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Real space renormalization group and totalitarian paradox of majority rule voting

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Abstract

The effect of majority rule voting in hierarchical structures is studied using the basic concepts from real space renormalization group. It shows in particular that a huge majority can be self-eliminated while climbing up the hierarchy levels. This majority democratic self-elimination articulates around the existence of fixed points in the voting flow. An unstable fixed point determines the critical threshold to full and total power. It can be varied from 50% up to 77% of initial support. Our model could shed new light on the last century eastern European communist collapse. © 2000 Elsevier Science B.V. All rights reserved.

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1. From physics to politics

In recent years, statistical physics has been applied to a wide range of different fields including economy, finance, population dynamics, traffic flow and biology [1]. Application to politics is yet very scarce. A spin glass like model has been suggested to describe the former Yugoslavia fragmentation [2].

In this paper we present an application of real space renormalization group techniques [3] to the study of the dynamics of representativity within a democratic hierarchical structure. Each level is elected from the one below using a local majority rule voting. At the top level is the president.

We show that majority rule voting produces critical thresholds to full power. Having an initial support above the critical threshold guarantees to win the top level. The value

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of the critical threshold to power is a function of the voting structure. It depends on both the size of voting groups and the number of voting hierarchical levels.

While renormalization group technics are a mathematical trick to evaluate long wave-length fluctuations in collective phenomena, here we are giving a real meaning to each renormalization step. Instead of integrating out short-range fluctuations, we are building a voting procedure which associates an elected person to each local cell. Moreover in contrast to critical phenomena where the focus is at the unstable fixed point, here we are studying the dynamics in reaching the stable fixed points.

We are thus applying some physical tools to build a new quantitative model to describe the dynamics of political voting. Emphasis is on other aspects of the renormalization group transformation than in physics. We are not just making a metaphor between physics and politics.

The rest of the paper is organized as follows. In Section 2 we present the main frame of the voting model. The study is restricted to two tendency A–B free competing model. Neither strategy nor interactions are included. We first start from the case of 3-person cells in Section 3. A critical threshold to power is found. It equals 50% of initial support at the bottom level. The self-elimination of the minority occurs within only a few voting levels. Section 4 introduces some natural bias in the voting rule; like a tip to the ruling party. It is illustrated in the case of even 4-person cells. The tie case 2A–2B is then put in favor of the ruling party. Such a bias is found to shift the value of the critical threshold to power from 50 to 77% for the opposition. Cell size effects are then analyzed in Section 5. Analytic formula are derived in Section 6. Given an A initial support p_0, the number of voting levels necessary to their self-elimination is calculated. The results are then turned in a more practical perspective for hierarchical organizations in Section 7. Some visualization of a numerical simulation is shown in Section 8. Extension to 3 competing group is sketched in Section 9. Last section discusses the historical last century collapse of eastern european communist parties.

2. Setting the frame

We consider a population with A and B individuals. These A and B characters represent two opposite tendencies. The respective proportions of this bottom level (denoted level-0) are p_0 and $1 - p_0$. Each member does have an opinion. The frame may be a political group, a firm, a society.

Like in renormalization group scheme we start from small size local cells. Here the degrees of freedom are individual opinions. Each cell is constituted randomly from the population. Once formed, it elects a representative, either an A or a B using a local majority rule within the cell.

These elected people (the equivalent of the super spin rescaled to an Ising one in real space renormalization) constitute the first hierachical level of the hierarchy called level-1. Here this level is real and not fictitious like in the renormalization group scheme. The process is then repeated again and again. Each time starting from elected people at one level, to form new cells which in turn elect new representatives to build up a higher level. At the top level is the president.

3. A wonderful world

We start with the simplest case. Cells are built of 3 persons randomly selected from the population. It could correspond to home localization or working place. Each cell then elects a representative using a local majority rule. Cells with either 3A or 2A elect an A. Otherwise it is a B who is elected. Therefore, the probability to have an A elected from the bottom level is

$$p_1 \equiv P_3(p_0) = p_0^3 + 3p_0^2(1 - p_0), \tag{1}$$

where $P_3(p_n)$ denotes the voting function, here a simple majority rule.

