Kolmogorov complexity of 2D sequences

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Kolmogorov complexity

- Goal: to measure the complexity
  of an individual object
- (Shannon) theory of information:
  measures the complexity of a random variable
- A theory of optimal compression

“the size of the smallest program that generates the object”

Examples

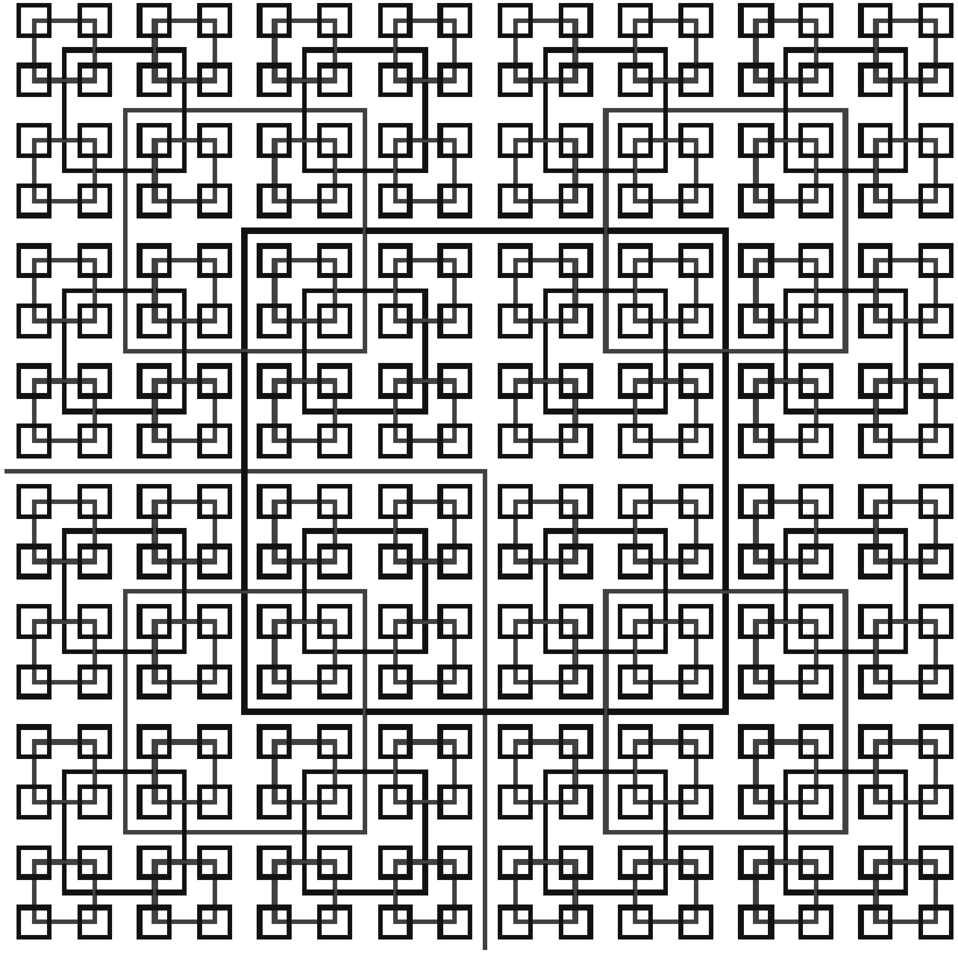
- $K(n) < \log(n) + c$
- $K(2^n+17) < \log(n) + c$
- $K(x^y) < \log(x) + \log(y) + c$
- $K(x^y|y) < \log(x) + c$
- Strings with low complexity are rare

Two theorems

- The set of prime numbers is infinite
- If $K(x_0,x_1,\ldots x_n|n) < c$, then x_i is computable

Exemple of 2D infinite objects

A finitary drawing

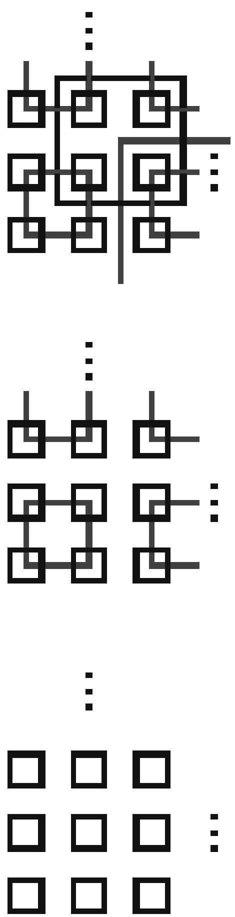


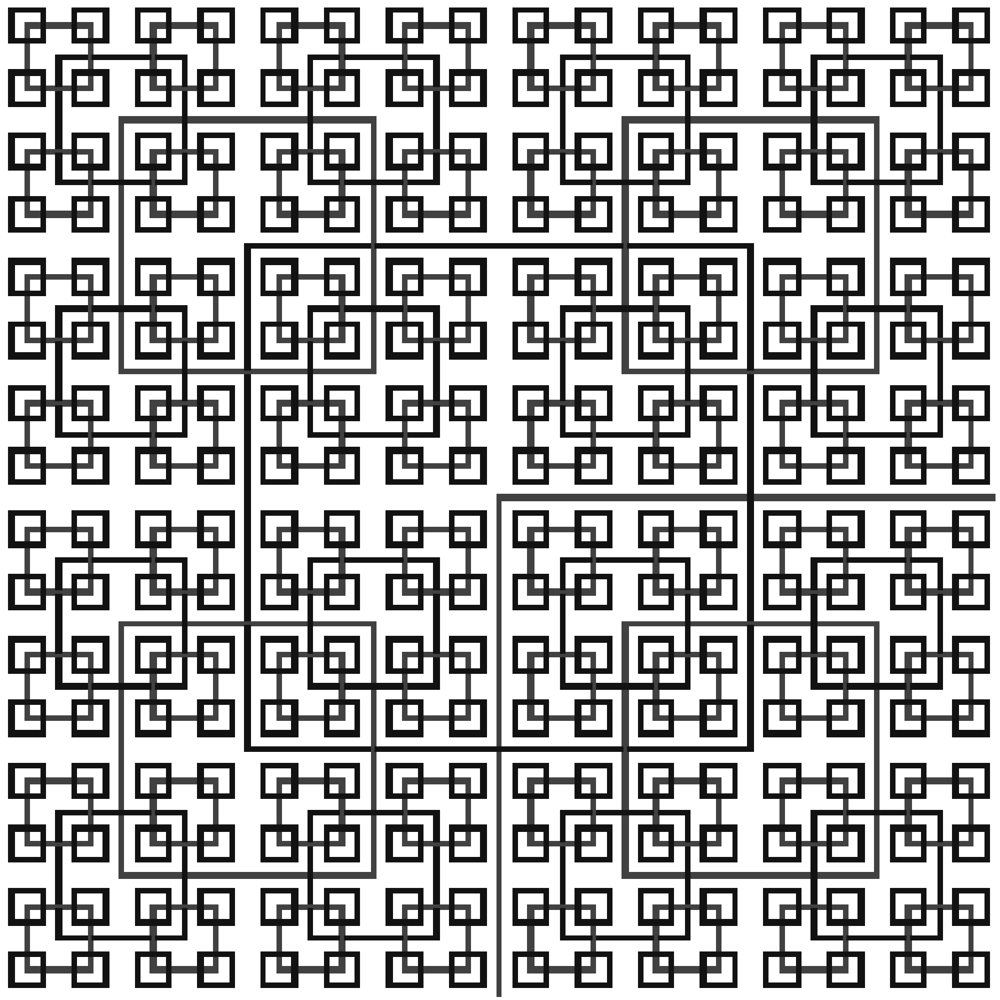
This infinite object is “simple”

*n*x*n* squares

have log(*n*)

complexity





Complex infinite objects

- Flip a coin for each cell
- No structure
- Theorem (Levin Schnorr 1971):

  random configurations have maximal complexity. The complexity of all their nxn-*squares* centered in (0,0) is n^2.

