Combinatorial optimization, analysis of algorithms … … and statistical physics

Rémi Monasson — *Ecole Normale Supérieure, Paris*
Univ. Louis Pasteur, Strasbourg

Collaborators :

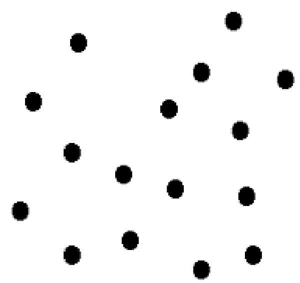
Giulio Biroli — *Rutgers + Saclay*
Simona Cocco — *Chicago + Strasbourg*
Scott Kirkpatrick — *IBM + H.U. Jerusalem*
Bart Selman — *Cornell*
Martin Weigt — *Göttingen*
Riccardo Zecchina — *ICTP, Trieste*

Lyon, December 02

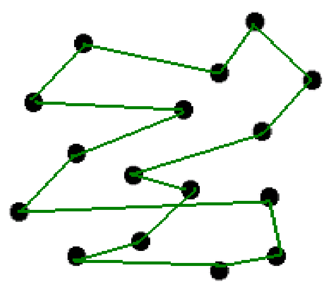
Two examples of optimization problems:

Traveling salesman

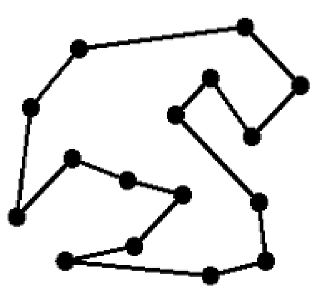
N cities
+ distances
between cities



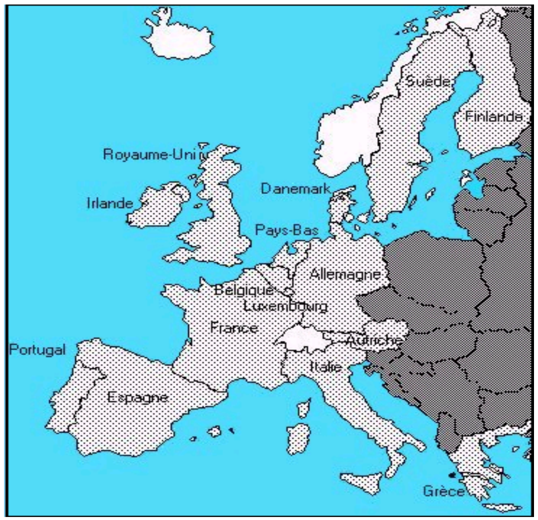
Tour =
visit of every city
once and return to
initial city



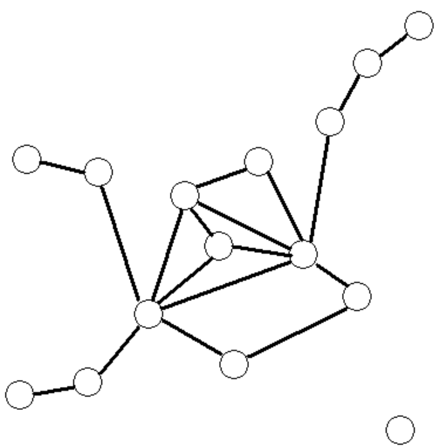
Shortest tour?



Graph coloring



European
union



non planar
graphs?

Relationship with statistical physics *I. Equilibrium*

- minimum of a cost function = ground state of a classical Hamiltonian
  (quasi-solutions = excited states ...)

- distribution of instances = quenched disorder in interactions

- list of problems:
  - traveling salesman (non Euclidean)
  - graph partitioning
  - optimal matching
  - neural networks

From 1984 to 1990
Anderson, De Dominicis, Fu, Kirkpatrick, Krauth, Mézard, Orland, Parisi, Sherrington, Sourlas, Toulouse …

Hopfield, Amit, Gutfreund, Sompolinsky, Gardner …

- extremal distribution of correlated variables?

Replicas …

Example : *Edwards-Anderson model on square lattice:*

N spins S_i
2 N random couplings J_{ij}

$$H[J,S] = -\sum_{\langle i,j \rangle} J_{ij}\, S_i\, S_j$$

2^N correlated energy levels!

low temperature :
distribution of minimum,
quasi-minima ...

Relationship with statistical physics *II. Dynamics*

- Algorithm = sequence of computation rules º dynamical evolution of the instance

Example : sorting

L = 6, 1, 18, 7, 10, 2, 3, 15
1; 6, 18, 7, 10, 2, 3, 15
1, 2; 6, 18, 7, 10, 3, 15 …….

