Generating functions in algebraic complexity

A criterion for VNP-completeness

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A *graph property* is a set of graphs closed by isomorphism.

Definition

Let R be a graph, $\mathcal{P}$ a graph property and $\omega : V_E \to \mathcal{V} \cup \mathbb{Q}$ a weight function. The *generating function* on these parameters is:

$$\mathrm{GF}_\omega(\mathcal{P}, R) = \sum_{\substack{E \subset E_R \\ (V_E, E) \in \mathcal{P}}} \prod_{e \in E} \omega(e)$$

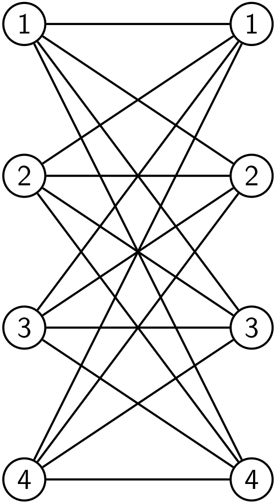
$$\mathrm{FG}_{\mathcal{V}^d}(\mathcal{P}, R) = \sum_{\substack{E \subset E_R \\ (V_E, E) \in \mathcal{P}}} \prod_{\langle i,j \rangle \in E} x_{\langle i,j \rangle} \text{ and } \mathrm{FG}_{\mathcal{V}}(\mathcal{P}, R) = \sum_{\substack{E \subset E_R \\ (V_E, E) \in \mathcal{P}}} \prod_{(i,j) \in E} x_{(i,j}$$

Let $\mathcal{PM}$ the property of being a perfect matching: every connected component has exactly two vertices. Let $k_{n,n}$ the complete balanced bipartite graph. The

$$\mathrm{FG}_{\mathcal{V}^d}(K_{n,n}, \mathcal{PM}) = \mathsf{per}^*$$

Where

$$\mathsf{per}_n^* = \sum_{\substack{\pi:[n]\to[n] \\ \text{partial bijection}}} \prod_{i \in \mathrm{def}(\pi)} x_{i,\pi(i)}$$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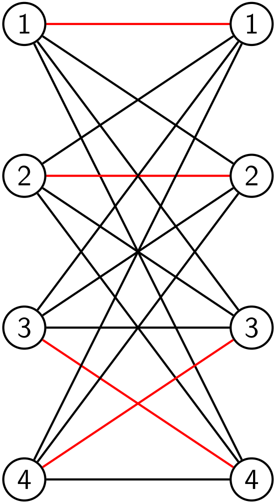
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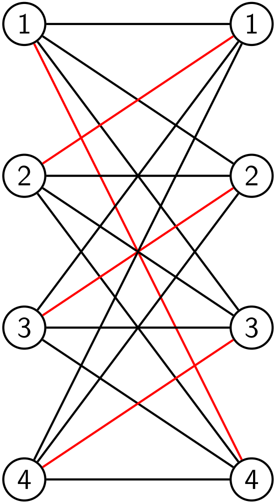
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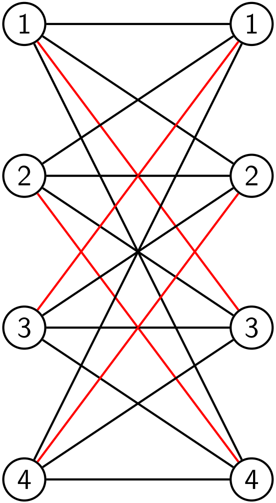
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A clique C in a graph R is a subset of its vertices such that its induced subgraph is complete. Let $\mathcal{C}$ the property of being a clique (i.e. of being complete). Then the family,

$$(\mathrm{FG}_{\mathcal{V}}(\mathcal{C}, K_n))_{n\in\mathbb{N}}$$

is VNP-complete

A complete bipartite graph G is a graph which vertices can be split into two subsets $V_1 \cup V_2 = E_G$ and which edges are $V_1 \times V_2$. Let $\textsc{BipComp}$ the property of being a bipartite complete graph. Then the family

$$(\mathrm{GF}(\textsc{BipComp}, K_n))_{n\in\mathbb{N}}$$

is VNP-complete.

Proposition (Valiant criterion)

If $\mathcal{P}$ is a graph property such that the test of knowing is a graph G is in $\mathcal{P}$ is in $P/poly$, then the family of generating functions $(\mathrm{GF}(\mathcal{P}, K_n))_{n\in\mathbb{N}}$ is in VNP.

conjecture (Lyaudet)

If the property $\mathcal{P}$ can be written in a logic $\mathcal{L}$, then the generating function of $\mathcal{P}$ is VNP-complete if and only if $\mathcal{P}$ satisfies a condition C.

A property $\mathcal{P}$ is *homomorphisable* is there is a graph H such that for any graph G,

$$G \in \mathcal{P} \text{ if and only if } G \text{ is homomorphic to } H$$

Theorem

If $\mathcal{P}$ is a property homomorphisable, then the family $(\mathrm{FG}_{\mathcal{V}}(\mathcal{P}, K_n))_{n\in\mathbb{N}}$ is VNP-complete.