The same process of cell forming is repeated within level-1. The elected persons (from level-0) form cells which in turn elect new higher representatives. The new elected persons constitute level-2. The process can then be repeated again and again. The probability to have an A elected at level $(n+1)$ from level-n is

$$p_{n+1} \equiv P_3(p_n) = p_n^3 + 3p_n^2(1 - p_n), \tag{2}$$

where p_n is the proportion of A elected persons at level-n.

The analysis of the voting function $P_3(p_n)$ exhibits the existence of 3 fixed points $p_l = 0$, $p_{c,3} = \frac{1}{2}$ and $p_L = 1$. The first one corresponds to where no A was elected. The last one $p_L = 1$ represents the totalitarian situation where only A are elected. Both p_l and p_L are stable fixed points. At contrast $p_{c,3}$ is unstable. It determines indeed the threshold to flowing towards either full power (with p_L) or to total disappearance (with p_l). Starting from $p_0 < \frac{1}{2}$ leads to the first case while $p_0 > \frac{1}{2}$ drives to the second one.

On this basis, we see that majority rule voting produces the self-elimination of any A proportion as long as the initial support is less than 50% ($p_0 < \frac{1}{2}$). However, this democratic self-elimination requires a sufficient number of voting levels to be completed.

At this stage, the instrumental question is to determine the number of levels required to ensure full leadership to the initial larger tendency. To make sense the level number must be small enough, most organizations have only few levels (less than 10).

To illustrate the above voting dynamics, let us calculate the representativity flow starting, for instance, from p_0=0.45. We get successively p_1=0.42, p_2=0.39, p_3=0.34, $p_4 = 0.26$, $p_5 = 0.17$, $p_6 = 0.08$ down to $p_7 = 0.02$ and $p_8 = 0.00$. Within 8 levels 45% of the population is self-eliminated.

Nevertheless, the overall process preserves the democratic character of majority rule voting. It is the bottom leading tendency (more than 50%) which eventually gets for sure the full leadership of the organization top level. It is worth noticing the symmetry of situation with respect to A and B tendencies. The threshold to full power is the same (50%) for both of them.

4. Ruling is good for you

From the above analysis, to turn down a top leadership requires to have more than 50% at the bottom which is a fair constraint. However, from real life situations, to produce an alternative leadership is much more difficult. A simple majority often appears to be not enough.

Many institutions are indeed built to strengthen some stability. Frequent political changes are not perceived as good for the organization. On this basis, a tip is often given to the ruling party. To be in charge, gives some additional power which breaks the symmetry between the two tendencies. For instance, giving one additional vote to the committee president or allowing the president to designate some committee members.

Using the democratic statement "to change things, you need a majority" produces indeed a strong bias in favor of current rulers. To exemplify it, we consider even size cells. Again restricting to the simplest case, it means 4 people cells. Assuming B as the ruling party, A need either 4 or 3A in a given cell to take over. The tie case 2A–2B votes for a B.

In going from 3 to 4 person size cells, the salient new feature is indeed the existence of 2A–2B configurations for which there exists no majority. In most social situations no decision implies de facto, no change. There exists a bias in favor of the rulers which now makes the voting function unsymmetric. The probability to get an A elected at level $n+1$ is

$$p_{n+1} \equiv P_4(p_n) = p_n^4 + 4p_n^3(1-p_n)\,, \tag{3}$$

where p_n is as before the proportion of A elected persons at level-n. In contrast for a B to be elected the probability is

$$1 - P_4(p_n) = p_n^4 + 4p_n^3(1-p_n) + 2p_n^2(1-p_n)^2\,, \tag{4}$$

where the last term embodies the bias in favor of B. From Eqs. (3) and (4) the stable fixed points are still 0 and 1. However, the unstable one is now drastically shifted to

$$p_{c,4} = \frac{1+\sqrt{13}}{6}\,, \tag{5}$$

for A. It makes A threshold to power at about 77%. Simultaneously, B threshold to stay in power is about 23%, making both situations drastically different. To take over power, A need to go over 77% of initial bottom support while to stick to power, B only need to keep their support above 23%.

