Equilibrium Path Selection in Socioeconomic Models

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Prepared for the WSCS

ÉNS Lyon

December 12, 2002

1 Introduction

- A key characteristic of socioeconomic models is that expectations about future endogenous variables impact on the level of current endogenous variables.
- Therefore, even simple models are likely to have more than one candidate for equilibrium path.
- The question is then to know how to select "the" equilibrium path, which economists think to be the relevant one
- Being able to find the relevant path is important because the model can then be used to make predictions, counter-factual experiments and forecasts.

- In this lecture, I will review economic models with simple (trivial) mathematical structure, but with no-trivial equilibrium selection problems.

2 Simple Linear One-Dimension Case

2.1 The "Model"

- We model the variable y as

$$y_t = aE[y_{t+1}|t] + cx_t \tag{1}$$

x is an exogenous real variable, a and c are real parameters, $E[y_{t+1}|t]$ is the expectation of y_{t+1} held at time t.

- Here is a specificity of socioeconomic models: the current value of y depends on its future expected value (price of a house, "pleasure" attached to a piece of clothing,...)

- We need a second equation, which tells us how do agents form their expectations.

- Many possibilities: adaptative expectations, learning, static.

- As a benchmark, we study Rational Expectations

Rational Expectations : We assume that agents have expectations equal to the mathematical expectation of y_{t+1} based on information available at time t.

- We make two further assumptions:

1. Individuals know the model (equation (1))
2. All individuals have the same information

Therefore, $E[y_{t+1}|t]$ is defined as a conditional expectation

$$E[y_{t+1}|t] = E[y_{t+1}|I_t] \tag{2}$$

, where $I_t = \{y_{t-i}, x_{t-i}, \quad i = 0, ..., \infty\}$

- In words, those assumptions imply that agents do not make systematic forecast errors. Their predictions are unbiased. They do not forecast to be wrong on average.

- We aim at solving the model composed of (1) and (2)

An example : Stock prices and arbitrage. Think of an investor that can put his money on 2 assets: a riskless saving account that serves an interest rate $r > 0$ and a stock, with before-dividend price p_t and dividend d_t.

- Assume that markets work without frictions (I can buy or sell without restrictions) and that investors care only about the expected return of an asset.

- The rate of return of the saving account is r

- The rate of return of the stock is $\frac{(E_t[p_{t+1}|I_t]+d_t)-p_t}{p_t}$.

- we can expect the following *arbitrage* condition to hold:

$$\frac{(E_t[p_{t+1}|I_t] + d_t) - p_t}{p_t} = r$$

or equivalently

$$p_t = aE_t[p_{t+1}|I_t] + ad_t \qquad (\star)$$

with $a = 1/(1+r) < 1$.

- $(\star)$ is an equation of the type (1).

2.2 Solutions to equation (1) when $|a| < 1$

2.2.1 The "Fundamental" Solution

• Here we use the *law of iterated expectations*, which states that, if ω is a subset of the information set Ω, then

$$E\left[E[x|\omega]|\Omega\right] = E[x|\omega]$$

• In our case, as $I_t \subset I_{t+1} \subset \cdots \subset I_{t+j} \cdots$, we have

$$E\left[E[x|I_{t+1}]|I_t\right] = E[x|I_t]$$

In words, today's expectation of the next period's expectation of the variable x is the same as today's expectation of x.

- Let us do some forward substitution in (1): write (1) at $t+1$:

$$y_{t+1} = aE[y_{t+2}|I_{t+1}] + cx_{t+1}$$

Take expectations conditional on I_t:

$$E[y_{t+1}|I_t] = aE[E[y_{t+2}|I_{t+1}]|I_t] + cE[x_{t+1}|I_{t+1}]$$

and using the law of iterated expectations:

$$E[y_{t+1}|I_t] = aE[y_{t+2}|I_t] + cE[x_{t+1}|I_{t+1}]$$

Replace in (1)

$$y_t = a^2E[y_{t+2}|I_t] + acE[x_{t+1}|I_{t+1}] + cx_t$$

- Solving recursively up to time T:

$$y_t = c \sum_{i=0}^{T} a^i E[x_{t+1}|I_t] + a^{T+1} E[y_{t+T+1}|I_t]$$

- We shall assume that the first sum converges as T goes to infinity (the dividend growth rate is smaller than the interest rate)

- Then, if

$$\lim_{T \to \infty} a^{T+1} E[y_{t+T+1}|I_t] = 0 \tag{3}$$

the following is a solution

$$y_t^\star = c \sum_{i=0}^{\infty} a^i E[x_{t+i}|I_t] \tag{4}$$

Note that (4) satisfies (3), and is indeed a solution to (1).

- We refer to this solution as the *fundamental* one. This terminology comes from the asset pricing example.

Back to the stock price example : First, let's recall that with positive discounting, we are in the case $|a| < 1$