Question of the day

What is the complexity induced by a finite set of local constraints?

Motivations: molecule arrangements, etc.

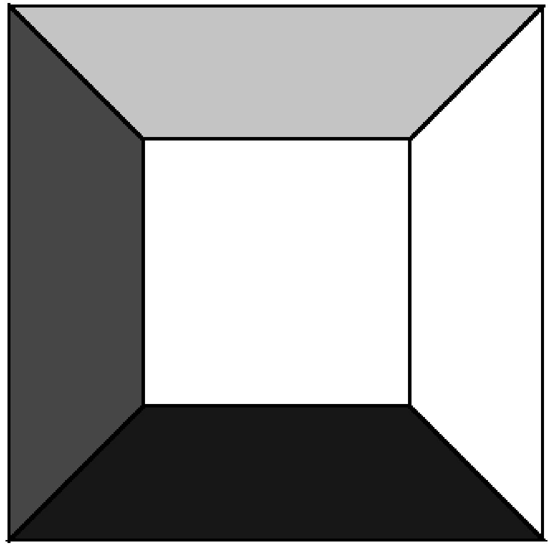
- Hilbert's 18th problem
- Hilbert *das Entscheidungsproblem*

Tile sets

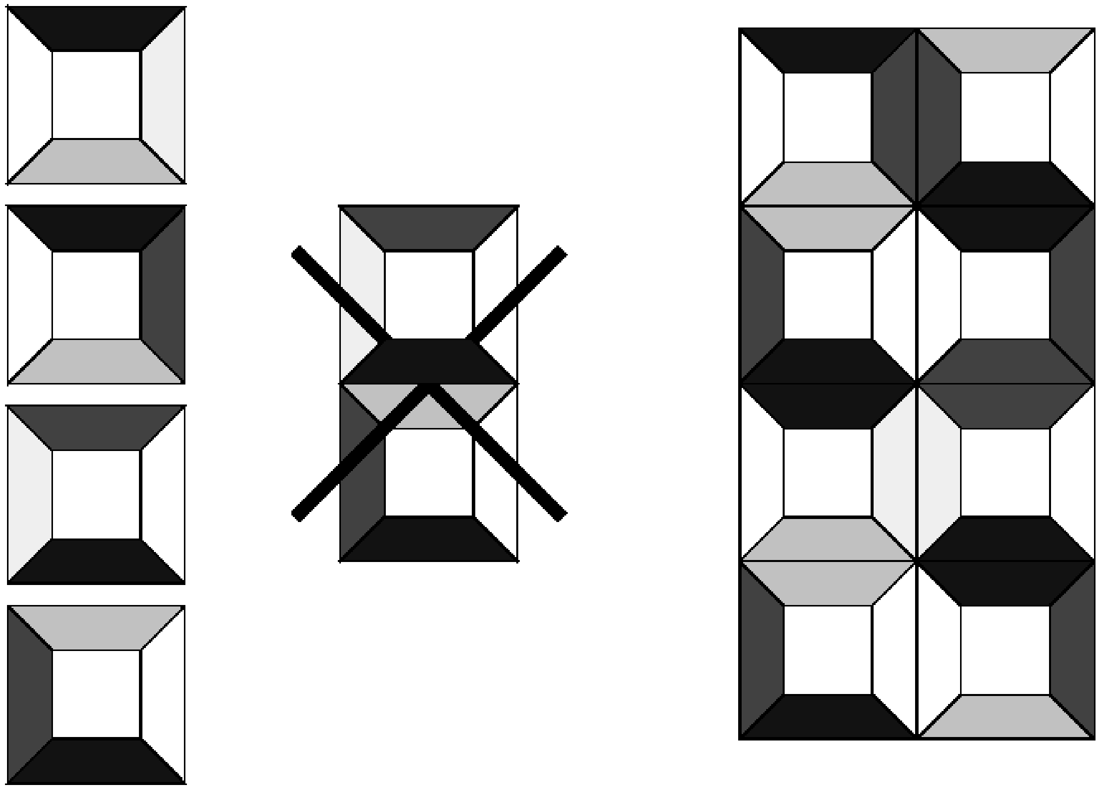
- Wang tiles
  - Squares with colored borders
- Tiles with arrows
  - Arrows and colors
- Polygons -- rational coordinates
  - Correct arrangement
- ✓ No irrational coordinates (Penrose)

Wang tiles

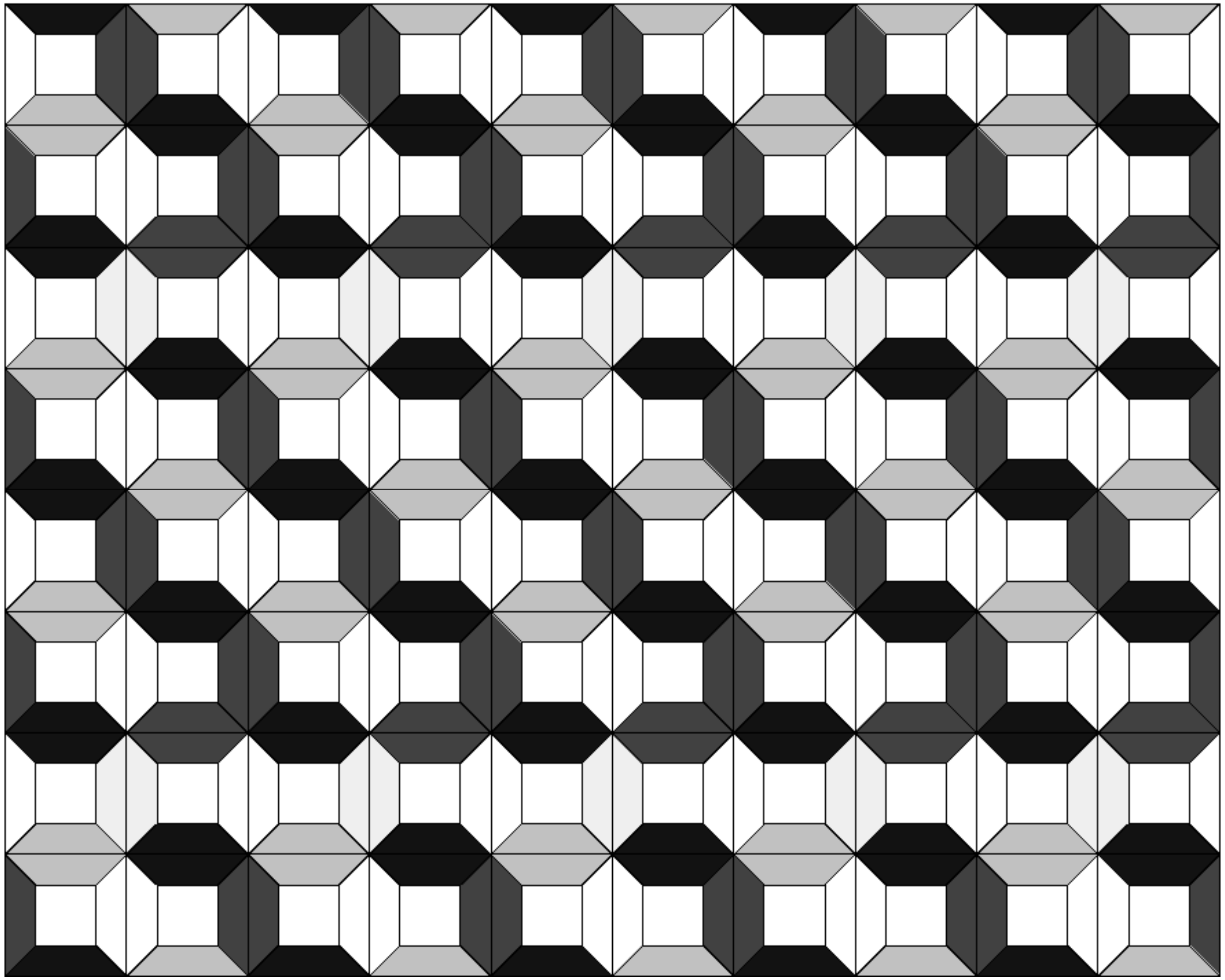
- Squares of unit size
- Colored borders
- No rotations
- Finite number
- Matching colors