In addition to the asymmetry the bias makes the number of levels to democratic self-elimination even smaller than in the precedent case (3-cell size). Starting again from $p_0 = 0.45$ we now get $p_1 = 0.24$, $p_2 = 0.05$ and $p_3 = 0.00$. Instead of 8 levels, 3 are enough to make A to disappear.

To illustrate how strong is this effect, let us start far above 50% with for instance $p_0 = 0.70$. The associated voting dynamics becomes $p_1 = 0.66$, $p_2 = 0.57$, $p_3 = 0.42$, $p_4 = 0.20$, $p_5 = 0.03$, and $p_6 = 0.00$. Within only 6 levels, 70% of a population is thus self-eliminated.

Using an a priori reasonable bias in favor of B turns a majority rule democratic voting to a totalitarianism outcome. To get to power A must pass over 77% of the overall support which is almost out of reach in any normal democratic two tendency situation.

5. Going big

Up to now we have considered very small size cells. But many organizations have larger size cells. Extending above cases to any size r is indeed straightforward in setting the problem. Only equations get a bit more complicated. However, the main features remain unchanged under size changes.

For an r-size cell the voting function $p_{n+1} = P_r(p_n)$ becomes

$$P_r(p_n) = \sum_{l=r}^{l=m} \frac{r!}{l!(r-l)!} p_n^l (1 + p_n)^{r-l} , \tag{6}$$

where $m = (r+1)/2$ for odd r and $m = (r+1)/2$ for even r which thus accounts for A-bias.

The two stable fixed points $p_l = 0$ and $p_L = 1$ are unchanged. They are size independent. In the case of odd sizes, the unstable fixed point is also unchanged with $p_{c,r} = \frac{1}{2}$. On the contrary, for even sizes, the asymmetry between the threshold values for respectively rulers and nonrulers weakens with increasing sizes. For A threshold it is $p_{c,4} = (1 + \sqrt{13})/6$ for size 4 and it decreases asymptotically towards $p_{c,r} = \frac{1}{2}$ for $r \to \infty$. But it stays always larger than $\frac{1}{2}$ still making the barrier hard to pass for the opponents. It is known that in democratic countries a few percent difference between two candidates is seen as huge.

In parallel increasing cell sizes reduces the number of levels necessary to get to the stable fixed points.

6. Let us be practical

Given an initial support p_0, we want to calculate p_n the corresponding value for A support after n voting levels as a function of p_0. Accordingly, we expand the voting function $p_n = P_r(p_{n-1})$ around the unstable fixed point $p_{c,r}$,

$$p_n \approx p_{c,r} + (p_{n-1} - p_{c,r})\lambda_r , \tag{7}$$

where $\lambda_r \equiv \mathrm{d}P_r(p_n)/\mathrm{d}p_n|_{p_{c,r}}$ with $P_r(p_{c,r}) = p_{c,r}$. Rewriting the last equation as

$$p_n - p_{c,r} \approx (p_{n-1} - p_{c,r})\lambda_r , \tag{8}$$

we can then iterate the process to get

$$p_n - p_{c,r} \approx (p_0 - p_{c,r})\lambda_r^n , \tag{9}$$

from which we get

$$p_n \approx p_{c,r} + (p_0 - p_{c,r})\lambda_r^n . \tag{10}$$

From Eq. (10) two different critical numbers of levels n_c^l and n_c^L can be obtained. The first one corresponds to $p_{n_c^l} = 0$ and the second to $p_{n_c^L} = 1$. Putting $p_n = p_{n_c^l} = 0$ in Eq. (10) gives

$$n_c^l \approx \frac{1}{\ln \lambda_r} \ln \frac{p_{c,r}}{p_{c,r} - p_0} , \tag{11}$$

which is defined only for $p_0 < p_{c,r}$ showing that only below $p_{c,r}$ can the proportion decrease to zero. On the other hand, putting $p_n = p_{n_c^L} = 1$ in the same Eq. (10) gives

$$n_c^L \approx \frac{1}{\ln \lambda_r} \ln \frac{p_{c,r} - 1}{p_{c,r} - p_0} , \tag{12}$$

which is now defined only for $p_0 > p_{c,r}$ since $p_{c,r} < 1$, showing that only above $p_{c,r}$ can the proportion increase to one.