In this case,

$$p_t^\star = \frac{1}{1+r} \sum_{i=0}^{\infty} \frac{E[d_{t+i}|I_t]}{(1+r)^i}$$

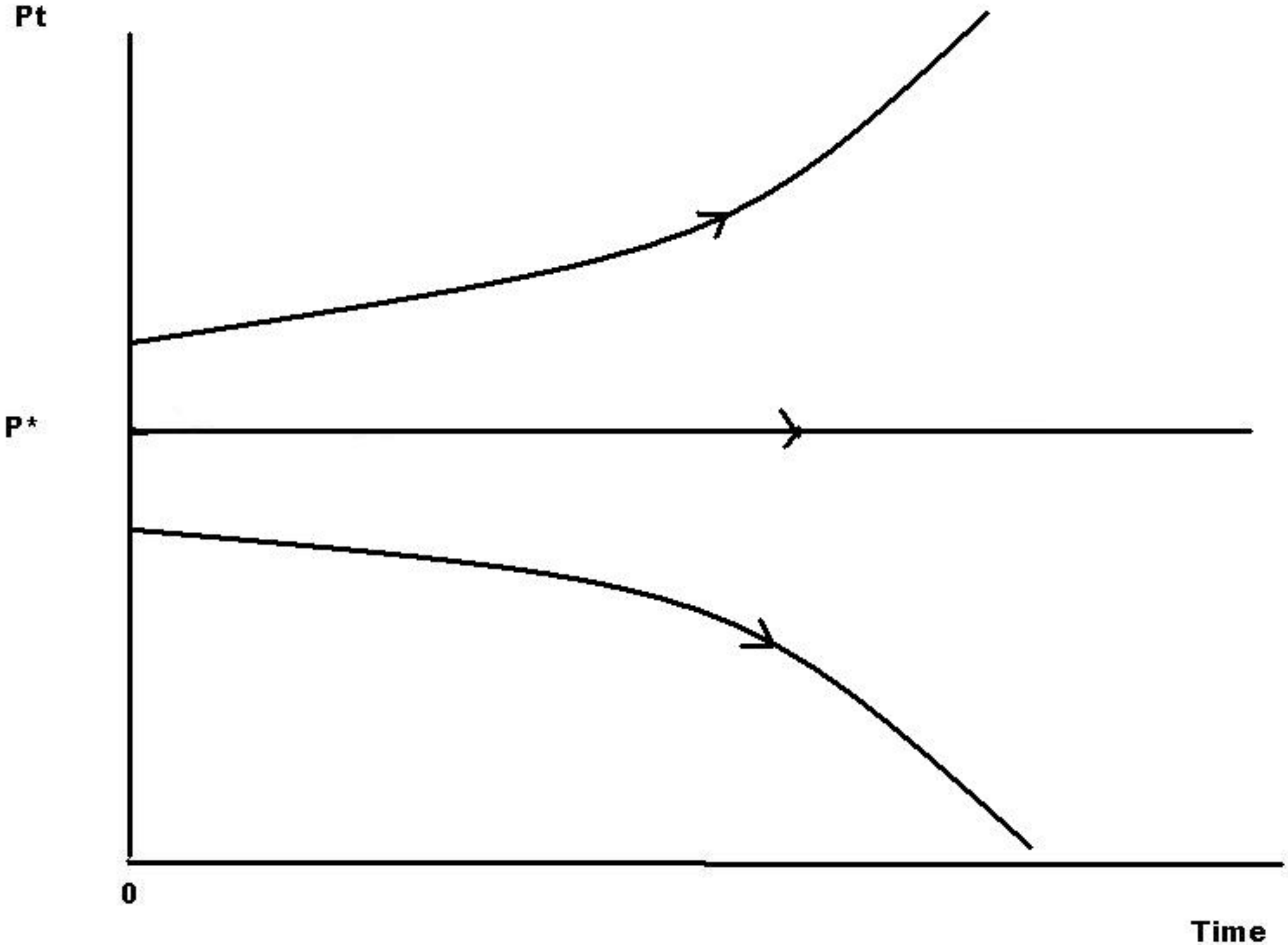
• This has a straightforward economic meaning: the fundamental value (price) of an asset is equal to the discounted flow of its expected dividend. It is pinned down by the no explosion assumption (3).

• Note that we have a difference equation of order one with no initial condition $\rightsquigarrow$ the terminal condition (3) is used to pinned down the fundamental solution.

- Comment: $p_t^\star$ is a *forward* variable. It does not depend on what happened yesterday, but only on expectations about future dividends.

- Assume that $d_t = d$. Then the fundamental value of the stock is the constant

$$p_t^\star = d \sum_{i=0}^{\infty} \left(\frac{1}{1+r} \right)^{i+1} = \frac{d}{r}$$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- The fundamental solution might depend on current state of the economy, but only because the current sate is conveying some information about the future.

- If for example d_t follows the stationary linear first-order stochastic process:

$$d_t - \overline{d} = \rho(d_{t-1} - \overline{d}) + \varepsilon_t$$

where $|\rho| < 1$, $\varepsilon_t \in I_t$ and $E[\varepsilon_t | I_{t-1}] = 0$. It is easy to check that

$$E[d_{t+i} | I_t] = \overline{d} + \rho^i (d_t - \overline{d})$$

so that the fundamental solution is

$$p_t^\star = \overline{p^\star} + \frac{1}{1 + r - \rho}(d_t - \overline{d})$$

with $\overline{p^\star} = \frac{1+r}{r}$

- The forward looking aspect of the solution has very stark implications that we might not find elsewhere than in socioeconomic systems (but I do not know a lot about *elsewhere*): the economy can react to *news*.

News : Assume that we are in the case where d is constant, so that

$$p_t = \frac{d}{r} = p^\star$$

- In period t_0, agents receive the following information : from period $T > t_0$ to ∞, the dividend will be $\widehat{d} > d$.

- What will be the behavior of p_t?

- Before t_0, $p_t = p^\star$

- After T, we are in the same case than before t_0, except that $d = \widehat{d}$, so that

$$p_t = (1+r)\frac{\widehat{d}}{r} = \widehat{p^\star}$$

- Let us now solve backward, starting from period $T-1$. By arbitrage condition, we should have in period $T-1$:

$$\frac{(p_T + d) - p_{T-1}}{p_{T-1}} = r$$

so that

$$p_{T-1} = \frac{d}{1+r} + \frac{p_T}{1+r}$$

Given that $p_T = \frac{\widehat{d}}{r}$, we have

$$p_{T-1} = \frac{d}{1+r} + \frac{\widehat{d}}{r(1+r)}$$

that we can write as

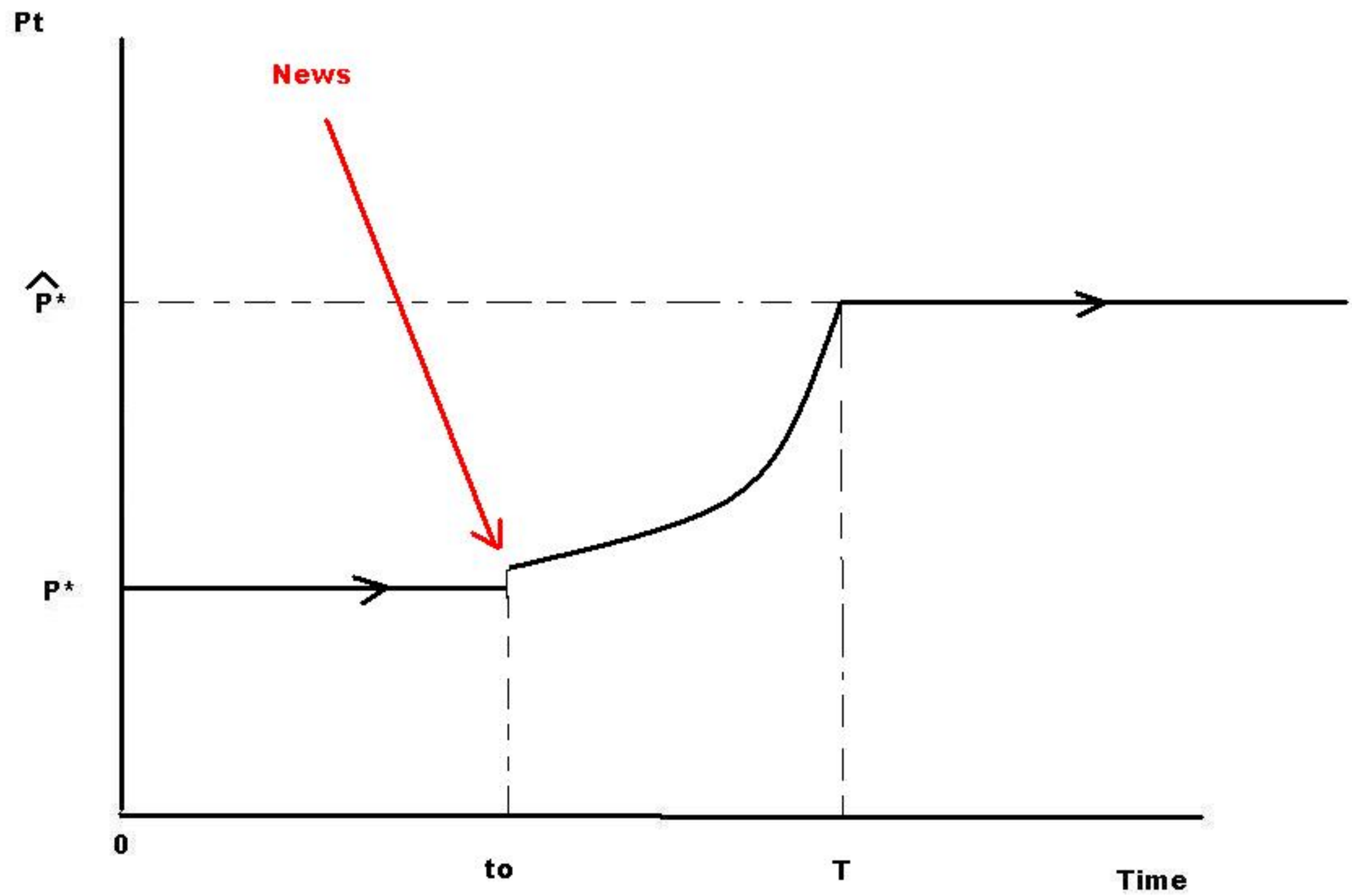
$$p_{T-1} = \frac{d}{r} + \left(\frac{d}{1+r} - \frac{d}{r}\right) + \frac{\widehat{d}}{r(1+r)}$$