Example - Wang tiles



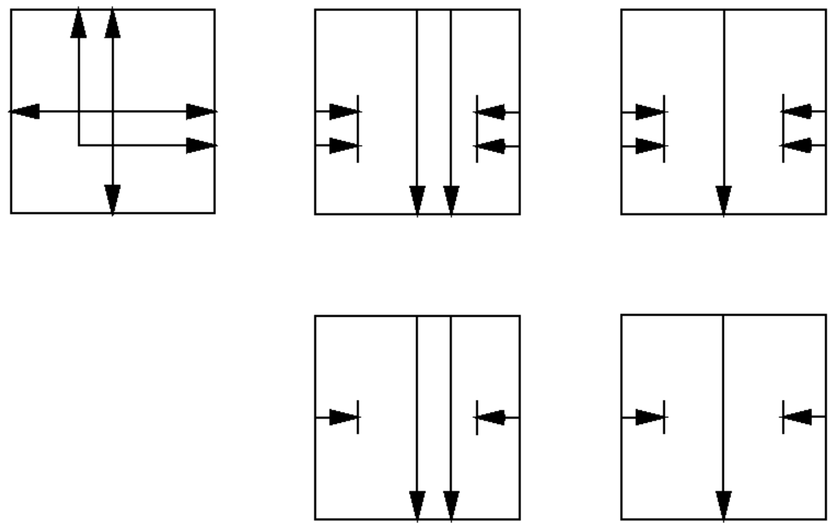
Periodic tiling obtained



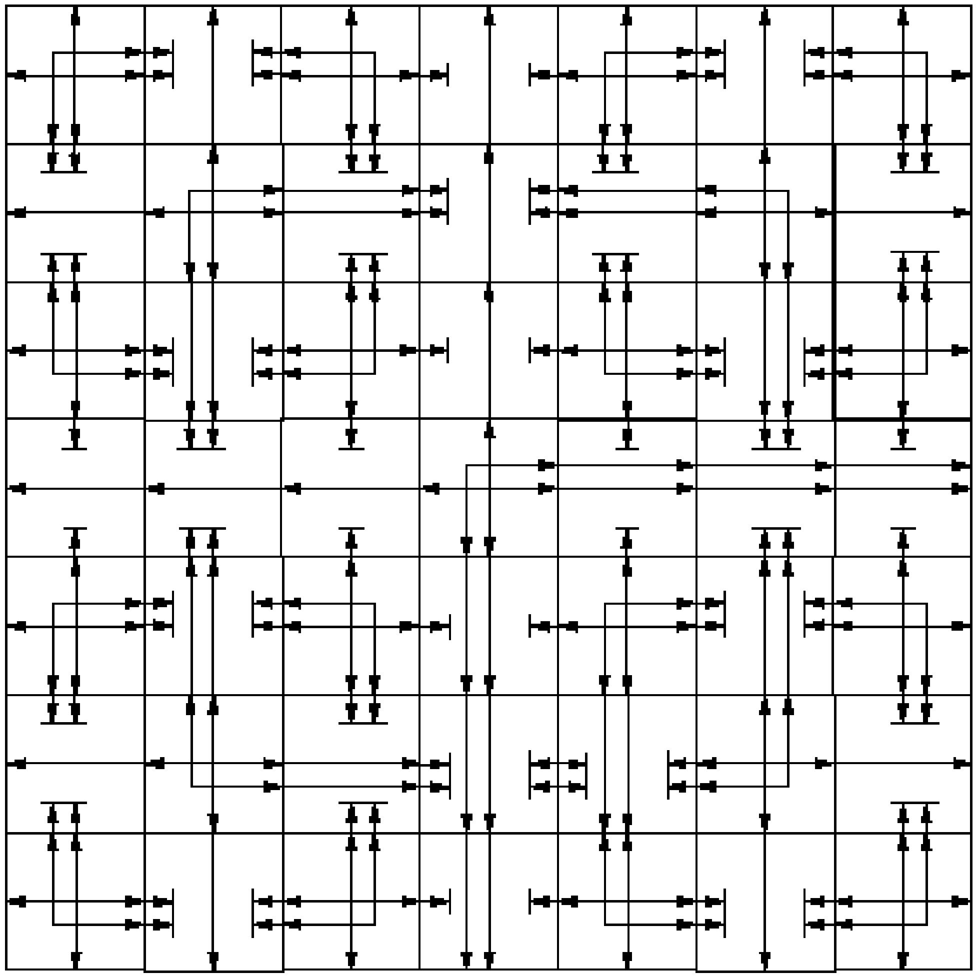
2x4

Tiles with arrows

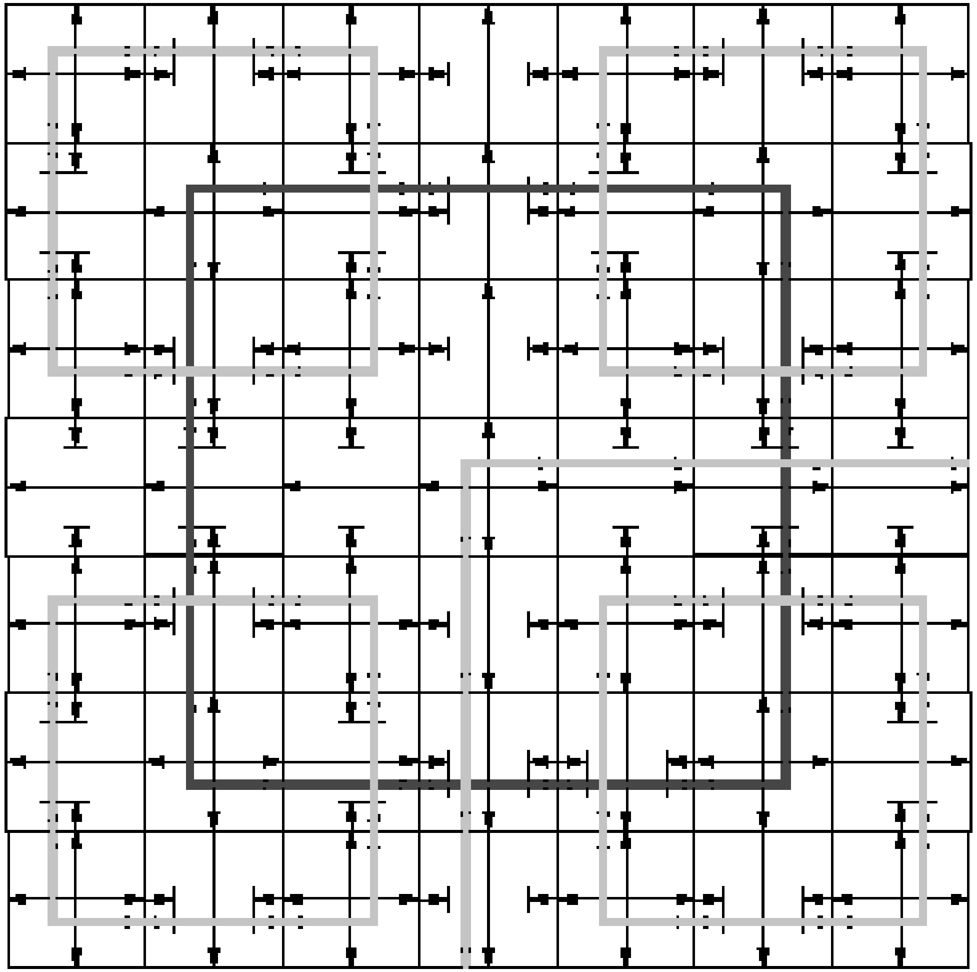
- Squares of unit size
- Arrows on borders
- Rotations allowed
- Finite number
- Arrows must match



