# Generalising the 1-2-3 Conjecture to signed graphs

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## Abstract

In distinguishing labelling problems, one aims, given a graph, at labelling its edges so that certain sets of vertices are distinguished through some function computed from their incident labels. The 1-2-3 Conjecture is one such specific problem [KLT04], asking whether almost all graphs can be labelled with 1,2,3 so that no two adjacent vertices are incident to the same sum of labels. To date, this conjecture has been verified for several classes of graphs, including complete graphs and 3-colourable graphs [Sea12]. It is also known that almost all graphs can be labelled in the desired way with labels 1,2,3,4,5 [KKP10], and that, in general, finding such a labelling with labels 1,2 is NP-complete [DW11].

A common line of research in graph theory, is generalising existing problems defined on undirected graphs to more general structures. In the context of the 1-2-3 Conjecture, this was recently done for digraphs [BBS15,BBPW17,BL20] and hypergraphs [KKP16].

## Objectives of the internship

The goal of this internship will be to investigate possible generalisations of the 1-2-3 Conjecture to signed graphs. Recall that a 2-edge-coloured graph is a graph in which each edge can be either positive or negative. A signed graph is a 2-edge-coloured graph in which vertices can be resigned at will, where resigning a vertex means changing the polarity of all its incident edges. In some sense, signed graphs stand as a dynamic version of 2-edge-coloured graphs.

1. Investigate possible ways to generalise the 1-2-3 Conjecture to 2-edge-coloured graphs and signed graphs. Note, for instance, that, here, every vertex is incident to two types of edges, the negatives ones and the positive ones. This yields a negative colour and a positive colour by any labelling for every vertex, thus two distinguishing parameters to play with. There are thus several ways to define a signed 1-2-3 Conjecture, some of which might be more interesting than others.

2. For every resulting signed variant of the 1-2-3 Conjecture, the next goal will be to provide results towards it. A first task will be to support the conjecture by proving it for common classes of graphs. A

second task will be to provide bounds towards the conjecture. To that aim, one way to proceed will be to understand and adapt existing proofs from the literature.

3. For every variant, we will also investigate algorithmic aspects. For instance, is it hard to determine the smallest set 1,...,k such that every 2-edge-coloured/signed graph can be labelled? Another interesting algorithmic question is about finding particular graph signatures. For instance, is it hard to decide whether, for some variant, a graph can have its edges signed so that label 1 can eventually be assigned to every edge?

4. The resigning operation opens the way for algorithmic problems that are very close to reconfiguration problems in essence. For instance, given a signed graph with a labelling, can we always make this labelling proper by switching vertices? Is there an efficient algorithm to achieve this?

## References

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