or

$$p_{T-1} = p^\star + \frac{1}{1+r}\left(\widehat{p^\star} - p^\star\right)$$

- We can now compute p_{T-2}, then p_{T-3}, ...and we get for $t_0 \leq t < T$:

$$p_t = p^\star + \left(\frac{1}{1+r}\right)^{T-t}\left(\widehat{p^\star} - p^\star\right)$$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2.2.2 Spanning the all set of solutions : bubbles

- (4) is obviously not the only solution.
- Let us relax now (3)
- Any solution of the model can now be written as

$$y_t = y_t^\star + b_t$$

when some restrictions have to be imposed on b_t so that y is a solution of (1).

- Plugging the expression of y_t in (1):

$$y_t^\star + b_t = aE[y_{t+1}^\star | I_t] + aE[b_{t+1}|I_t] + cx_t$$

and using the definition of the fundamental solution, this reduces to

$$b_t = aE[b_{t+1}|I_t] \tag{5}$$

or

$$E[b_{t+1}|I_t] = \frac{1}{a} b_t$$

- for any b that satisfies (5), $y_t = y_t^\star + b_t$ is a solution of (1).
- Since $|a| < 1$, b_t explodes in expected value:

$$\lim_{i \to \infty} E[b_{t+1}|I_t])a^{-i}b_t = \begin{cases} +\infty & \text{if} \quad b_t > 0 \\ -\infty & \text{if} \quad b_t < 0 \end{cases} \tag{6}$$

- b_t is called a *bubble*

The Stock Price Example :

• Obvious reference to the asset pricing example: the difference between the actual price and the fundamental value is a bubble.

• It is a rational bubble: I am willing to buy the asset for more that its fundamental value because I will sell it for even more.

• Note that there are no initial nor terminal condition for the bubble $\rightsquigarrow$ $b_t = b_0(1+r)^t$ is a rational bubble for any arbitrary b_0.

• If there were a terminal condition stating for example that the bubble will be constant after some date T, then no one will want to buy the asset at period T at a price $p_T^\star + b_T$

• Why? Because, by arbitrage condition, I am willing to buy at price $p^{\star}_{T} + b_T$ only if I can sell at a price at least equal $p^{\star}_{T+1} + (1+r)b_T$, so that $b_{T+1} = (1+r)b_T$, which violates the assumption of no more growing bubble.

• Then if it so, no one will want to buy in period $T-2$ at price $p^{\star}_{t-1} + b_{T-2}$, ... etc

$\rightsquigarrow$ only the fundamental path would be a solution.

• One could doubt of the relevance of a solution in which the price of the asset goes to infinity in a deterministic way.

• But we can construct other bubbly solutions where the bubble process is more complex than $b_t = b_0(1+r)^t$

Example of a Bursting Bubble : Assume that b follows the following process

$$b_{t+1} = \begin{cases} \frac{1+r}{q} b_t + \varepsilon_{t+1} & \text{with probability} \quad q \\ \varepsilon_{t+1} & \text{with probability} \quad 1-q \end{cases}$$

and $E[\varepsilon_{t+1}|I_t] = 0$

- It can be easily checked that this process satisfies (5), so that $p_t^\star + b_t$ is indeed a solution of (1).

- In this solution, the fact that the bubble can burst with probability q is compensated by a higher return ($\frac{1+r}{q}$=.

- The noise ε permits news bubbles to form after a crash.

2.3 Solutions when $|a| > 1$: Indeterminacy

$$y_t = aE[y_{t+1}|t] + cx_t \tag{1}$$

- Let's recap: In the case $|a| < 1$, there is a infinity of bubbly solutions. But if we want (have good reasons, that might come from the model, as we shall see later) to impose a non explosive solution, then we are left only one solution, the fundamental one.

- If $|a| > 1$, the fundamental solution is no longer well defined, as the sum in (4) is unlikely to converge in general

$$y_t^\star = c \sum_{i=0}^{\infty} a^i E[x_{t+i}|I_t] \tag{4}$$

- In the case, $x_t = 1$, it is easy to check that the set of solutions is given by

$$y_t = \frac{c}{1-a} + b_t$$

where

$$b_t = \frac{1}{a} b_{t-1} + \varepsilon_t \qquad , \qquad E[\varepsilon_t | I_{t-1}] = 0 \tag{7}$$

- The bubbles are now all shrinking bubbles

- In the case where $\varepsilon \equiv 0$, y converges to $\frac{c}{1-a}$ for any initial value of $y_0 \rightsquigarrow$ hard to have good stories to select a particular path.

- We cannot use rules-that-exclude-explosive-path as selection devices $\rightsquigarrow$ need some selection device. Without such a device, the path is indeterminate.

- We shall come back on this issue later.

2.4 A Key Assumption: Rational Expectations

- The forward-looking aspect of the dynamics is really the consequence of the RE assumption.

- We could assume other type of expectations.

- Or we could assume that agents learn in real-time about the model.