- Analysis = calculation of the running time

Knuth ’60

Nb. of comparisons to find min of k numbers = k-1
Nb. of comparisons = (N-1)+(N-2)+…+1 = N(N-1)/2

- Different classes of optimization algorithms:
  - local search
    similarity with physical dynamics (Monte Carlo, simulated annealing, … cf. vitreous transition)
    incomplete (cannot prove the absence of solution)
  - global search
    no physical origin (designed by computer scientists to be **complete**)
    non Markovian (memory effects), *non local* (jumps in phase space)

What is the Satisfiability problem?

CAN'T GET NO SATISFACTION

Brian Hayes

American Scientist,
Volume 85, Number 2,
Pages 108-112,
March-April 1997.

http://www.amsci.org

You are chief of protocol for the embassy ball. The crown prince instructs you either to invite Peru or to exclude Qatar. The queen asks you to invite either Qatar or Romania or both. The king, in a spiteful mood, wants to snub either Romania or Peru or both. Is there a guest list that will satisfy the whims of the entire royal family?

This contrived little puzzle is an instance of a problem that lies near the root of theoretical computer science. It is called the satisfiability problem, or SAT, and it was the first member of the notorious class known as NP-complete problems. These are computational tasks that seem intrinsically hard, but after 25 years of effort no one has yet proved that they are necessarily difficult. It remains possible (though unlikely) that we are simply attacking them by clumsy methods, and if we could dream up a clever algorithm they would all turn out to be easy. Settling this question is the most conspicuous open challenge in the theory of computation.

SAT also has practical importance. In artificial intelligence various methods of logical deduction and theorem-proving are related to SAT. And similar issues arise in computer software for one phase almost all the propositions can be satisfied, but in another phase almost none can. The cases that are hardest to resolve lie near the transition between these regimes.

The connection between SAT and the physics of phase transitions strikes me as a surprising one—a classic who'd-have-thunk-it result. We are accustomed to using mathematics as a tool for interpreting the physical world, but not the other way around. And yet the phase-transition model of SAT works so well that it cannot be a mere metaphor, much less a coincidence.

P and NP

The problem of the embassy ball is small enough to be solved by even the most plodding of methods. The problem is represented by the formula:

$$(p \text{ OR } \sim q) \text{ AND } (q \text{ OR } r) \text{ AND } (\sim r \text{ OR } \sim p)$$

Here p, q and r are Boolean variables, whose only possible values are *true* or *false*. The ~ symbol indicates negation, so that $\sim p$ is read "not p." The logical OR operation is defined so that (p OR q) has the value *true* if either p or q is *true*, whereas (p AND q) evaluates to *true* only if both p and q are *true*.

p = true if
Peru ambassador
is invited,
false otherwise

Satisfiability of (random) Boolean constraints

(w or NOT x or y)
and
(NOT w or x or z)
and
(x or y or NOT z)

?

3-SAT NP-complete
(and >3)