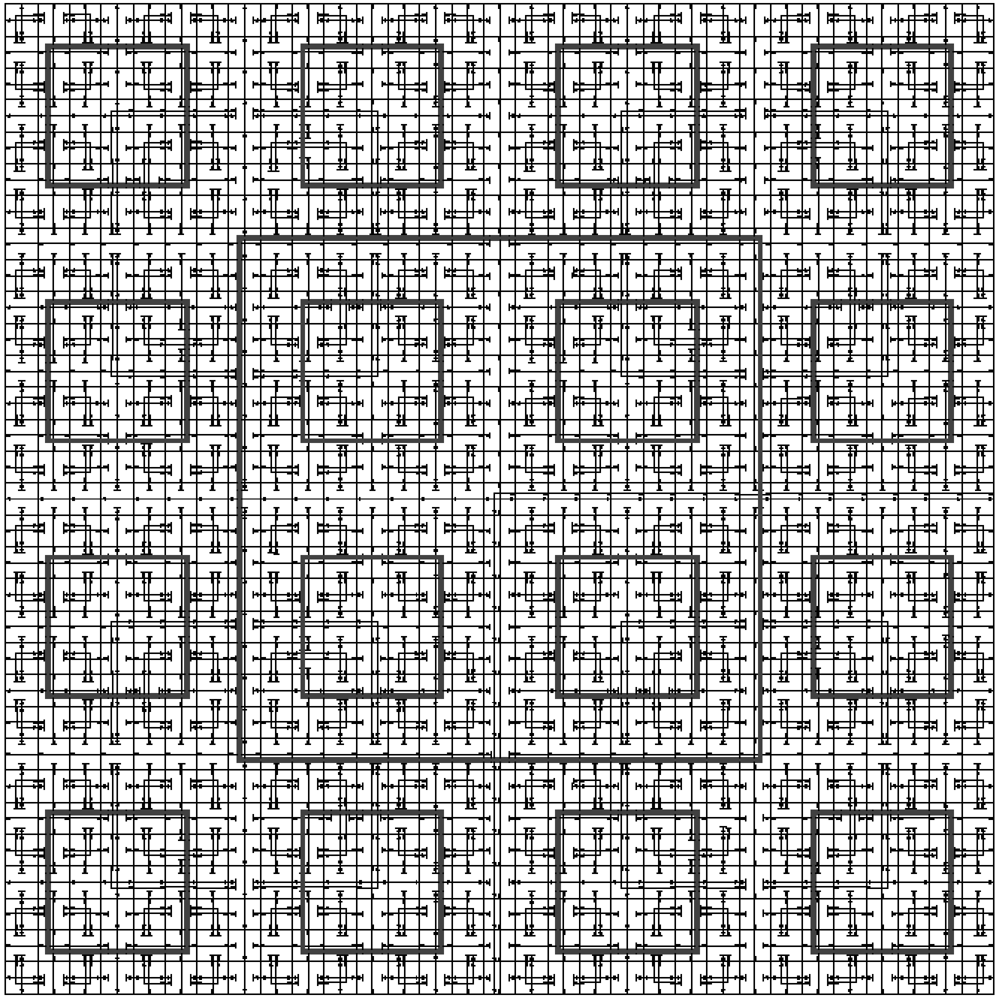
Example - tiles with arrows



See something and…

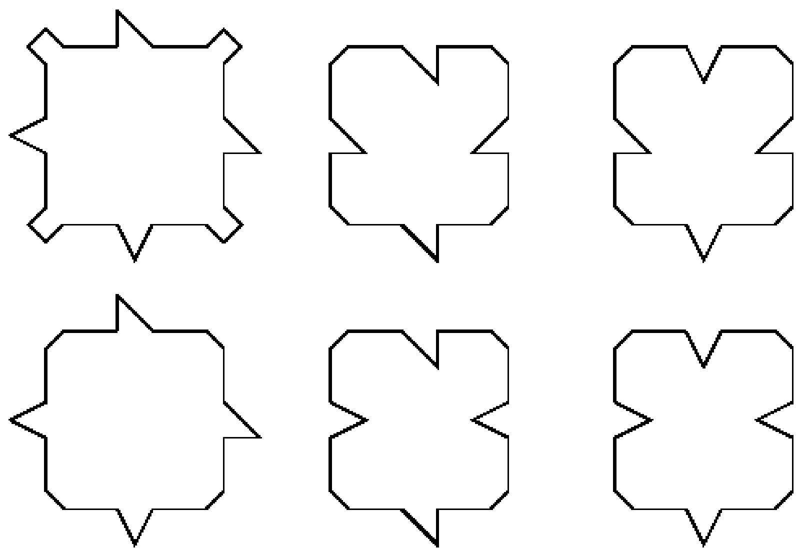


…imagine more

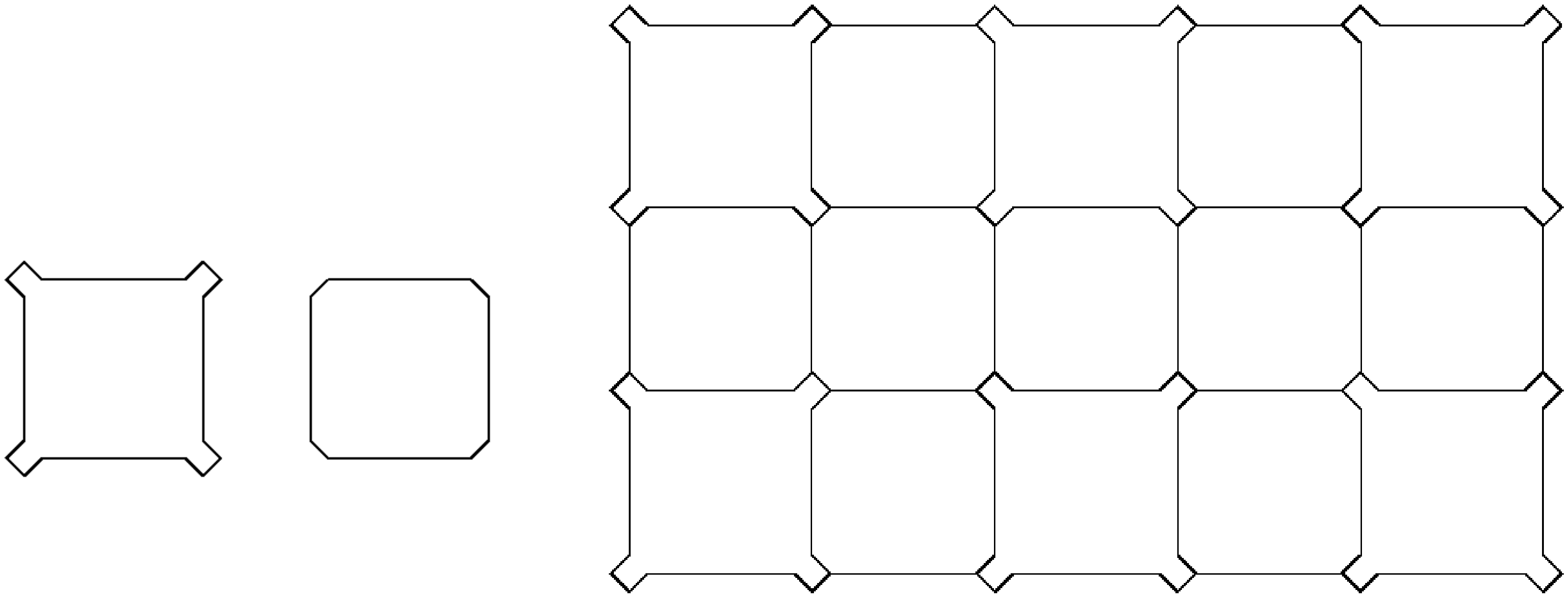


Polygons -- rational coordinates

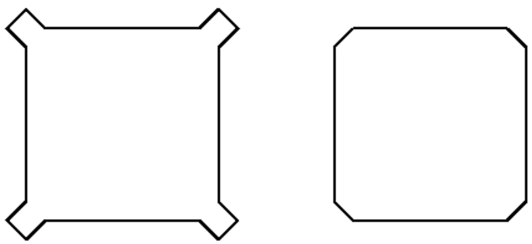
- Polygon on a grid
- Polygon simple
- No rotations
- Finite number
- Correct arrangement

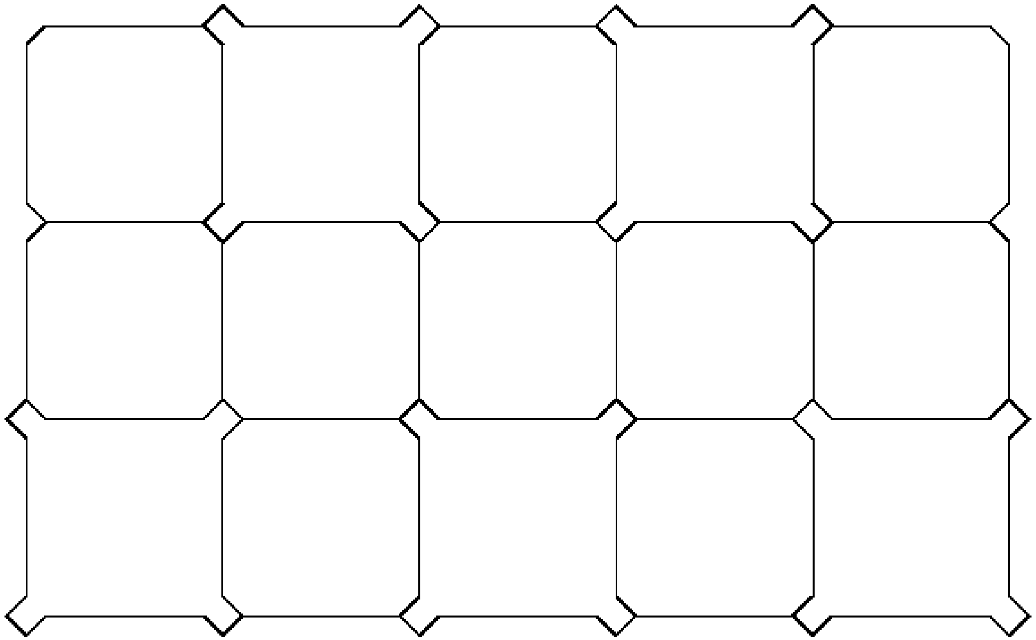


Elementary example



And also…



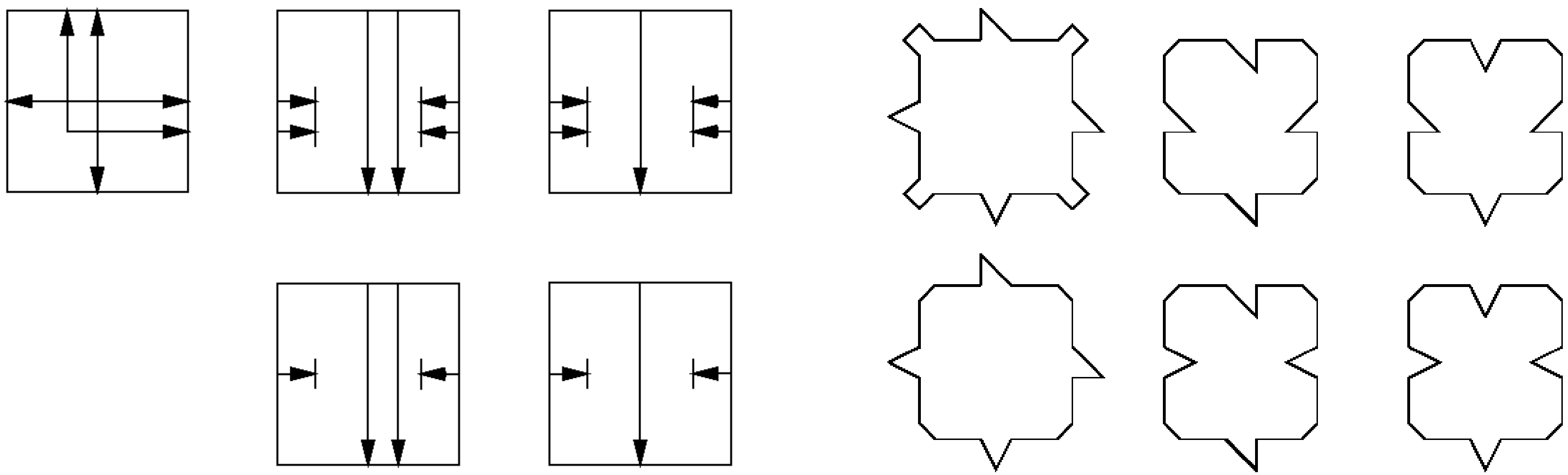


Tiling of a region

- The matching constraint must be ok inside the region
- No constraint on the border
- Examples:
  - Tiling of a rectangle
  - Tiling of a half-plane
  - Tiling of the plane

Simulations

- These models are equivalent for tilability of a region.
- Some theory is needed here (skipped)



A more general model: Local constraints

- Planar configurations of 0’s and 1’s
- A configuration is a tiling

  if and only if

- a local and uniform constraint is verified
  - Local : neighborhood
    Uniform : same rule in each cell