2.4.1 Adaptative expectations

- Assume for example that expectations are adaptative:

$$E[y_{t+1}|t] = y_t + \lambda\left(E[y_t|t-1] - y_t\right)$$

with $\lambda \in [0,1]$

- In such a case, the model becomes fully backward. For a given y_0, there is one and only one equilibrium path

- but

1. Agents male systematic forecast errors if they know the model (the can predict that there expectation is biased),

2. News have no impact on the dynamics of the system.

2.4.2 Learning

- We can assume that agents know the parametric form of the model (equation (1)), but not the true value of parameters a and c.

- Then, they can start with some priors about a and c, say $\widehat{a}_0$ and $\widehat{c}_0$, act in the economy one period, observe the outcome and update their believes about a and b, $\widehat{a}_1$ and $\widehat{c}_1$.

- If x_t is a known random process, they can use statistical theory to update *optimally* (in some sense) their believes.

- Under some regularity conditions (typically $|a| < 1$, x stationary), they will eventually learn the model, and will behave as in the model with rational expectations.

3 A Two-Dimension Economic Example

3.1 The Model

• The model is known as the optimal growth model.

• It is assumed that there is a representative agent (strong assumptions, that closes many doors), and that there is no uncertainty

• The problem is the following:

- The agent enters period t with a given amount of potatoes plants (capital K_t).

- In any period, potatoes plants produce $Y_t = F(K_t)$ (production) (F increasing and concave)

- Production can be consumed (C_t), and give utility $U(C_t)$ (U increasing and concave) or planted ("Invested", K_{t+1})

- The economy start with an initial number of potatoes plants K_0

- Preference are intertemporal, with some preference for the present :

$$W(K_0) = \sum_{t=0}^{\infty} \beta^t U(C_t)$$

• The representative agent faces the following intertemporal problem:

$$\max_{K_t, C_t} \sum_{t=0}^{\infty} \beta^t U(C_t)$$

subject to

$$K_{t+1} + C_t \leq F(K_t)$$

$$K_0 \quad \text{given}$$

3.2 Solution

- The problem is well behaved, and FOC are N&S. They are given by, for all $t \geq 1$,

$$U'(C_t) = \beta U'(C_{t+1}) F'(K_{t+1}) \tag{8}$$

$$K_{t+1} + C_t = F(K_t) \tag{9}$$

$$\lim_{i \to \infty} \beta^{t+i+1} U'(C_{t+i+1}) K_{t+i+1} = 0 \tag{10}$$

$$K_0 \quad \text{given} \tag{11}$$

Comment : The problem is about how much to consume today, how much to consume tomorrow, given that consumption can be transferred intertemporally though savings (planting). A key feature is that utility is concave, so that the agent wants to smooth its intertemporal stream of potatoes consumption.

- (9) is a resource (feasibility) constraint

- (8) is intertemporal arbitrage equation. If I decide to consume one less potato (one ε less potatoes), I will loose $U'(C_t)$ utility today. At the solution, this should be exactly offset by the extra utility coming from the increase in savings. This extra potato will be planted, will give $F'(K_{t+1})$ more potatoes tomorrow. As one more potato consumed brings the utility $\beta U'(C_{t+1})$, optimality requires (9).

- (10) can be understood as follows. If the world was to end at period T, it would be optimal for the agent not to save at period T. The value of K_T in terms of the intertemporal utility would be zero. This value is $\beta^T U'(C_{T+1})K_{T+1}$. Now we let T goes to infinity, and we obtain (10).

3.3 The Model's Dynamics

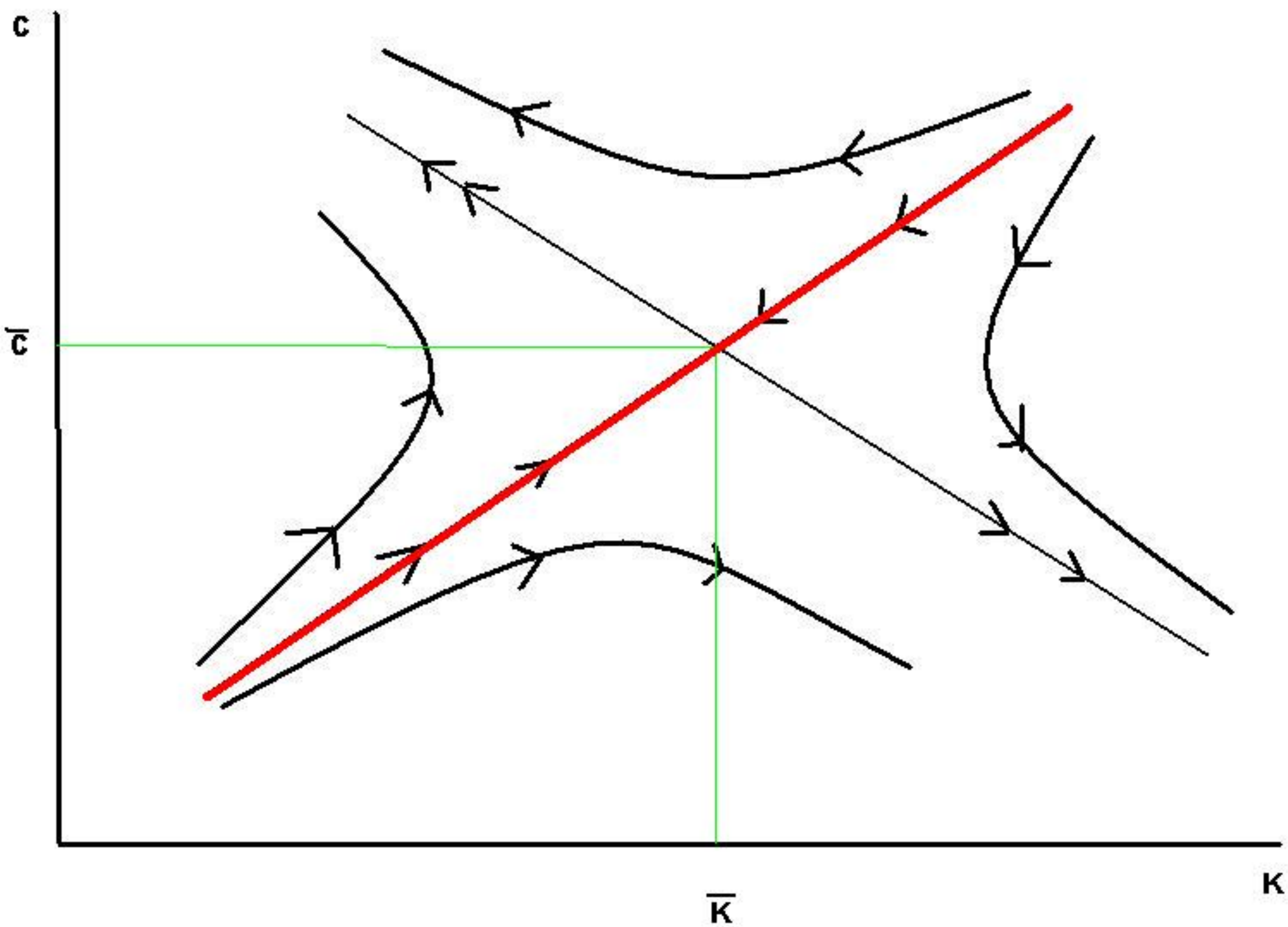
- It can be shown easily that the model has a unique steady state (note: we keep the term *"equilibrium"* for the solution path $\rightsquigarrow$ economic equilibria do evolve with time!).