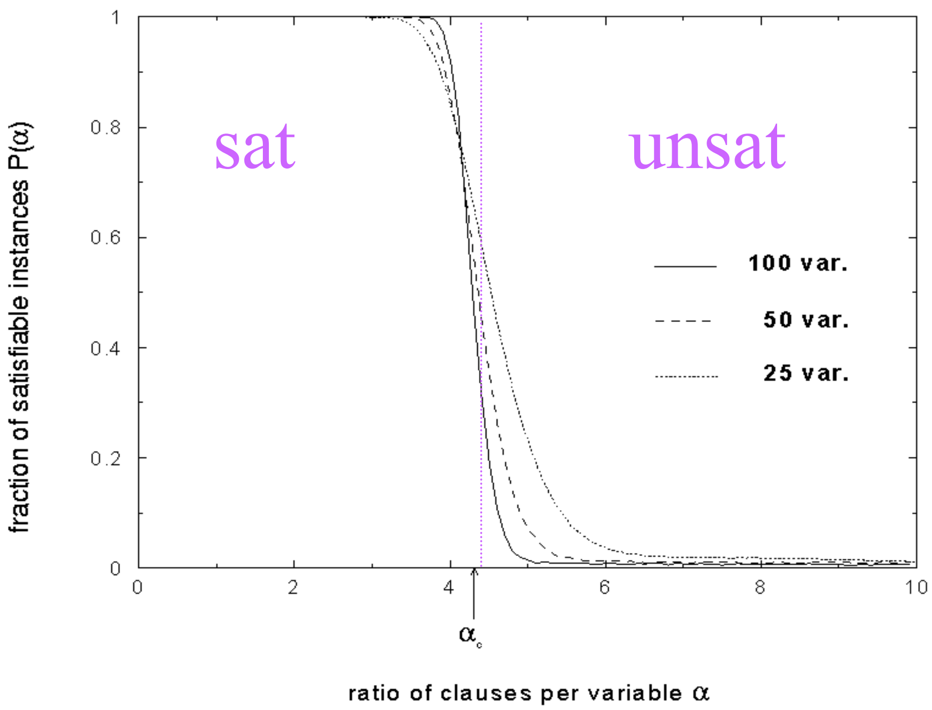
2-SAT P

$$\alpha = \frac{\text{nb. of clauses}}{\text{nb. of variables}}$$

Mitchell, Selman, Levesque '92
Crawford, Auton '93
Gent, Walsh '94

Chao, Franco '86, '90
Chvatal, Szmeredi '88

The Phase transition of 3-SAT



$\alpha_C \approx 4.3$

phase transition!

Rigorous results

- $\alpha_C > 3.26$
- $\alpha_C < 4.51$
- transition region width $\rightarrow 0$

SAT,
a disordered
spin system
(*at zero temperature*)

Dictionary

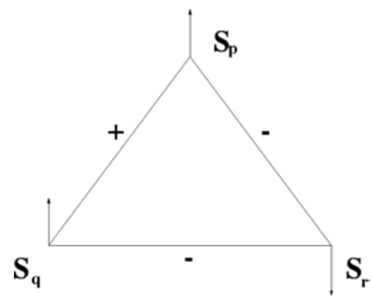
K-SAT	statistical physics
Boolean variable x = True , False	*Ising spin* $s_x = +1, -1$
Clauses	*Couplings and fields acting on spins*
Number of clauses violated by a logical configuration	*Energy E of the spins configuration*
some examples:	
2 SAT x or $\bar{y}$	$E = \frac{1}{4}(1-s_x)(1+s_y)$
$(x$ or $\bar{y})$ and $(\bar{x}$ or $z)$	$E = \frac{1}{4}(1-s_x)(1+s_y) + \frac{1}{4}(1+s_x)(1-s_y)$
3 SAT x or $\bar{y}$ or z	$E = \frac{1}{8}(1-s_x)(1+s_y)(1-s_z)$
Minimal number of violated clauses	*Ground state energy*
The problem is satisfiable	*Ground state energy* $= 0$
The problem is not satisfiable	*Ground state energy* > 0

Spin glasses on random graphs

* Ball problem

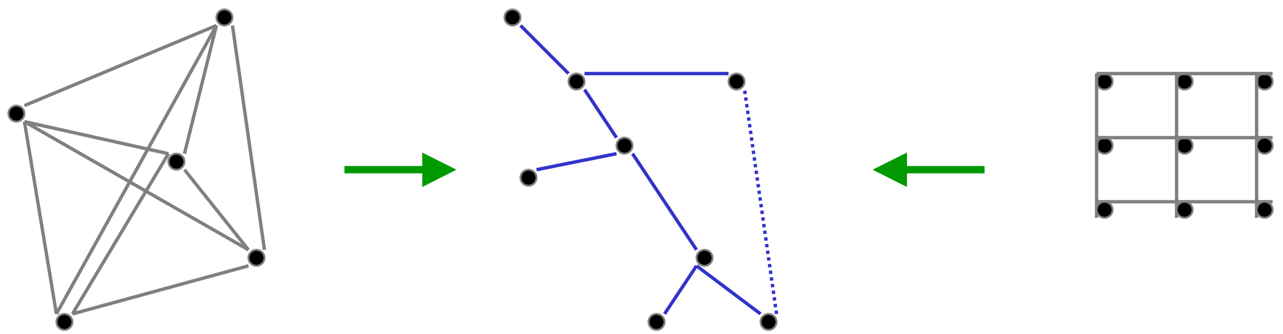
$$\left(p \vee \bar{q}\right) \wedge \left(q \vee r\right) \wedge \left(\bar{r} \vee \bar{p}\right)$$

$$E = \frac{1}{4}\left(-S_p S_q + S_q S_r + S_r S_p + 3\right)$$



frustration ...

*



$|J| = N^{-1/2}$
no geometry

$|J| = 1$
infinite D geometry

$|J| = 1$
finite D geometry

* Multi-spins interactions (K-SAT = K-body)