Palettes

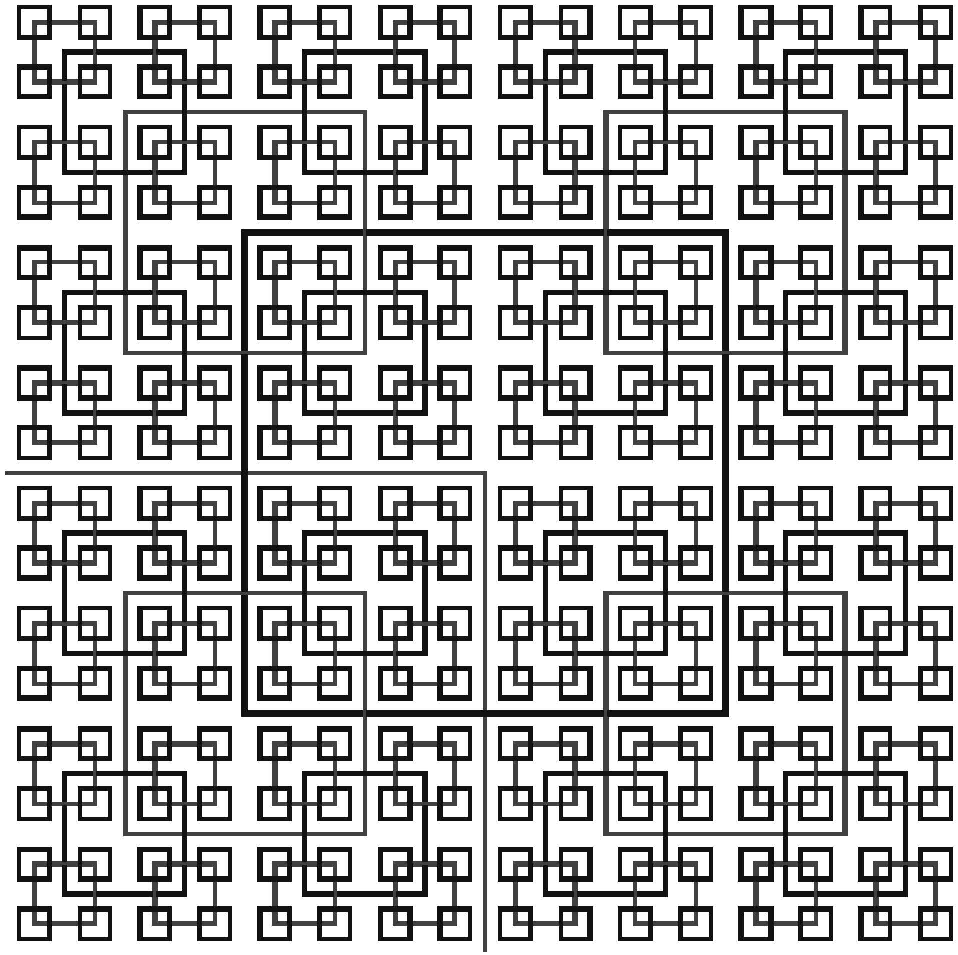
- A local constraint is a palette if and only if it can tile the plane (L. Levin)
  - Idem : Wang tiles
  - Idem : tiles with arrows
  - Idem : polygons

« computation - geometry »

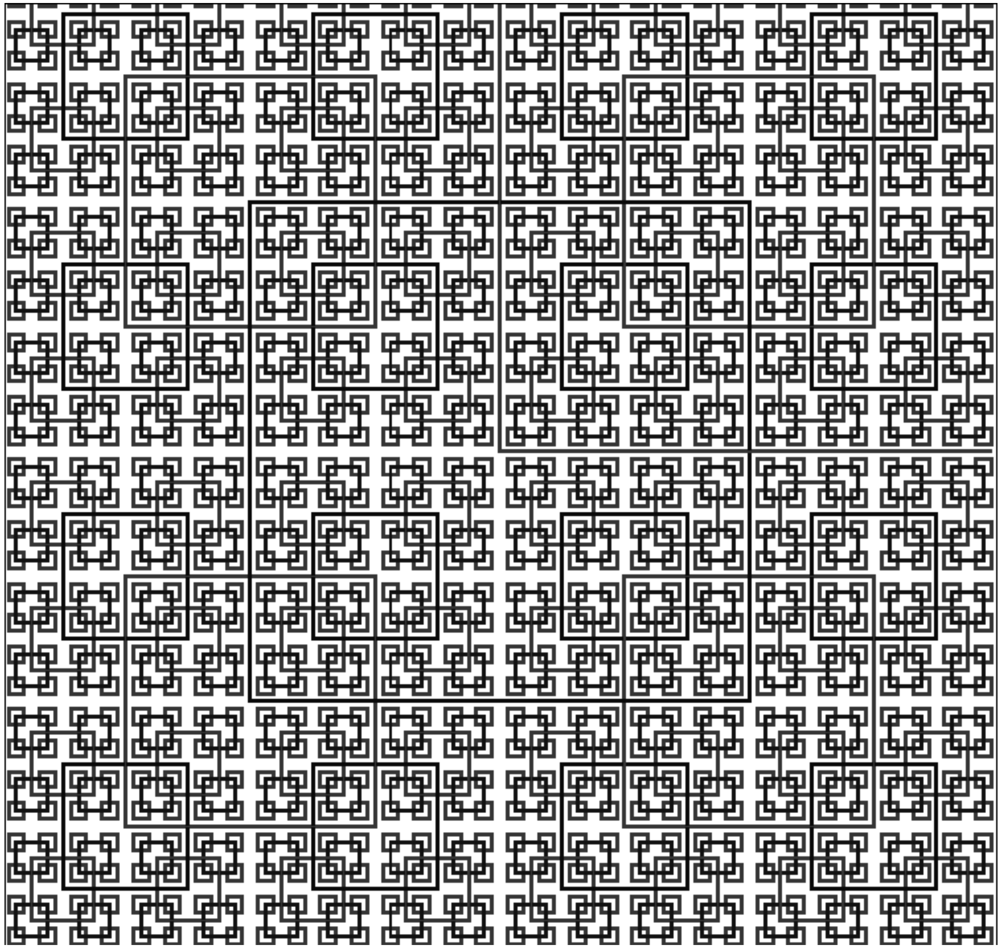
- Decision problem : « domino problem »
  - Input : a local constraint T
  - Question : is T a palette ?
- This problem is undecidable (Berger 1966)

Break translational symmetry

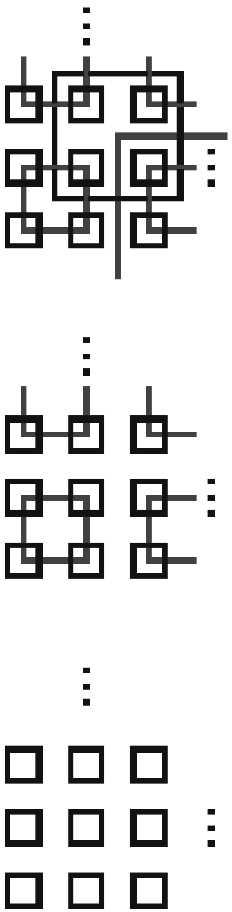
Nice configuration
(little cheating…)

