Proposal for L3 Research Internship for ENS Lyon for 2023

Continuous time Analog machines and models of computation: Universal Ordinary Differential Equations.

Olivier Bournez

Keywords: Analog Machines and Models. Computation Models. Complexity.

Superviser: Olivier Bournez.

Administrative Position: Professor (of Computer Science) at Ecole Polytechnique.

Location: Laboratoire d'Informatique de l'X, LIX, Ecole Polytechnique , 91128 Palaiseau Cedex

Phone: +33 (0)1 77 57 80 78

Email: olivier.bournez@lix.polytechnique.fr.

Www: http://www.lix.polytechnique.fr/~bournez.

Language: French or English. This proposal is intentionally written in English for Non-French speaking students who may be interested.

General Presentation

Today's theoretical computer science, and in particular classical computability and complexity consider mostly computations over a discrete space with a discrete space. This aims at modeling today's computers, which are *digital* computers working over bits. This covers today's machines, and today's classical models such as Turing machines working over words over a finite alphabet with a discrete time.

But machines where time is continuous can be considered and can indeed be built. Such machines are analog, and are working over continuous quantities like voltage. Notice that first ever built programmable computers were actually analog machines. This includes for example the differential analysers that were first mechanical machines working over quantities encoded by angles of shafts, and later on electronic machines working over quantities like continuous voltages. Such machines were typically used to solve ordinary differential equations. Realizing today analog computer does not yield to major difficulties: see example [6] for a book about the history of computing, not forgetting analog computing history as in most of the books.

It turns out that the corresponding computability and complexity theory has not received so much attention: even if models of computation where space could be continuous and time remains discrete have been considered (see e.g. [1], or [7]), these models are still discrete time.

Notice that there is no hope to get a unification of the type of Church-Turing thesis for all such models. However, there is indeed something similar in many aspects for **continuous time and space** models.

The purpose of this internship is to focus on these latter models: i.e. truly analog (continuous time and space) models of computation such as differential analysers or electronic analog computers. In this context, for many reasons, the equivalent of a Turing machine can be considered as a *polynomial ordinary differential equation (pODE)*. Indeed, a (projection of a) solution of such a pODE, that could be considered as the analog of *computable functions* enjoy many stability properties similar to stability properties of computable functions. All common analytic functions are in this class, an observation similar to the fact that all common functions in mathematics are computable. Such functions are stable by most of the operations (addition, multiplication, subtraction, division, inverse, composition, ODE solving, etc. . .). Some analytic computable functions are known not to be in this class. However, if a modern definition of computability is considered for pODEs, then computable functions for Turing machines and by pODEs coincide. Etc . . .

Context of the work

We recently proved that there is a very natural and well-founded way of measuring time of computation for such functions: time corresponds to the length of the trajectory.

This surprising result, was awarded the ICALP'2016 best paper award (Track B), and led to the (European PhD) Ackermann Award 2017 to Amaury Pouly. Amaury Pouly was cosupervised by Daniel Gra c ca in Portugal and myself in Palaiseau, France. We used this result in bioinformatics, and we proved that kinetic mechanisms can simulate Turing machines. We received the CMSB'2017 best paper award for this result, and the "prix du journal La Recherche" 2019 (for computer science) in 2019 for this result.

On one hand, this demonstrates that analog models of computation (in particular old and first ever considered models of machines) are equivalent to modern models both at the computability (we established this fact about 10 years ago [2]) and complexity level (this was an open problem, and part of the awarded facts [5, 3]).

More importantly, on the other hand, this opens new lights on classical computability and complexity: Indeed, this proves that polynomial time (class P or FP for respectively decision problems and functions) can be defined very easily using only very simple concepts from analysis: polynomial ordinary differential equations and length of curves. Indeed, this states for example the following rather unexpected fact: a function over the real is computable in polynomial time iff it can be uniformly approximated by solutions of a polynomial ordinary differential equation of polynomial length.

Description of the work

We recently used our "technology" to prove that there is a universal ordinary differential equation [4].

There exists a **fixed** polynomial vector p in d variables such that for any functions $f \in C^0(\mathcal{R})$ and $\varepsilon \in C^0(\mathcal{R}, \mathcal{R}^*_+)$, there exists $\alpha \in \mathcal{R}^d$ such that there exists a unique solution $y : \mathcal{R} \to \mathcal{R}^d$ to $y(0) = \alpha$, $y' = p(y)$. Furthermore, this solution satisfies that $|y_1(t) - f(t)| \leq \varepsilon(t)$ for all $t \in \mathcal{R}$, and it is analytic.

The objective will be to improve this result, in particular, by trying to improve our solution (that were not at all optimised) to reduce the dimension of the equations. An alternative will be to use this result, to generalize some concepts from theoretical computer science: For example, by replacing the concept of "simplest program" by simplest ordinary differential equation/initial condition, one may expect theorems similar to the one known in the classical settings for Kolmogorof complexity.

We have some preliminary statements in these directions, but the purpose is to reach a nice(r) and elegant characterizations.

Comments

The true topic of the work is related to computability and complexity theory. This requires only common and basic knowledge in ordinary differential equations. Most of the intutions of our today's constructions come from classical computability and complexity.

There is no specific prerequisite for this internship. Possibilities of funding according to the administrative situation of candidates.

The supervisor is also open on variations on these questions according to the preferences of the candidate (focusing on algorithmic aspects " how to derive an efficient algorithm using continuous methods", logical aspects "what is the associated proof theory", etc.).

References

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