- Let us consider the linearized version of (8) and (9), around its steady state

- The equilibrium dynamics of the model can be written as

$$\begin{pmatrix} C_{t+1} - \overline{C} \\ K_{t+1} - \overline{K} \end{pmatrix} = A \times \begin{pmatrix} C_t - \overline{C} \\ K_t - \overline{K} \end{pmatrix}$$

- In such an economy, it can be shown that A has one eigenvalue with modulus inside the unit interval and one outside the unit interval $\rightsquigarrow$ for an arbitrary initial condition (C_0, K_0), the economy explodes, and do not converge to its steady state.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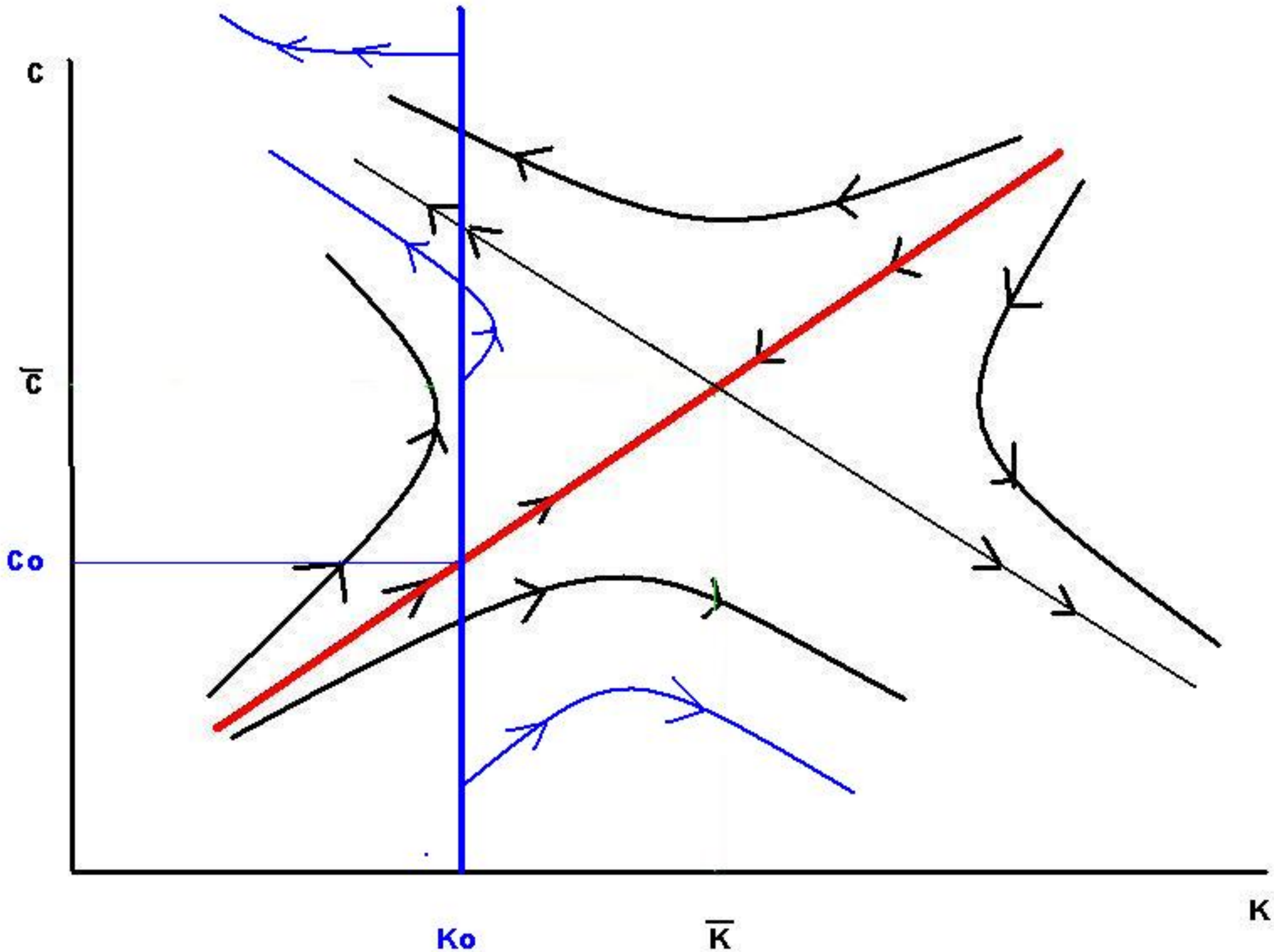
• Is that a problem? No, because the model does not give us 2 but one initial condition, K_0. Initial consumption C_0 is a jump variable, that is pinned down by a terminal condition (the transversality condition), which is satisfied only if the economy converges towards its steady state.

