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Factoring integers with CADO-NFS

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Why factor?

- Cryptography:
  - Integer factorization is a (supposedly) difficult problem, but integer multiplication is not
  - E.g., basis for the security of the RSA public-key cryptosystem:
    - $\rightarrow$ private key: large primes p and q
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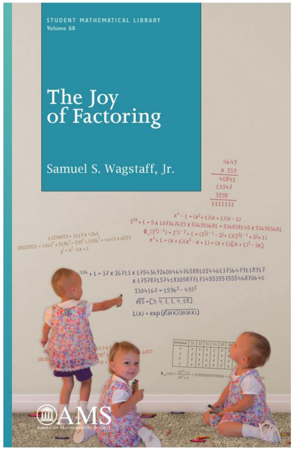

- For fun ☺

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  - (G)NFS (General Number Field Sieve) [Buhler, Lenstra, & Pomerance, 1993]:

$$O\left(\exp\left(\sqrt[3]{\frac{64}{9}}\,(\log N)^{1/3}\,(\log\log N)^{2/3}\right)\right)$$

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- Quantum computer:
  - 2012: fact. of 56153 (a whopping 5 digits!)

Free (as in free speech) factorization software

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  - NFS@home [Childers]:
    `http://escatter11.fullerton.edu/nfs/`
  - Msieve [Papadopoulos]:
    `http://www.boo.net/~jasonp/qs.html`
  - CADO-NFS:
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- Website: `http://cado-nfs.gforge.inria.fr/`

The Number Field Sieve

- Based on Fermat's factoring method (congruence of squares):
  - Find two integers x and y such that $x^2 \equiv y^2 \pmod{N}$
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  - Such a pair is called a relation: add (a, b) to $\mathcal{R}$ (set of relations)

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The Number Field Sieve

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  - Sieving domain: coprime pairs (a, b) in $[-A, A]\times]0, A]$
  - Factor base $\mathcal{B}_i$: prime ideals $\mathfrak{p} = (p, r)$ of $\mathbb{Z}[\alpha_i]$ with $p \leq B_i$
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  - If both norms are smooth, then (a, b) is a relation
- We need more relations than elements of the factor bases:

$$\#\mathcal{R} > \#\mathcal{B}_1 + \#\mathcal{B}_2$$

The Number Field Sieve

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The Number Field Sieve

1. **Polynomial selection**: find suitable polynomials f_1 and f_2
2. **Factor base generation**: build factors bases $\mathcal{B}_1$ and $\mathcal{B}_2$
3. **Relation collection** (a.k.a. sieving): build set of relations $\mathcal{R}$
4. **Filtering**: build and simplify matrix from relations
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8. **Profit!**

Back to CADO-NFS

- Each step is handled by a specific binary/script

❶ **Polynomial selection**	`polyselect/polyselect2l`
❷ **Factor base generation**	`sieve/makefb`
❸ **Relation collection**	`sieve/{freerel,las}`
❹ **Filtering**	`filter/{dup1,dup2,purge,merge,replay}`
❺ **Linear algebra**	`linalg/bwc/bwc.pl`
❻ **Characters**	`linalg/characters`
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- `cadofactor.py`: Python script to run whole factorization
  → All NFS parameters in a single parameter file

Python script + parameter file
`scripts/cadofactor/cadofactor.py`

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Back to CADO-NFS

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- `cadofactor.py`: Python script to run whole factorization
  $\rightarrow$ All NFS parameters in a single parameter file
- `factor.sh`: Bash script for simple factorizations

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Bash script `factor.sh`

Python script + parameter file `scripts/cadofactor/cadofactor.py`

Let's play!

- Requirements:
  - GNU/Linux (or Mac OS X + Xcode)
  - GCC 4.4 or later
  - GMP 5 or later
  - GNU Make and CMake 2.6.3 or later
  - Python 3.2 or later
  - SQLite 3, including Python bindings
  - GNU Wget or cURL
  - GNU Gzip
  - GNU Bash

Let's play!

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$ cp local.sh.example local.sh
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- Optional: tweak build configuration (esp. for Mac OS X):

```
$ cp local.sh.example local.sh
$ vi local.sh
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- Build:

```
$ make
```

A toy factorization

- Let's factor this 59-digit composite integer:

  $$c_{59} = 90377629292003121684002147101760858109247336549001090677693$$

  (you can just copy-paste it from `http://www.loria.fr/~detreyje/cado-nfs.txt`)

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- Run:

```
$ export CADO_DEBUG=1
$ mkdir /tmp/c59
$ t=/tmp/c59 ./factor.sh 903...693 -t 2
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- Run:

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- Get factors!

```
...
Info:Complete Factorization: ...
588120598053661 260938498861057
760926063870977 773951836515617
OK
```

