Generating functions in algebraic complexity
A criterion for VNP-completeness

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A graph property is a set of graphs closed by isomorphism.

**Definition**

Let $\mathcal{P}$ be a graph property and $\omega : V_E \to \mathcal{V} \cup \mathcal{Q}$ a weight function. The generating function on these parameters is:

$$GF_\omega(\mathcal{P}, R) = \sum_{E \subseteq E_R} \prod_{e \in E} \omega(e)$$

$$FG_{\mathcal{V}d}(\mathcal{P}, R) = \sum_{E \subseteq E_R} \prod_{\langle i, j \rangle \in E} x_{\langle i, j \rangle}$$

and

$$FG_{\mathcal{V}}(\mathcal{P}, R) = \sum_{E \subseteq E_R} \prod_{(i, j) \in E} x_{(i, j)}$$
Let \( \mathcal{PM} \) the property of being a perfect matching: every connected component has exactly two vertices. Let \( k_{n,n} \) the complete balanced bipartite graph. The

\[
FG_{\mathcal{PM}}(K_{n,n}, \mathcal{PM}) = \text{per}^*
\]

Where

\[
\text{per}_n^* = \sum_{\pi : [n] \to [n]} \prod_{i \in \text{def}(\pi)} x_{i,\pi(i)}
\]

partial bijection
Generating functions in algebraic complexity

Definition
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Definition

\[ \text{Diagram with connections between nodes} \]
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Definition
A clique $C$ in a graph $R$ is a subset of its vertices such that its induced subgraph is complete. Let $C$ the property of being a clique (i.e. of being complete). Then the family,

$$(FG_V(C, K_n))_{n \in \mathbb{N}}$$

is VNP-complete
A complete bipartite graph $G$ is a graph which vertices can be split into two subsets $V_1 \cup V_2 = E_G$ and which edges are $V_1 \times V_2$. Let $\text{BipComp}$ the property of being a bipartite complete graph. Then the family

$$(\text{GF}(\text{BipComp}, K_n))_{n \in \mathbb{N}}$$

is VNP-complete.
Proposition (Valiant criterion)

If $\mathcal{P}$ is a graph property such that the test of knowing is a graph $G$ is in $\mathcal{P}$ is in $P/poly$, then the family of generating functions $(GF(\mathcal{P}, K_n))_{n \in \mathbb{N}}$ is in VNP.
conjecture (Lyaudet)

If the property $\mathcal{P}$ can be written in a logic $\mathcal{L}$, then the generating function of $\mathcal{P}$ is VNP-complete if and only if $\mathcal{P}$ satisfies a condition $\mathcal{C}$. 
A property $\mathcal{P}$ is homomorphisable is there is a graph $H$ such that for any graph $G$,

$$G \in \mathcal{P} \text{ if and only if } G \text{ is homomorphic to } H$$

**Theorem**

*If $\mathcal{P}$ is a property homomorphisable, then the family $(FG_V(\mathcal{P}, K_n))_{n \in \mathbb{N}}$ is VNP-complete.*
Corollary

- Let $Col_k$ the property of being $k$-colourable. Then the family \((FG(V(Col_n, K_n)))_{n \in \mathbb{N}}\) is VNP-complete for any $k > 1$.
- Let $Bip$ the property of being a bipartite graph, then the family \((FG(V(Bip, K_n)))_{n \in \mathbb{N}}\) is VNP-complete for any $k > 1$. 
Definition

Let $\mathcal{P}$ be a graph property. The enumerating function associate to this property is

$$EF(\mathcal{P}, n) = \sum_{G \in \mathcal{P}, |G| = n} \prod_{e \in E_G} x_e$$

Let $\mathcal{PM}$ the property of being a perfect matching. Then $EF(\mathcal{PM}, n) = \text{per}_n$. 
Generating functions in algebraic complexity

Enumerating functions

\[ G = (V_G, E_G) \]
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Enumerating functions

\[ E \subseteq E_G \]
\((V_E, E)\) is a perfect matching and is in \(\text{GF}(\mathcal{PM}, K_n)\)
\((V_G, E)\) is not a perfect matching and is not in \(\text{EF}(\mathcal{PM}, n)\)
A cut in a graph \( G \) is a partition of its vertices \( A \cup B = V_G \). A graph is a cut if its vertices can be split into a partition \( A \cup B = V_G \) and its directed edges are \( A \times B \).

**Proposition**

Let \( \text{Cut} \) the property of being a cut. Then the family

\[
\text{EF}(\text{Cut}, n))_{n \in \mathbb{N}}
\]

is VNP-complete.

**Theorem (Bürgisser)**

Over \( \mathbb{F}_2 \), the family \( (\text{EF}(\text{Cut}, n))_{n \in \mathbb{N}})_{n \in \mathbb{N}} \) is neither in VP nor VNP-complete, provided that \( \text{Mod}_2 \text{NP} \not\subseteq \text{P}/\text{Poly} \).
Theorem (Engels 2014)

For a graph $H$ and a class of graphs $\mathcal{E}$, let $\mathcal{P}(H, \mathcal{E})$ the property of being homomorphic to $H$ and in $\mathcal{E}$. Then for $\mathcal{E}$ the set of cycles, of cliques, of trees, of outerplanar graphs, of planar graphs or of graph of genus $k$, the family

$$(\text{GF}(\mathcal{P}(H, \mathcal{E}), K_n))_{n \in \mathbb{N}}$$

is VNP-complete in most of the cases.
Definition

Let $\mathcal{C}$ be a class of circuit, $(f_n)_{n \in \mathbb{N}}$ and $(g_n)_{n \in \mathbb{N}}$ two families of polynomials. We say that $(f_n)_{n \in \mathbb{N}}$ is a $\mathcal{C}$ of $(g_n)_{n \in \mathbb{N}}$ if there exists a family of circuit $C_n$ of polynomial size such that:

- $C_n$ is in $\mathcal{C}$
- $C_n$ use oracle gates $g_n$
- $C_n$ computes $f_n$.

For examples, classical $c$-reductions are VP-reductions. Projections are VP$_{depth 2}$-reduction (the set of circuit of depth 2).
Merci !