• Therefore, the equilibrium path is *determinate* for economists: there is only one path that satisfies the system (8) to (11), and this path is selected by choosing, fir a given K_0, the unique value of C_0 that puts the economy on the saddle-path.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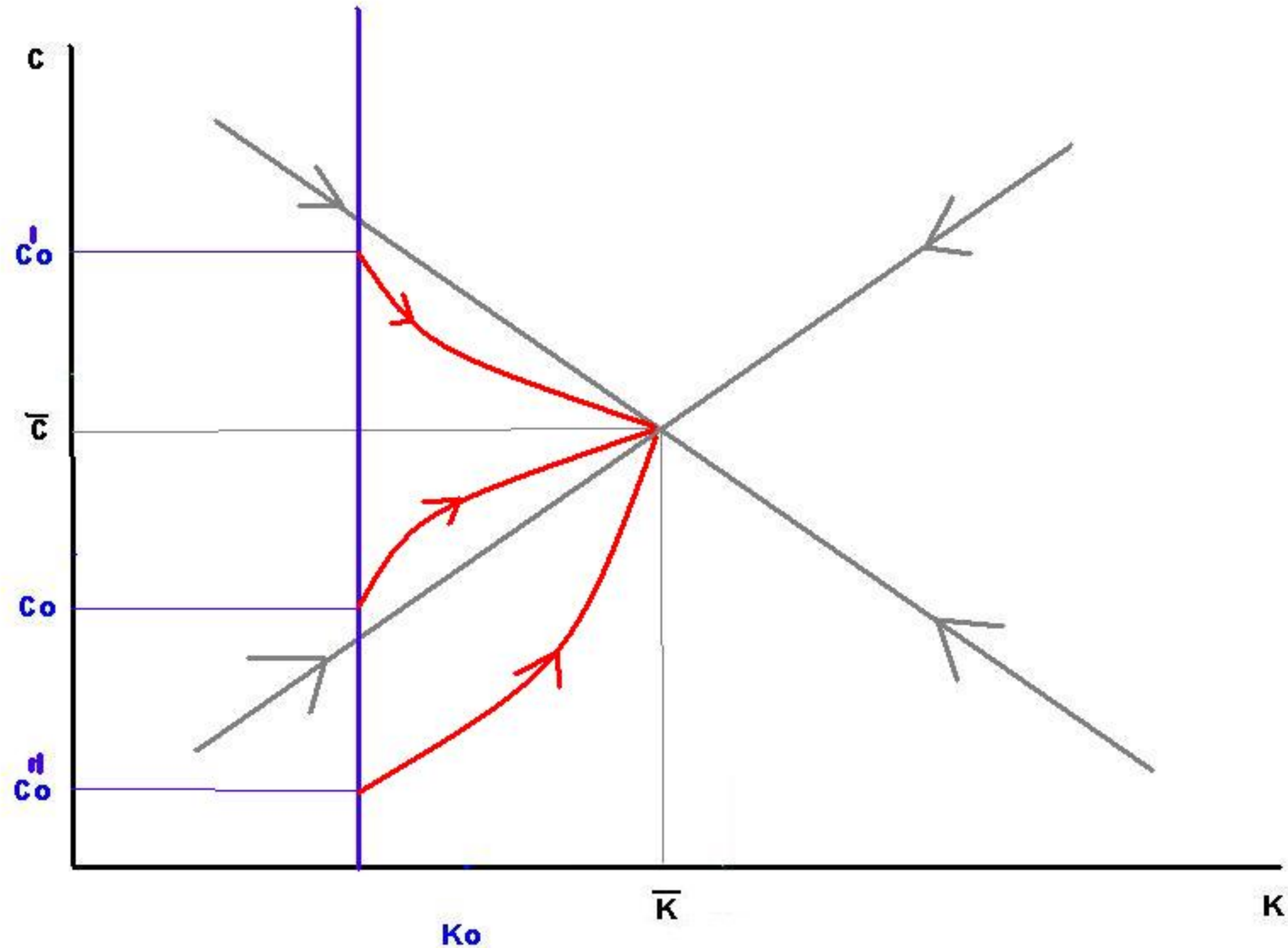
3.4 Indeterminacy

- It is possible to design some variations of this model (typically by adding some non convexities in the production function), for which steady state is (locally) stable (the two eigenvalues of A are inside the unit circle).

- Such a situation might be the "good" one for physicians (the system always converges to its rest point). For economists, the equilibrium path becomes indeterminate $\rightsquigarrow$ There is not enough structure to pin down C_0.

- There is a continuum of equilibrium path $\rightsquigarrow$ no predictive power of the theory.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- Any belief about what should be the equilibrium path (meaning how C_0 should be chosen) is indeed an equilibrium path $\rightsquigarrow$ self-fulfilling prophecies
- What could be the equilibrium selection device?

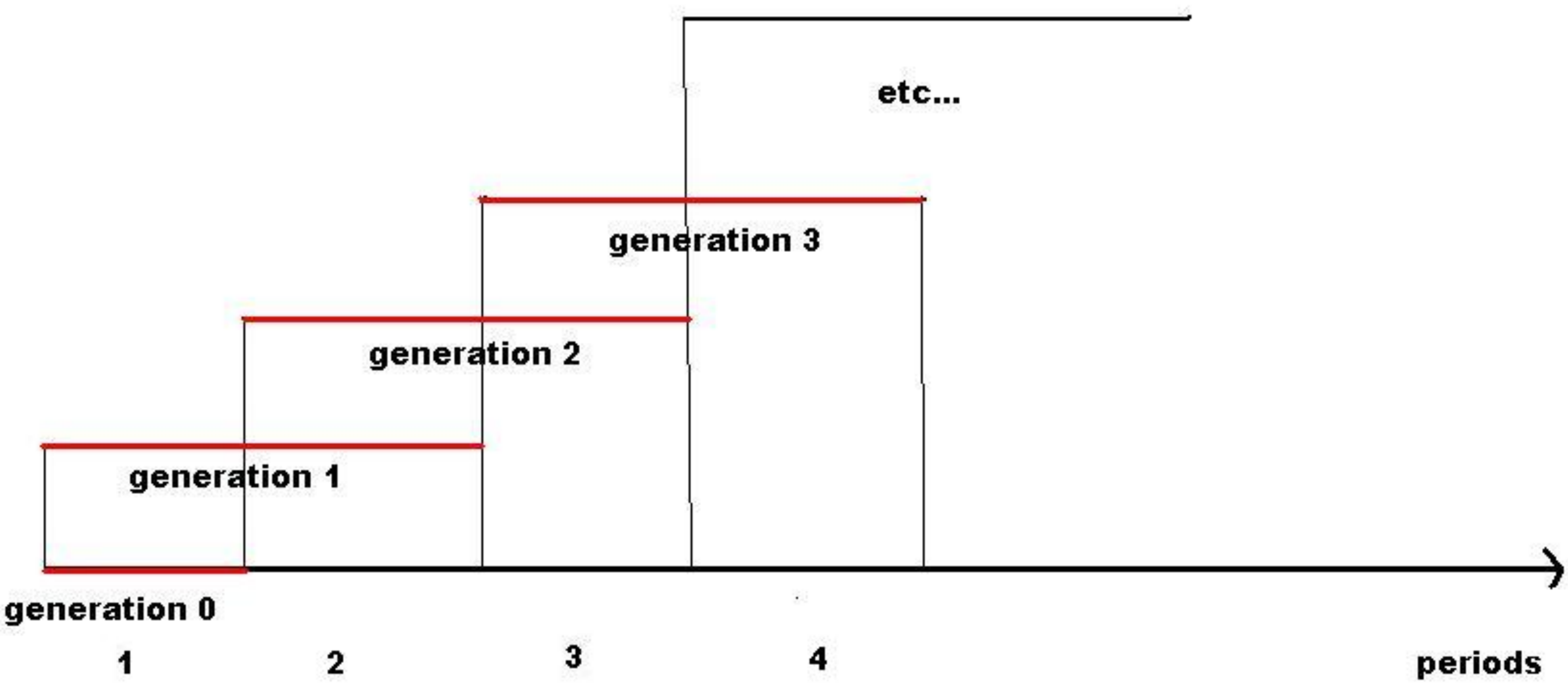
1. Sunspots : believes that some extrinsic variables are correlated with economic outcome (compatible with rational expectations)
2. Learning : Assume that agent do not know perfectly the model, and learn about it using past observations $\rightsquigarrow$ the economy becomes fully backward.

4 Non-Linear Dynamics, Indeterminacy and Sunspots

• I present here a simple deterministic model in which one can construct stochastic sunspot equilibria.