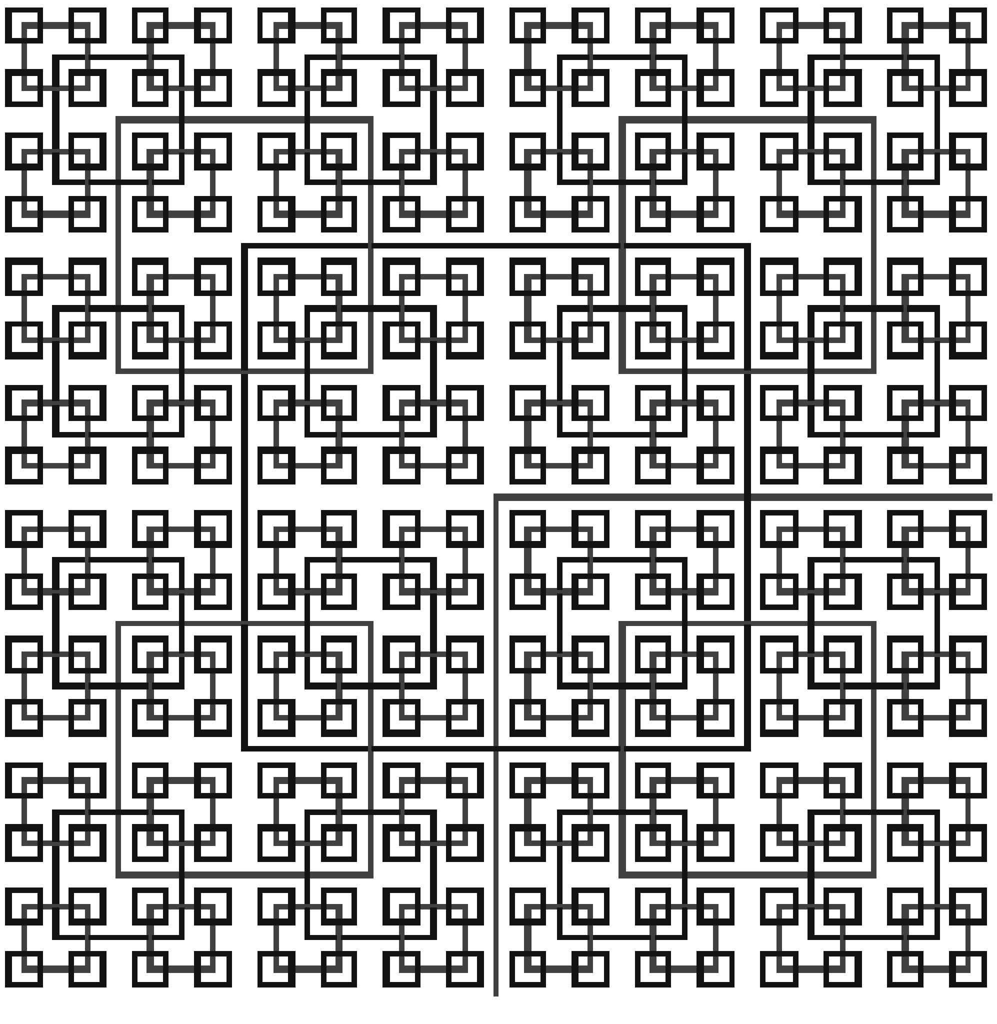


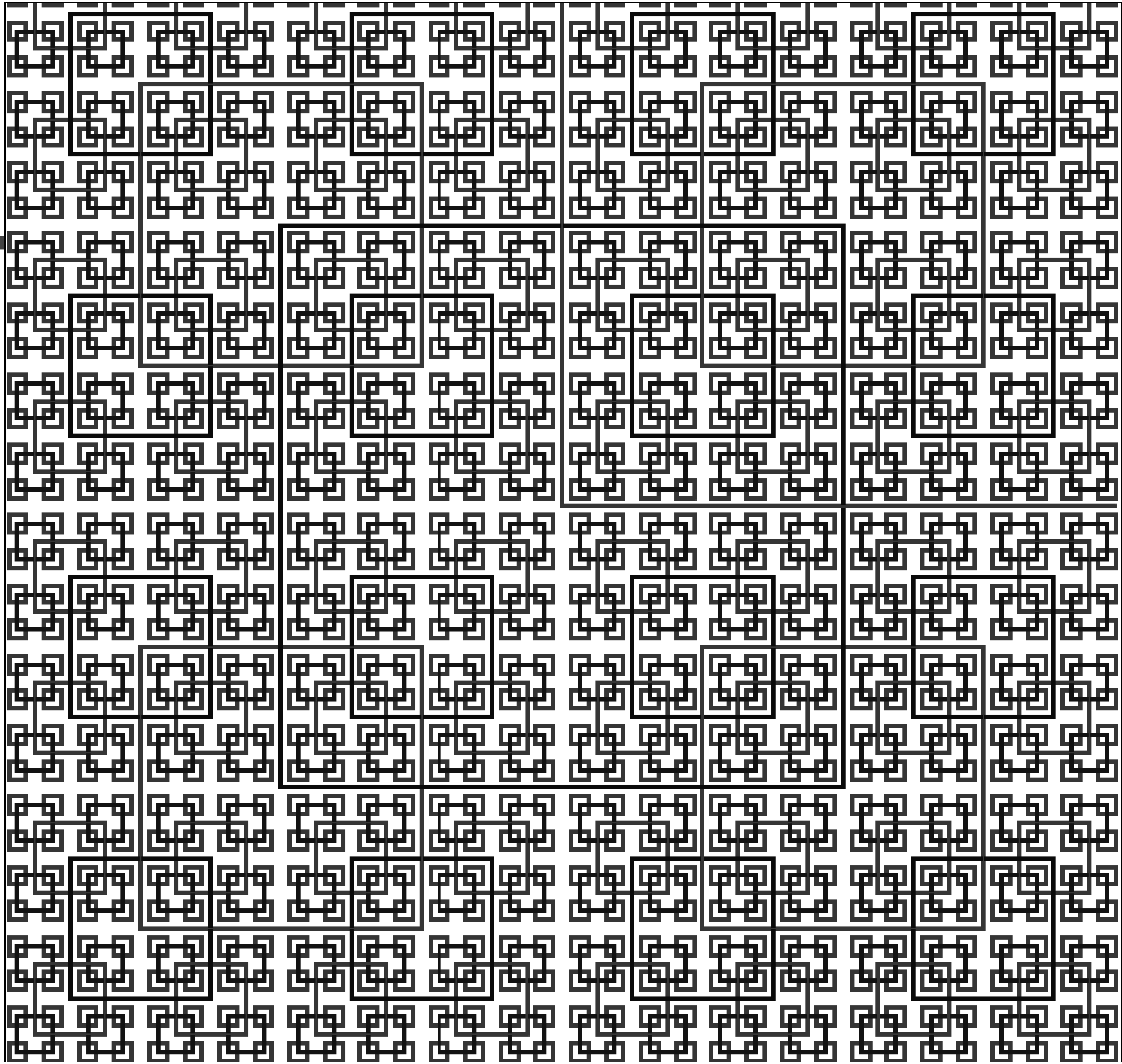
Still nicer : a carpet !



How to build such carpets…







How to express that

- Carpets can be produced by tilings

  or

- There exists a palette that produces carpets

  or

- In all tilings by a palette, carpets appear

Tilings enforced by a palette

A set of configurations that is

- Shift invariant
- Compact

What we hope to enforce

Let *c* be a configuration

- The set of configurations that contain the same finite patterns than *c*
- Id est :

$$\Gamma(c) = \overline{\bigcup_{i,j \in \mathbb{Z}} \{\sigma_h^i \circ \sigma_v^j(c)\}}.$$

