Kolmogorov complexity of 2D sequences

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Kolmogorov complexity

- Goal: to measure the complexity of an individual object
- (Shannon) theory of information:
  measures the complexity of a random variable
- A theory of optimal compression

“the size of the smallest program that generates the object”
Examples

- $K(n) < \log(n) + c$
- $K(2^n+17) < \log(n) + c$
- $K(x^y) < \log(x) + \log(y) + c$
- $K(x^y|y) < \log(x) + c$
- Strings with low complexity are rare
Two theorems

- The set of prime numbers is infinite
- If $K(x_0, x_1, \ldots, x_n | n) < c$, then $x_i$ is computable
Exemple of 2D infinite objects

A finitary drawing
This infinite object is “simple”

\[ nxn \text{ squares have } \log(n) \] complexity
Complex infinite objects

- Flip a coin for each cell
- No structure
- Theorem (Levin Schnorr 1971):
  random configurations have maximal complexity. The complexity of all their $n \times n$-squares centered in (0,0) is $n^2$. 
What is the complexity induced by a finite set of local constraints?

Motivations: molecule arrangements, etc.

- Hilbert’s 18th problem
- Hilbert *das Entscheidungsproblem*
Tile sets

- Wang tiles
  - Squares with colored borders
- Tiles with arrows
  - Arrows and colors
- Polygons -- rational coordinates
  - Correct arrangement
- ✓ No irrational coordinates (Penrose)
Wang tiles

- Squares of unit size
- Colored borders
- No rotations
- Finite number
- Matching colors
Example - Wang tiles
Periodic tiling obtained

2x4
Tiles with arrows

- Squares of unit size
- Arrows on borders
- Rotations allowed
- Finite number
- Arrows must match
Example - tiles with arrows
See something and...
...imagine more
Polygons -- rational coordinates

- Polygon on a grid
- Polygon simple
- No rotations
- Finite number
- Correct arrangement
Elementary example
And also…
Tiling of a region

- The matching constraint must be ok inside the region
- No constraint on the border
- Examples:
  - Tiling of a rectangle
  - Tiling of a half-plane
  - Tiling of the plane
Simulations

- These models are equivalent for tilability of a region.
- Some theory is needed here (skipped)
A more general model:
Local constraints

- Planar configurations of 0’s and 1’s
- A configuration is a tiling if and only if
  - a local and uniform constraint is verified
    - Local: neighborhood
    - Uniform: same rule in each cell
A local constraint is a palette if and only if it can tile the plane (L. Levin)

- Idem: Wang tiles
- Idem: tiles with arrows
- Idem: polygons
Decision problem: « domino problem »

- Input: a local constraint T
- Question: is T a palette?

This problem is undecidable (Berger 1966)
Break translational symmetry

Nice configuration
(little cheating…)
Still nicer: a carpet!
How to build such carpets…
How to express that

- Carpets can be produced by tilings
  or
- There exists a palette that produces carpets
  or
- In all tilings by a palette, carpets appear
Tilings enforced by a palette

A set of configurations that is
- Shift invariant
- Compact
What we hope to enforce

Let $c$ be a configuration

- The set of configurations that contain the same finite patterns than $c$

- Id est:

\[
\Gamma(c) = \bigcup_{i,j \in \mathbb{Z}} \{\sigma_h^i \circ \sigma_v^j(c)\}.
\]
The carpet is enforceable

Possible proofs:
1. Give explicitly a palette that enforces it
2. Give a construction method for such a palette
3. Prove that such a palette exists
1. A palette that enforces carpets

More or less...
2. Construction method: self-similarity of carpets

- Smallest squares are red and form a 2 steps grid
- Squares of same size are vertically and horizontally aligned
- In the center of a red square (resp. blue) lays a corner of a blue one (resp. red)
- Squares of same color are disjoined
3. Existence proof

A configuration $c$ is:

- of finite type if and only if there exists $n$ such that

$$\Gamma_n(c) = \Gamma(c)$$

- of potentially finite type if and only if it can be « enriched » into a configuration of finite type.
Finite types and tilability

- A configuration is of potentially finite type if and only if it is enforced by a tiling.

- Theorem: the carpet is of potentially finite type.
  Constructive proof ($n=2$)
Question of the day (bis)

Consider all tilings obtained with a considered palette. How complex is the simplest one?
Theorems

- Undecidability of the « domino problem »… Applications in logics. (Berger 1966, Robinson 1971, Gurevich and Koriakov 1972)
- There exists a palette that produces only non-recursive tilings (Hanf and Myers 1974) Cannot be improved (Albert Muchnik)
- Complexity bound: Any palette can form at least a tiling in which squares of size $n$ contain at most $O(n)$ bits of information. (BD, Leonid Levin and Alexander Shen 2001)
- There exists a palette s.t. for all tiling, any square of size $n$ contains about $n$ bits of information. (same paper - long version in preparation - ready November 2067) Checks that the infinite sequence is complex
- Extensions to configurations that tolerate tiling errors?
Complex tilings constructed

- Aperiodic tile sets
- Areursive tile sets \((x,y) \sqsubseteq T(x,y)\)
- Complex tilings:
  - in all \(n \times n\)-squares there are \(n\) bits of a random sequence (optimal)
Complexity lemma

- An infinite sequence $x$ is uniformly $c$-random if and only if there exists $N$ such that for all $k > N$ for all $i$ $K(x_i...x_{i+k}) > ck$

- Lemma: For all $c < 1$ there exists a uniformly $c$-random sequence

- works for bi-infinite sequences - no arbitrary large subsequences of 0’s