Though the above expansions are a priori valid only in the vicinity of $p_{c,r}$, they turn out to be rather good estimates even down to the two stable fixed point 0 and 1 rounding always to the larger figure while taking the integer part of Eqs. (11) and (12).

7. How to operate

However, once organizations are set, they are usually not modified. Therefore, within a given organization the number of hierarchical levels is a fixed quantity. Along this line, to make practical above analysis the question of "How many levels are needed to eliminate a tendency?" must be addressed to,

"Given n levels what is the minimum overall support

to get full power and for sure?" .

Or, alternatively, for the ruling group,

"Given n levels what is the critical overall support of the competing

tendency below which it always-eliminates totally?" ,

which means no worry what so ever about the current ruling policy.

To implement this operative question, we rewrite Eq. (10) as

$$p_0 = p_{c,r} + (p_n - p_{c,r})\lambda_r^{-n} . \tag{13}$$

It yields two critical thresholds. The first one is the disappearance threshold $p^n_{l,r}$ which gives the value of support under which A disappears for sure at the top level of the n-level hierarchy. It is given by Eq. (13) with $p_n = 0$,

$$p^n_{l,r} = p_{c,r}(1 - \lambda_r^{-n}) . \tag{14}$$

In parallel, $p_n = 1$ gives the second threshold $p^n_{L,r}$ above which A gets full and total power. From Eq. (13),

$$p^n_{L,r} = p^n_{l,r} + \lambda_r^{-n} . \tag{15}$$

There exists now a new regime for $p^n_{l,r} < p_0 < p^n_{L,r}$. In this, neither A disappears totally nor get to full power.

It is a coexistence region where some democracy is prevailing since results of the election process are only probabilistic. No tendency is sure of winning making alternate leadership a reality. However as seen from Eq. (15), this democratic region shrinks as a power law λ_r^{-n} of the number n of hierarchical levels. A small number of levels puts higher the threshold to a total reversal of power but simultaneously lowers the threshold for non-existence.

The above formulas are approximate since we have neglected corrections in the vicinity of the stable fixed points. However, they give the right quantitative behavior with $p^n_{l,r}$ fitting to the $n+1$ value and $p^n_{L,r}$ to the $n+2$ one. For more accurate formulas see [4,5].

To get a practical feeling of what Eqs. (14) and (15) mean, let us illustrate the case $r = 4$ where $\lambda = 1.64$ and $p_{c,4} = (1 + \sqrt{13})/6$. Considering 3, 4, 5, 6 and 7 level organizations, $p^n_{l,r}$ is equal to 0.59, 0.66, 0.70, 0.73 and 0.74, respectively. In parallel $p^n_{L,r}$ equals 0.82, 0.80, 0.79, 0.78 and 0.78. These series emphasize drastically the totalitarian character of the voting process.

8. Some vizualisation

To exhibit the strength of the phenomena, we show some snapshots of a numerical simulation [6]. The two A and B tendencies are represented, in white and black squares, respectively, with the bias in favor of the black ones, i.e., a tie 2–2 votes for a black square. A structure with 8 levels is shown. We can see in each picture, how a huge white square majority is self-eliminated. The written percentages are for the white representation at each level. The “Time” and “Generations” indicators should be discarded.

Fig.1 shows a case with 52.17% initial B (white) support. After 3 levels no more white square appears.

Fig. 2 shows a case with 68.62% initial B (white) support, far more than 50%. After 4 levels no more white square is found.

Fig. 3 shows a case with a huge 76.07% initial B (white) support. Now, 6 levels are needed to get a black square (B) elected. The initial B support (black) is only of 23.03%.