4.1 The Economy

• The economy is populated with overlapping generations. At each period, one agent is born, and live for two periods.


etc...
generation 3
generation 2
generation 1
generation 0
1
2
3
4
periods

- An agent work when young and consume when old. The only way he can save is by the mean of money. The produced good is perishable.

- The problem solve by an agent born in period t is

$$\max U = E_t \left[c_{t+1} + g - \frac{n_t^2}{2} \middle| \Omega_t \right]$$

subject to

$$y_t = n_t$$

$$M_t \leq p_t y_t \tag{1}$$

$$M_t \geq p_{t+1} c_{t+1}$$

and the two budget constraints can be collapsed into a single life cycle constraint:

$$p_{t+1} c_{t+1} - p_t n_t \leq 0$$

• M is the stock of money, p the price of consumption in terms of money, y the quantity of production, n is labor. g is some good that is provided by the government

• The young produces, sells to the old against money. The next period, when old, he uses the money to buy good from the new young.

• There is no intrinsic uncertainty in the model, i.e. nothing that affect technology of preferences. But we shall see that the equilibrium can be random. In such a case, Ω_t contains the actual distribution of future prices.

• We need some assumption for generation 0. We assume that the agent of generation 0 is born old (in period 1, is endowed with M_0, and consumes $c_1 = \frac{M_0}{p_1}$

- There is a government that prints new money to buy good g, according to the budget constraint

$$p_t g = M_t - M_{t-1}$$

which is also

$$\frac{p_{t-1}}{p_t} = \frac{(M_t/p_t) - g}{M_{t-1}/p_{t-1}} \tag{2}$$

4.2 Characterization of Model's Solution

- The problem of an agent can be written as

$$\max U = E_t \left[\frac{p_t}{p_{t+1}} n_t + g - \frac{n_t^2}{2} \middle| \Omega_t \right]$$

and its optimal behavior is given by

$$n_t = E_t \left(\frac{p_t}{p_{t+1}} \right) \tag{3}$$

- Here we have the simplest model with *intertemporal substitution*. If p_{t+1} is small, it is worth working hard today because money will buy a lot of goods tomorrow.

- $\frac{p_t}{p_{t+1}}$ is the real interest rate between two periods.

- To close the model, we need the good market clearing condition

$$y_t = n_t = c_t + g$$

- (1) can be written as

$$\frac{M_t}{p_t} = c_t \tag{4}$$

- Using (2), (3) and (4), we obtain

$$n_t^2 = E_t[n_{t+1} - g] \tag{5}$$

with the initial condition of the economy n_1 satisfying

$$n_1 = \frac{M_0}{p_1} + g$$

- The equation (5) is a nonlinear functional equation which must be satisfied by any sequence of probability distributions over employment. (as we will show that stochastic equilibria can be constructed)

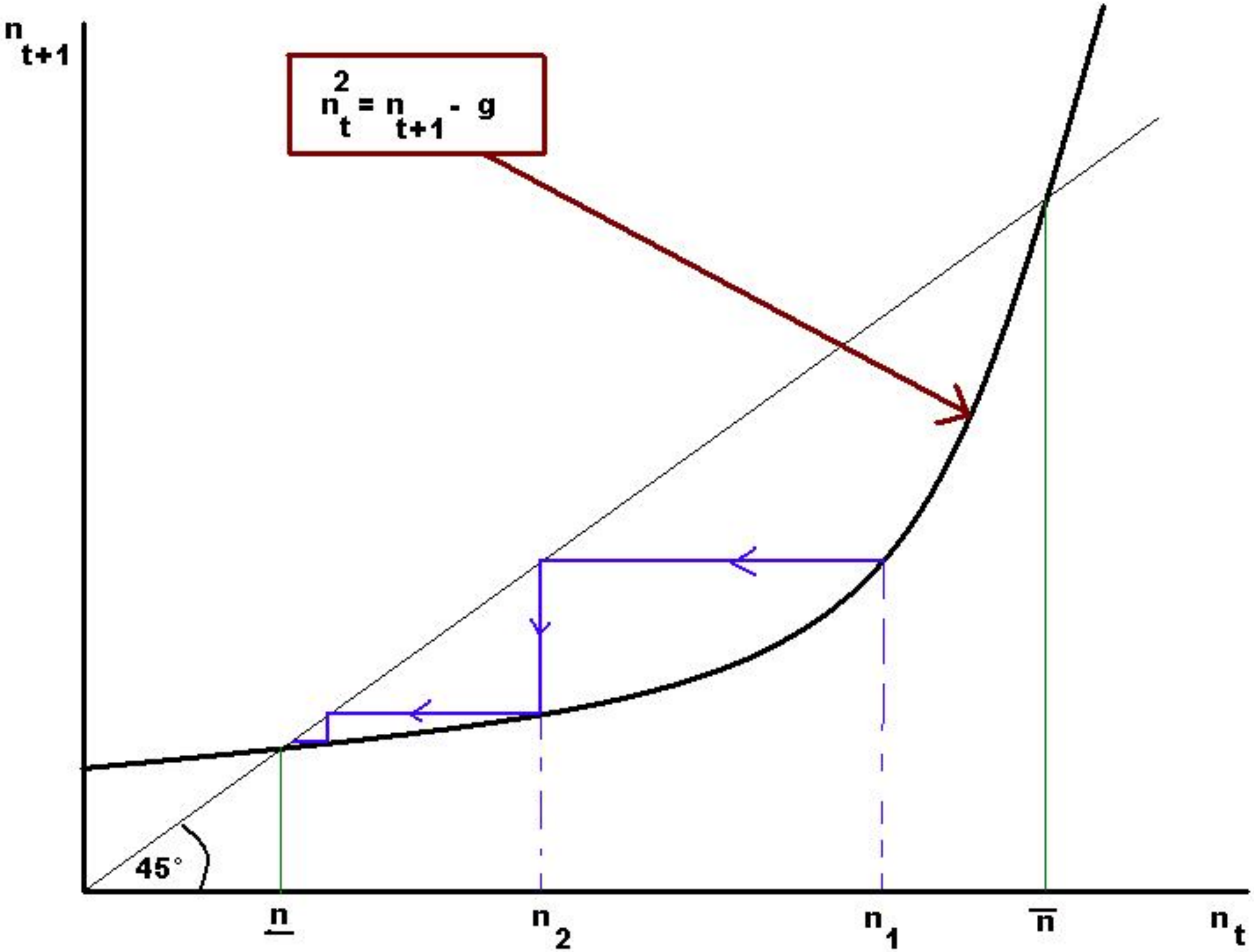
- This type of model is a model with typically multiple equilibria. Why? Because expectations (social norms, coordination,...) are crucial.

