in $\mathbb{Q}(\sqrt{2})[\xi]$	1 2
line and triple line, $\sigma_3=[3]$	line and triple line	$\begin{cases} xz+y^2=0 \\ yz=0 \end{cases}$	simple line in $\mathbb{Q}[\xi]$, triple line in $\mathbb{Q}[\xi]$	1 1
two concurrent double lines, $\sigma_3=[1(11)]$	point	$\begin{cases} z^2=0 \\ x^2+y^2=0 \end{cases}$	point in $\mathbb{Q}$	1
	two double lines	$\begin{cases} z^2=0 \\ x^2-y^2=0 \end{cases}$	two lines in $\mathbb{Q}[\xi]$	1
		$\begin{cases} z^2=0 \\ x^2-2y^2=0 \end{cases}$	two lines in $\mathbb{Q}(\sqrt{2})[\xi]$	2

Table 2: (continued)

complex type	real type	example	fi eld of defi nition	degree
quadruple line, $\sigma_3 = [(21)]$	quadruple line	$\begin{cases} y^2=0 \\ xy+z^2=0 \end{cases}$	line in $\mathbb{Q}[\xi]$	1
quadruple line, $\sigma_2 = [11]$	quadruple line	$\begin{cases} x^2=0 \\ y^2=0 \end{cases}$	line in $\mathbb{Q}[\xi]$	1

References

[1] J. Cremona and D. Rusin. Efficient solution of rational conics. *Mathematics of Computation*, 72(243):1417–1441, 2003.

[2] L. Dupont, D. Lazard, S. Lazard, and S. Petitjean. Near-optimal parameterization of the intersection of quadrics: I. The generic algorithm. Research Report n^o 5667, INRIA, September 2005.

[3] L. Dupont, D. Lazard, S. Lazard, and S. Petitjean. Near-optimal parameterization of the intersection of quadrics: II. A classification of pencils. Research Report n^o 5668, INRIA, September 2005.

[4] S. Lazard, L. M. Peñaranda, and S. Petitjean. Near-optimal parameterization of the intersection of quadrics: IV. An efficient and exact implementation. Research Report n^o 5670, INRIA, September 2005.

[5] The Magma Computational Algebra System. School of Mathematics and Statistics of the University of Sydney. `http://magma.maths.usyd.edu.au/magma`.

[6] B. Poonen. Computing rational points on curves. In *Number Theory for the Millenium*. A. K. Peters, Boston, 2001. Proc. of Millennial Conference on Number Theory, 2000.

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