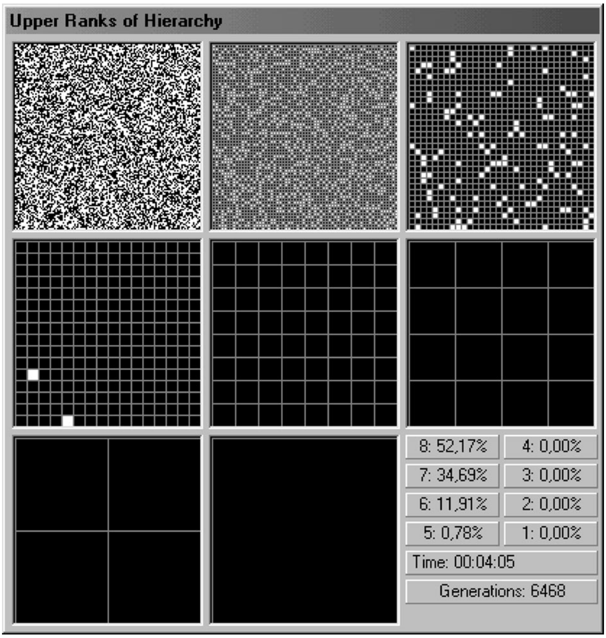

Fig. 1. A 8 level hierarchy for even groups of 4 persons. The two A and B tendencies are represented in white and black, respectively, with the bias in favor of the black squares, i.e., a tie 2–2 votes for a black square. Written percentages are for the white representation at each level. The “Time” and “Generations” indicators should be discarded. The initial white support is 52.17%.

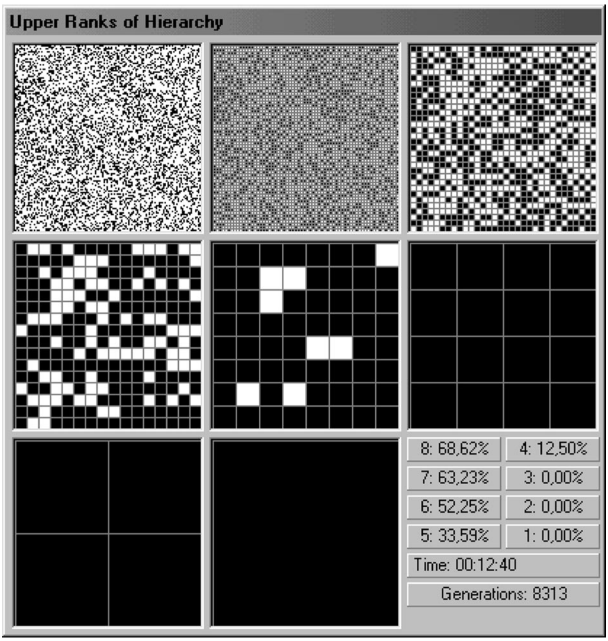

Fig. 2. The same as Fig. 1 with an initial white support of 68.62%.

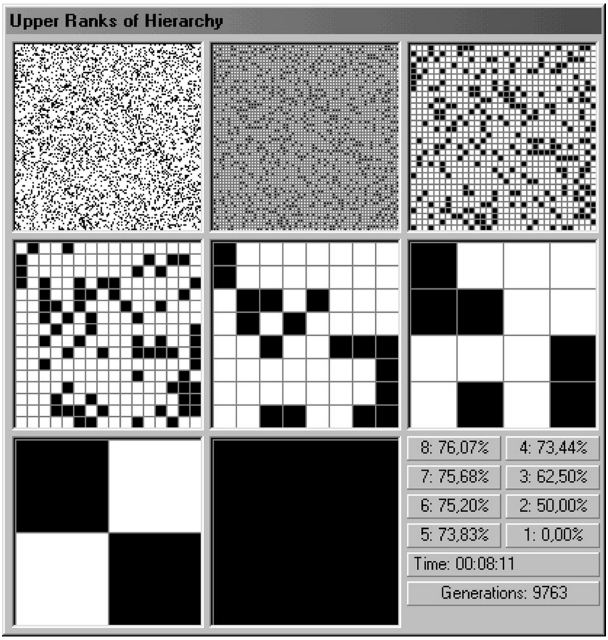

Fig. 3. The same as Fig. 1 with an initial white support of 76.07%.

Fig. 4 shows a case with 77.05% initial B (white) support. Finally, B (white) gets the presidency.

9. Extension to 3 competing groups

Up to now, we have treated very simple cases to single out some main trends produced by democratic voting, repeated over several hierarchical levels. In particular, the existence of critical thresholds was found to be instrumental in predicting the voting outcome. Moreover, these thresholds are not necessarily symmetric with respect to both competing tendencies. In the 4-cell case, it can be 0.77% for the opposition and 0.23% for the rulers. Such asymmetries are indeed always present one way or another.