  - If no one believe in the value of the money, money has no value

  - But if I believe that tomorrow's young will accept my money, I am ready to sell my potatoes against old's money today

4.3 The Set of Equilibria

- Let us depict equation (5).



n_{t+1}
$n_t^2 = n_{t+1} - g$
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n_2
n_1
$\overline{n}$
n_t

4.3.1 Deterministic Equilibria

- There are 3 stationary equilibria,

  1. one without money and production,

  2. two non trivial ones, one with low inflation and high production and one with high inflation and low production (recall that $n_t = p_t/p_{t+1}$). In those ones, money is a pure bubble (zero fundamental value).

- But $n_1 = \frac{M_1}{p_1}$ is not a predetermined variable, as p_1 is not predetermined. Therefore, there is also a continuum of deterministic non stationary equilibria, with $n_1 \in]0, \overline{n}]$, all converging towards $\underline{n}$.

4.3.2 Stochastic Equilibria

- There is no un certainty in this economy

- Nevertheless, there might be stochastic fluctuations if agents believe so.

- And those fluctuations are compatible with *rational expectations*, ie with the fact that agents do not make systematic forecast errors.

- Let us construct one (out of many) of those stochastic equilibria

- We can show that the following stochastic process is indeed an equilibrium path of the model:

$$n_{t+1} = n_t^2 + g + u_{t+1} \tag{6}$$

where u is an arbitrary stochastic process with a conditional mean of zero and a "small" support on an interval $[-a, b]$

- It is easy to see that the process (6) satisfies equation (5)

  the next figure depict the a possible stochastic equilibrium

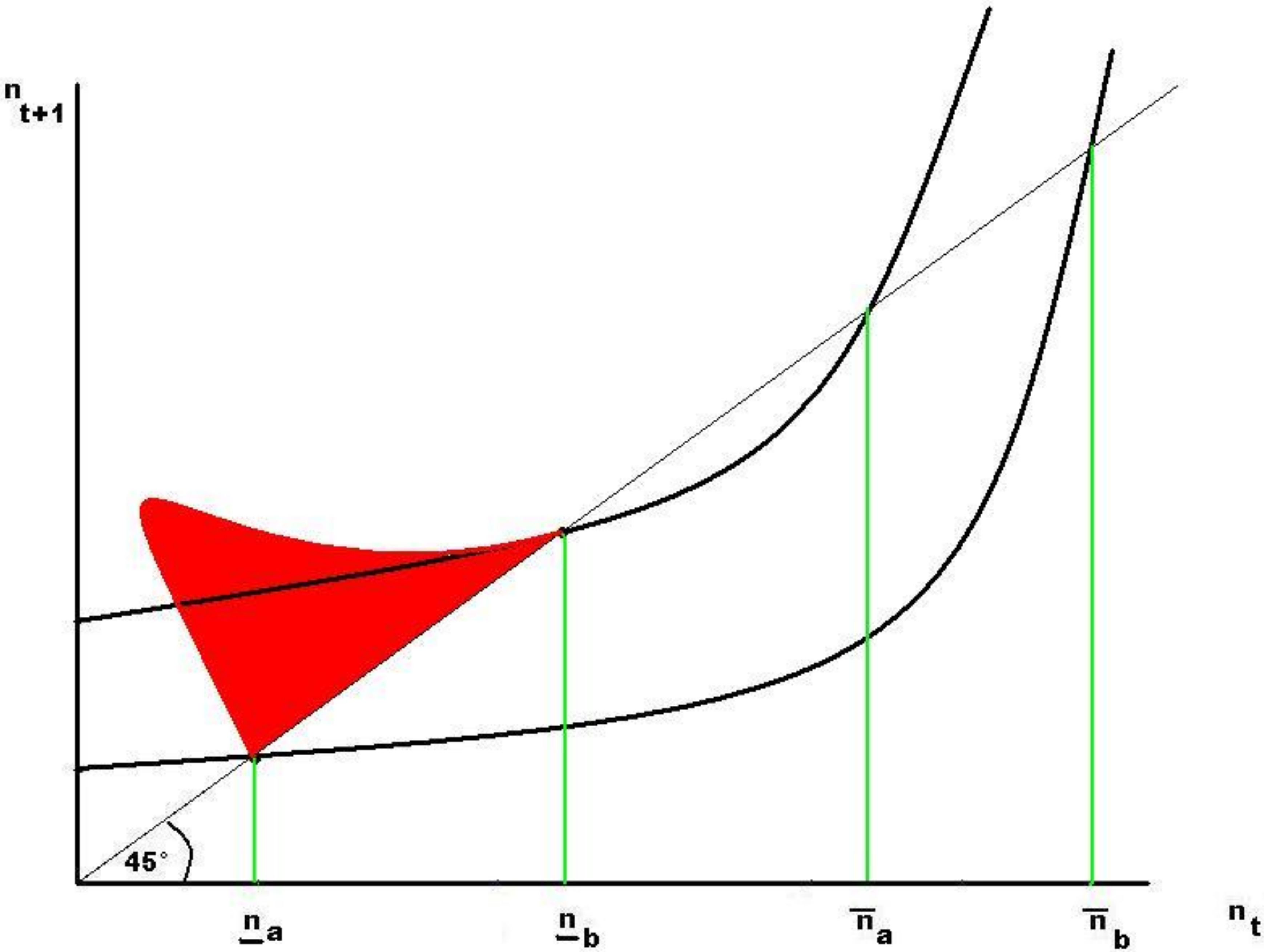
- If the economy starts at any point $n_1 \in]O, \overline{n}_b[$, the economy will move toward the interval $[\overline{n}_a, \overline{n}_b]$ with probability 1.

- At any time t, u_{t+1} is random and therefore n_{t+1} is random.

- The conditional probability distribution of n_{t+1} can be derived from the one of u_{t+1}.

- One can compute a sequence of conditional probability distributions from the process (6)

- Under mild regularity conditions, this sequence converges to an invariant distribution with support $[\overline{n}_a, \overline{n}_b]$.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n b
n a
n b
n t

- The stochastic equilibrium is constructed by randomizing across the non stationary equilibria. But the stochastic equilibrium is stationary (admits a stationary probability distribution) $\rightsquigarrow$ eventually, it can be "learned" by the agents, so that the rational expectation hypothesis (agents know that probability distribution) makes sense.

5 Conclusion

- A key characteristic of socioeconomic models is that expectations about future endogenous variables impact on the level of current endogenous variables
- This creates room for multiplicity of solutions, even with simple mathematical structures (linear difference equations)
- Of course, the "size" of the indeterminacy is likely to increase with the non linearity of the model.
- It is important to keep in mind that we have dealt only with rational expectations equilibrium path, in which agents do not make any systematic mistakes and are all alike

- Once non-rational expectations and/or heterogeneity are introduced, it is even harder to get robust predictions from socioeconomic models.

- There are other more "standard" problems associated with those models:

  1. Theorems about existence, (local) uniqueness of solutions

  2. Algorithms to solve those non-linear stochastic difference equations systems

- I would say (almost fully out of ignorance) that those problems are more in line with the problems that can be faced by other applied scientists.