Corollary

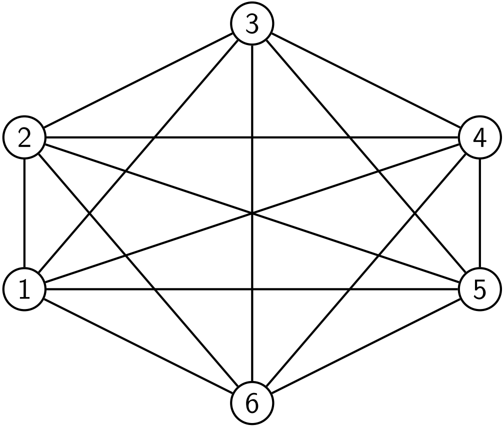
- *Let Col_k the property of being k-colourable. Then the family $(\mathrm{FG}_{\mathcal{V}}(Col_n, K_n))_{n\in\mathbb{N}}$ is VNP-complete for any $k > 1$.*
- *Let Bip the property of being a bipartite graph, then the family $(\mathrm{FG}_{\mathcal{V}}(Bip, K_n))_{n\in\mathbb{N}}$ is VNP-complete for any $k > 1$.*

Definition

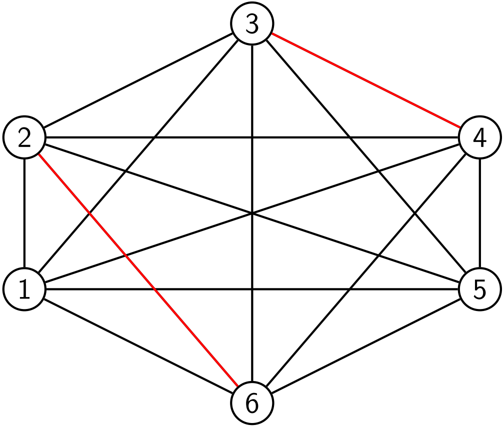
Let $\mathcal{P}$ be a graph property. The *enumarating function* associate to this property is

$$\mathrm{EF}(\mathcal{P}, n) = \sum_{G \in P, |G|=n} \prod_{e \in E_G} x_e$$

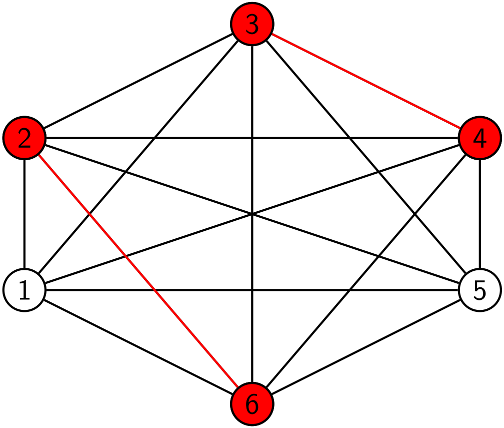
Let $\mathcal{PM}$ the property of being a perfect matching. Then $\mathrm{EF}(\mathcal{PM}, n) = \mathrm{per}_n$.



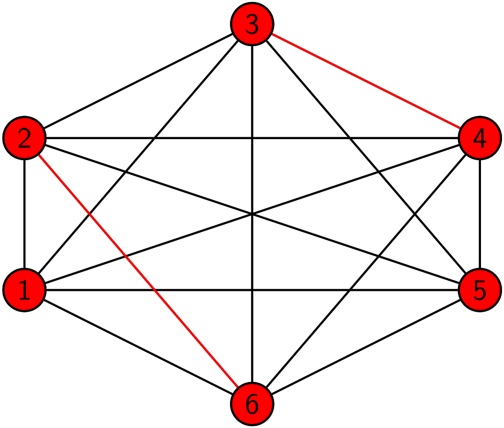
$$G = (V_G, E_G)$$



$$E \subseteq E_G$$



(V_E, E) is a perfect matching and is in $\mathrm{GF}(\mathcal{PM}, K_n)$



(V_G, E) is not a perfect matching and is not in $\mathrm{EF}(\mathcal{PM}, n)$

A *cut* in a graph G is a partition of its vertices $A \cup B = V_G$. A graph is a cut if its vertices can be split into a partition $A \cup B = V_G$ and its directed edges are $A \times B$.

Proposition

Let $\textsc{Cut}$ *the property of being a cut. Then the family*

$$\mathrm{EF}(\textsc{Cut}, n))_{n \in \mathbb{N}}$$

is VNP*-complete.*

Theorem (Bürgisser)

Over $\mathbb{F}_2$*, the family* $(\mathrm{EF}(\textsc{Cut}, n))_{n \in \mathbb{N}})_{n \in \mathbb{N}}$ *is neither in* VP *nor* VNP*-complete, provided that* $Mod_2NP \not\subseteq P/Poly$*.*

Theorem (Engels 2014)

For a graph H and a class of graphs $\mathcal{E}$, let $\mathcal{P}(H,\mathcal{E})$ the property of being homomorphic to H and in $\mathcal{E}$. Then for $\mathcal{E}$ the set of cycles, of cliques, of trees, of outerplanar graphs, of planar graphs or of graph of genus k, the family

$$(\mathrm{GF}(\mathcal{P}(H,\mathcal{E}), K_n)_{n\in\mathbb{N}}$$

is VNP-complete in most of the cases.

Definition

Let $\mathcal{C}$ be a class of circuit, $(f_n)_{n\in\mathbb{N}}$ and $(g_n)_{n\in\mathbb{N}}$ two families of polynomials. We say that $(f_n)_{n\in\mathbb{N}}$ is a $\mathcal{C}$ of $(g_n)_{n\in\mathbb{N}}$ if there exists a family of circuit C_n of polynomial size such that:

- C_n is in $\mathcal{C}$
- C_n use oracle gates g_n
- C_n computes f_n.

For examples, classical c-reductions are VP-reductions.
Projections are VP_{depth2}-reduction (the set of circuit of depth 2).

Merci !