In addition, most real situations involve more than two competing groups. Let us consider for instance the case of three competing groups A, B and C. Assuming a 3-cell case, the ABC configuration is now unsolved since it has no majority same as the case with the two tendency AABB 4-cell configuration. For the AABB case we made the bias in favor of the ruling group. For multi-group competition typically the bias results from parties agreement.

Usually, the two largest parties, say A and B are hostile to each other, while the smallest one C would compromise with either one of them. Along this line, the ABC configuration elects a C with a coalition with either A or B. In such a case, we need 2A or 2B to elect, respectively, an A or a B. Otherwise a C is elected.

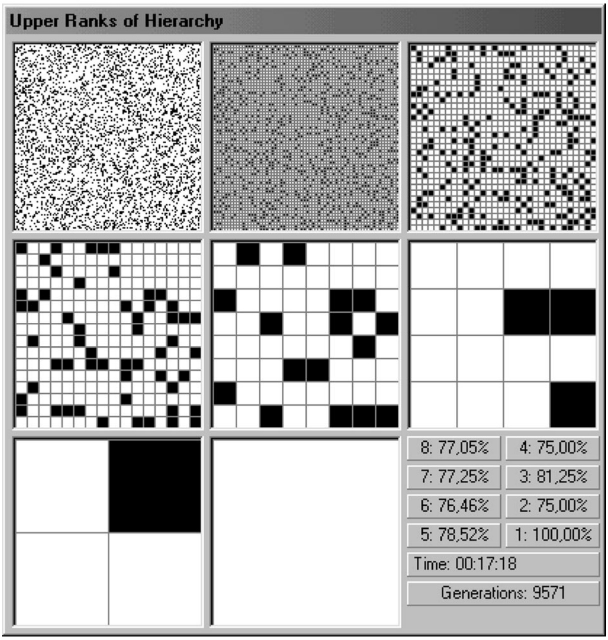

Fig. 4. The same as Fig. 1 with an initial white support of 77.05%.

Therefore, the elective functions for A and B are the same as for A in the two tendency 3-cell model. It means that the critical threshold to full power to A and B is 50%. In other words for initial support to both A and B, less than 50% C get full power. The required number of levels is obtained from the 3-cell formula.

Let us illustrate what happens for initial support, for instance of 39% for both A and B. The C are thus left with 22%. We will have the series for, respectively, A, B and C: 34%, 34%, 32% at the first level; 26%, 26%, 48% at the second level; 17%, 17%, 66% at the third level; 8%, 8%, 84% at the forth level; 2%, 02%, 96% at the fifth level; and 0%, 0%, 100% at the sixth level giving total power to the C minority within only 6 levels.

It is possible to generalize the present approach to as many groups as wanted. The analysis becomes much more heavy but the mean features of voting flows towards fixed point are preserved. Moreover power will go even more rarely to the largest groups as seen with the above ABC case.

10. Conclusion

To conclude we comment on some possible explanation to last century generalized auto-collapse of eastern European communist parties. Up to this historical and drastic event, communist parties seemed eternal. Once they collapsed many explanations were based on opportunistic change within the various organizations, within addition for the non-Russian ones, the end of the Soviet army threat.

Maybe part of the explanation is indeed related to our hierachical model. Communist organizations are based, at least in principle, on the concept of democratic centralism which is nothing else than a tree-like hierarchy. Suppose then that the critical threshold to power was of the order of 80% like in our 4-cell case. We could than consider that the internal opposition to the orthodox leadership did grew a lot and massively over several decades to eventually reach and pass the critical threshold. Once the threshold was passed, we had the associated sudden outrise of the internal opposition. Therefore, the at once collapse of eastern European communist parties would have been the result of a very long and solid phenomena inside the communist parties. Such an explanation does not oppose additional constraints but emphasizes some trends of the internal mechanism within these organizations.

At this stage it is of importance to stress that modeling social and political phenomena is not aimed to state an absolute truth but instead to single out some basic trends within a very complex situation. All models can be put at stake, and changed eventually.

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