Diving into details – Polynomial selection

- Find polynomials f_1 and $f_2 \in \mathbb{Z}[X]$ such that
  - f_1 and f_2 are irreducible and coprime over $\mathbb{Q}$
  - they have a common root $m \in \mathbb{Z}/N\mathbb{Z}$:

$$f_1(m) \equiv 0 \pmod{N} \quad \text{and} \quad f_2(m) \equiv 0 \pmod{N}$$

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- In practice:
  - Take a linear polynomial for f_2: this is called the "rational side"
  - Take a degree-d polynomial for f_1, with $d \in \{4, 5, 6\}$: this is called the "algebraic side"

$$f_1(X) = f_{1,d}X^d + f_{1,d-1}X^{d-1} + \cdots + f_{1,1}X + f_{1,0}$$

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- Look for a polynomial f_1 of degree d:
  - such that norms $N_1(a - b\alpha_1) = f_1(a/b)b^d$ are as small as possible for pairs (a, b) in the sieving domain
  - which has many roots modulo small primes

Diving into details – Polynomial selection

- Two main steps:
  - Size optimization: find polynomials with small norm
  - Root optimization: translate/rotate candidates so that they have many roots modulo small primes

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- CADO-NFS parameters (`tasks.polyselect.*`):
  - `degree`: degree d of polynomial f_1
  - `admin` (0): minimum value for leading coefficient $f_{1,d}$
  - `admax`: maximum value for leading coefficient $f_{1,d}$
  - `incr` (60): force $f_{1,d}$ to be a multiple of this smooth number
  - `nrkeep`: how many candidates to keep after first step

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- Best polynomial stored in:

$\langle name \rangle$`.polyselect2.poly`

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  - Fix a prime ideal $\mathfrak{q} = (q, \rho)$ of $\mathbb{Z}[\alpha_1]$
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Diving into details – Relation collection

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  - Polynomials:

$$\begin{aligned} f_1(X) = {} & 60 \cdot X^4 + 164823 \cdot X^3 + 2561101187 \cdot X^2 \\ & - 4872316534587 \cdot X - 9288039622841198 \\ f_2(X) = {} & 4827001309 \cdot X - 19261601140604 1 \end{aligned}$$

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  - Special-$\mathfrak{q}$: $(q, \rho) = (200003, 74941)$

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  - Sieving position: $(a, b) = (-876877, 31)$

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$$\begin{aligned} N_1(a - b\alpha_1) &= 340397725772199663711302 85 \\ N_2(a - b\alpha_2) &= -1020378278041926 4 \end{aligned}$$

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- Is (a, b) a relation? Factor its norms
  - Remove small factors by sieving techniques (up to bound B_i')

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$$\begin{aligned} N_1(a - b\alpha_1) &= \quad 3^2 \cdot 5 \cdot 43 \cdot 53 \cdot 59 \cdot 61 \cdot 151 \cdot 3053757221 \cdot q \\ N_2(a - b\alpha_2) &= -2^6 \cdot 67 \cdot 311 \cdot 617 \cdot 709 \cdot 17491 \end{aligned}$$

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- Is (a, b) a relation? Factor its norms
  - Remove small factors by sieving techniques (up to bound B'_i)
  - Co-factor remaining parts only if not too large

$$\begin{aligned} N_1(a - b\alpha_1) &= 3^2 \cdot 5 \cdot 43 \cdot 53 \cdot 59 \cdot 61 \cdot 151 \cdot 3053757221 \cdot q \\ N_2(a - b\alpha_2) &= -2^6 \cdot 67 \cdot 311 \cdot 617 \cdot 709 \cdot 17491 \end{aligned}$$

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- Example from c_{59}:
  - Polynomials:

$$\begin{aligned} f_1(X) = &\quad 60 \cdot X^4 + 164823 \cdot X^3 + 2561101187 \cdot X^2 \\ &- 4872316534587 \cdot X - 9288039622841198 \\ f_2(X) = &\quad 4827001309 \cdot X - 19261601140604 1 \end{aligned}$$

  - Special-$\mathfrak{q}$: $(q, \rho) = (200003, 74941)$
  - Sieving position: $(a, b) = (-876877, 31)$

- Is (a, b) a relation? Factor its norms
  - Remove small factors by sieving techniques (up to bound B_i')
  - Co-factor remaining parts only if not too large

$$\begin{aligned} N_1(a - b\alpha_1) &= \quad 3^2 \cdot 5 \cdot 43 \cdot 53 \cdot 59 \cdot 61 \cdot 151 \cdot 22447 \cdot 136043 \cdot q \\ N_2(a - b\alpha_2) &= -2^6 \cdot 67 \cdot 311 \cdot 617 \cdot 709 \cdot 17491 \end{aligned}$$

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  - `alim` / `rlim`: the maximum norm of sieved primes (B_i')
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Thank you for your attention

Happy factoring!