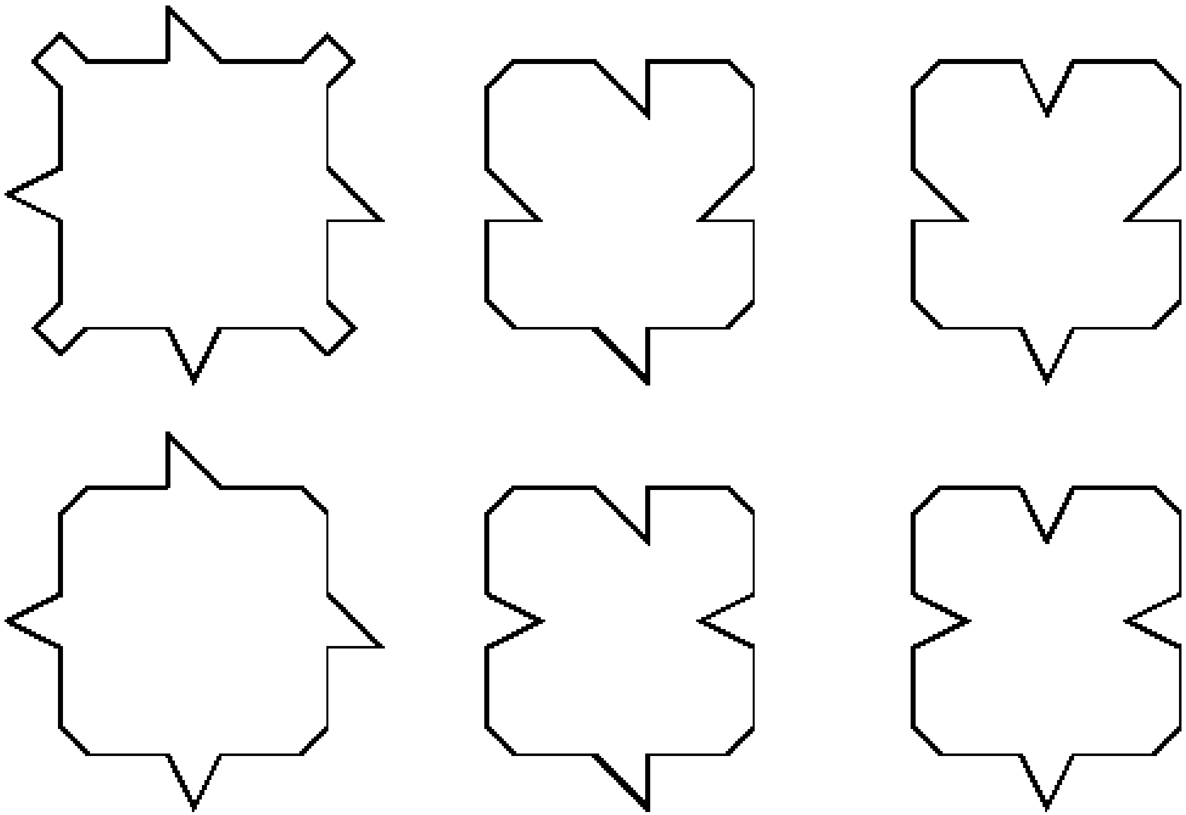
The carpet is enforceable

Possible proofs:

1. Give explicitly a palette that enforces it
2. Give a construction method for such a palette
3. Prove that such a palette exists

1. A palette that enforces carpets

More or less…



2. Construction method: self-similarity of carpets

- Smallest squares are red and form a 2 steps grid
- Squares of same size are vertically and horizontally aligned
- In the center of a red square (resp. blue) lays a corner of a blue one (resp. red)
- Squares of same color are disjoined

3. Existence proof

A configuration c is :

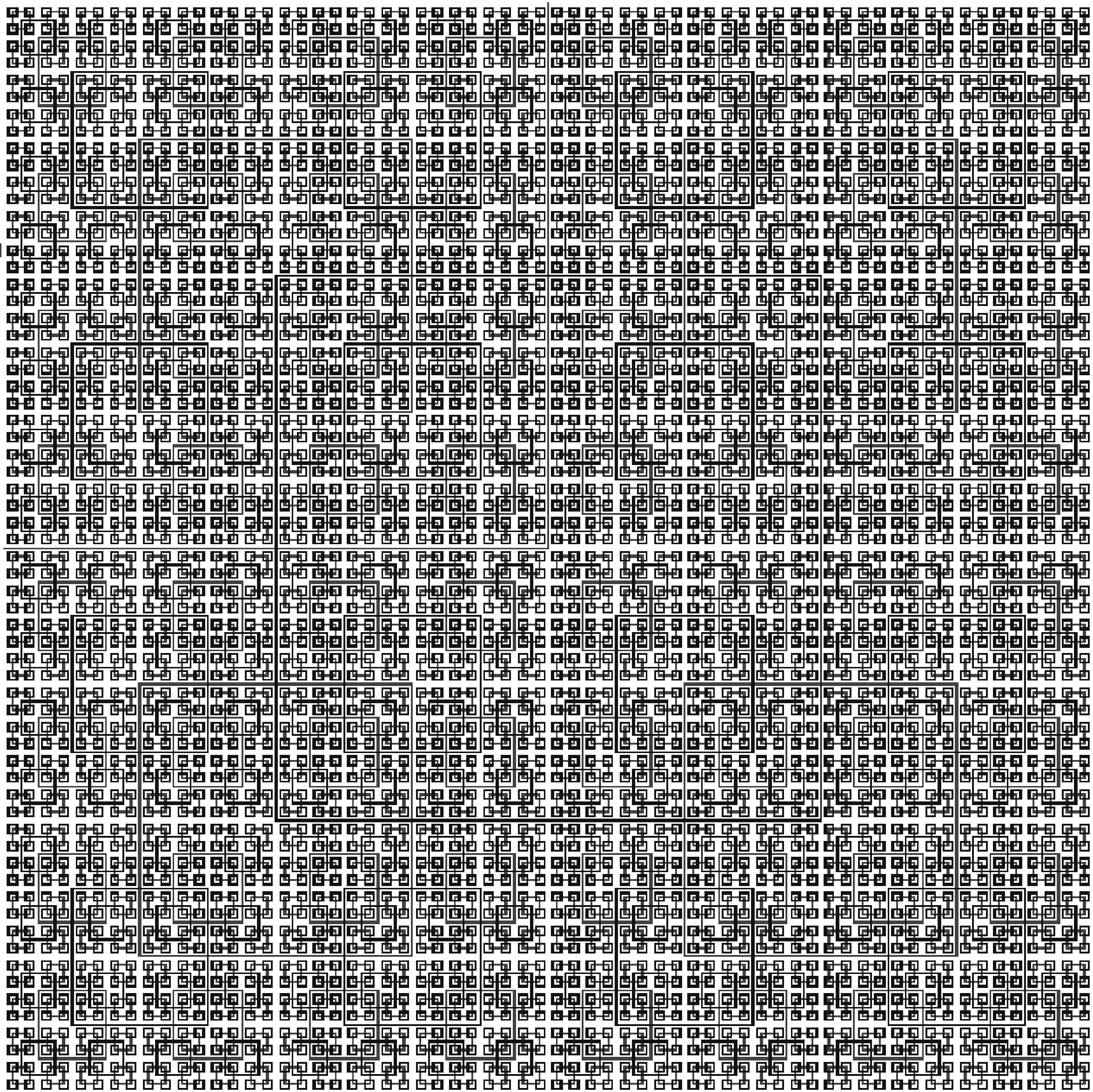
- of finite type if and only if there exists n such that

$$\Gamma_n(c) = \Gamma(c)$$

- of potentially finite type if and only if it can be « enriched » into a configuration of finite type.

Finite types and tilability

- A configuration is of potentially finite type if and only if it is enforced by a tiling.
- Theorem: the carpet is of potentially finite type.
  Constructive proof (n=2)



Question of the day (bis)

Consider all tilings obtained with a considered palette.

How complex is the simplest one?

Theorems

- Undecidability of the « domino problem »… Applications in logics. (Berger 1966, Robinson 1971, Gurevich and Koriakov 1972)
- There exists a palette that produces only non-recursive tilings (Hanf and Myers 1974) Cannot be improved (Albert Muchnik)
- Complexity bound: Any palette can form at least a tiling in which squares of size n contain at most $O(n)$ bits of information. (BD, Leonid Levin and Alexander Shen 2001)
- There exists a palette s.t. for all tiling, any square of size n contains about n bits of information. (same paper - long version in preparation - ready November 2067) Checks that the infinite sequence is complex
- Extensions to configurations that tolerate tiling errors?

Complex tilings constructed

- Aperiodic tile sets
- Arecursive tile sets $(x,y) \to T(x,y)$
- Complex tilings:

  in all nxn-squares there are n bits of a random sequence (optimal)

The end

Complexity lemma

- An infinite sequence x is uniformly c-random if and only if there exists N such that for all $k > N$ for all i $K(x_i \ldots x_{i+k}) > ck$
- Lemma: For all $c < 1$ there exists a uniformly c-random sequence
- works for bi-infinite sequences - no arbitrary large subsequences of 0's

The true end