Notes of STINT Workshop

Col de la Croix Perrin, September 2014

1 Problems

1.1 Large acyclic subdigraph in oriented graphs

Presented by Ararat Harutyunyan.

Theorem 1 (Ajtai-Komlos-Szemeredi 80's)**.** *If G is a triangle-free graph with average degree d then* $\alpha(G) \geq \frac{1}{100}\frac{n \log d}{d}$.

There is analogue conjecture for oriented graphs.

Conjecture 2 (Aharoni, Berger, Kfir 2008)**.** If G is an oriented graph with average degree d, then $\alpha_{ac}(G) \geq C\frac{n \log d}{d}$ where $\alpha_{ac}(G)$ is the largest size of an acyclic induced subdigraph.

Nothing better than n/d is known.
Stéphan Thomassé wonders if the following cannot even be true:

Problem 3. Let G be an oriented graph with average degree d. Is there always an induced subgraph without 3-cycle of order $\frac{n \log d}{d}$?

Theorem 4. *Every tournament of order n has a transitive subtournament of order* $\log_2(n) + 1$.

For random tournament you can get $2\log_2(n)$.

Problem 5 (Erdős-Moser)**.** What is the largest integer $tt(n)$ such that every tournament of order n has a transitive subtournament of order $tt(n)$?

1.2 Triangle packing in tournaments

Presented by Louis Esperet.

Computing the minimum size of a feedback vertex set in a tournament is NP-complete.

Conjecture 6 (Bessy and Bang-Jensen)**.** [2]] Computing the maximum size of triangle packing in a tournament is NP-complete.

Stephan Thomassé observed that the following should be true.

Conjecture 7. Deciding if there is a covering by disjoint triangles is NP-complete.

1.3 b-continuity of line-graphs

Presented by Claudia Linhares Sales.

A *b-colouring* with k colours is a proper colouring such that every colour class contains a vertex that has a neighbour in every other colour classes. Such a vertex is called a *b-vertex*.

A graph G is *b-continuous* if the set of integers k such that G admits a b-colouring is an interval.

Problem 8 (Mais Alkhateeb, 2012)**.** Every line-graph is b-continuous.

The following conjecture might also be true.

Problem 9. Every claw-free graph is b-continuous.

Chordal graphs are b-continuous, so line-graphs of trees are b-continuous.

1.4 Colouring graphs with few vertices of degree more than 3

Presented by Nick Brettell.

Problem 10. Is the following problem XP and even FPT:
Instance: A graph G.
Parameter: Number of vertices of degree at least 4.
Instance: Is G 3-colourable ?

If there is no big vertices, Brooks' Theorem imply that is it is polynomial-time solvable. It is also polynomial time solvable for 1 big vertex. If it is polynomial-time solvable for 3 , then it will imply that it is FPT. Indeed, we can guess the colours of big vertices and contract the big vertices of each colour in one vertex.

This problem is true. The following more general theorem can be proved.

Theorem 11. *Let k be a fixed integer. Let G be a graph with at most p vertices of degree more than k. One can decide in $O(p^k)(n^2)$ time whether G is k-colourable.*

The proof relies on a generalisation of Brooks' Theorem established independently by Borodin [4] and by Erdős, Rubin and Taylor [5]. A graph G is *degree-choosable* if G is L-colourable for any list-assignment L such that $|L(v)| \geq d_G(v)$ for every $v \in V(G)$. A connected graph is said to be a *Gallai tree* if each of its blocks is either a complete graph or an odd cycle.

Theorem 12 (Borodin [4], Erdős, Rubin and Taylor [5])**.** *Let G be a connected graph. Then G is degree-choosable if and only if G is not a Gallai tree.*

The proof of the following easy lemma is left to the reader.

Lemma 13. *Let C be a cycle and L a 2-list assignment of C. One can decide in $O(|C|)$ time whether C is L-colourable or not.*

Theorem 14. *Let G be a Gallai tree. For any list assignment L such that $|L(v)| \geq d_G(v)$ for every $v \in V(G)$, one can decide in $O(n^2)$ time if G is L-colourable.*

Proof. Let us describe a procedure $\texttt{colour}(G,L)$ that returns 'Yes' if and only if the Gallai tree G is L-colourable, and returns 'No' otherwise.

We first check whether there is a vertex such that $|L(v)| \geq d_G(v) + 1$. If there is such a vertex v, then we return 'Yes'. This is valid because one can L-colour G greedily according to the reverse of a search ordering with root v.

Henceforth, we may assume that $|L(v)| = d_G(v)$ for every $v \in V(G)$. We then compute the bock decomposition of G, that is the set of blocks of G and its cut-vertices. We consider a leaf block in this decomposition, that is a block B containing at most one cut-vertex. If B contains no cut-vertex, then G is a complete graph or an odd cycle. One can easily check that G is L-colourable if and only if the lists of its vertices are all equal. So we return 'No' or 'Yes', depending on whether all the list are equal or not respectively, which can be checked in $O(|B|^2)$-time.

Assume now that B contains a cut-vertex x. Let $L^*(x)$ be the set of colours such that B admits an L-colouring c such that $c(x) = \alpha$. For each colour $\alpha \in L(x)$, one check whether $\alpha \in L^*(x)$ in $O(|B|^2)$ time as follows. If B is a complete graph, we have to check whether all the lists of $B - x$ are all equal and contain α. If this is the case, we return 'No', otherwise we return 'Yes'. If B is an odd cycle, then we do it as in Lemma 13.

Observe that $|L^*(x)| \geq L(x) - d_B(x) = d_{G^*}(x)$, where $G^* := G \setminus (B \setminus \{x\})$. Henceforth, we return $\texttt{colour}(G^*, L^*)$, where $L^*(v) = L(v)$ for any vertex in $V(G) \setminus B$.

Let us now examine the complexity of $\texttt{colour}$. It has to compute a block decomposition of G, each time a recursive call is made. However, once we have the block decomposition of G, one can easily get the one G^*. It suffices to remove B from the sets of blocks and x from the set of cut-vertices if it is only n one block distinct from B. Therefore, we only need to compute such a decomposition once, which can be made in $O(n^2)$ time. Then we only need to update it. All updates can be done in $O(n^2)$ time too.

Except form this block decomposition calculation, for each considered block, we have to compute in $O(|B|^2)$ the new list L^*. The total complexity for such operation is at most $O(n^2)$.

Finally, $\texttt{colour}$ may run a search and run the reedy algortihm according to the reverse order if it finds a vertex whose list is larger than its degree. This can be done in $O(n^2)$.

Therefore $\texttt{colour}$ runs in $O(n^2)$ □

Remark 15. It seems that the $O(n^2)$ could be replaced by $O(n+m)$.

Proof of Theorem 11. Let X be the set of vertices of degree more than k in G. We guess the what could be the colouring on those vertices. There are at most p^k possible colouring.

For each of them, we then check if it can be extended in a k-colouring of the whole graph. To do so we consider $H = G - X$, and for every vertex $v \in V(G) \setminus X$, we consider the list $L(v)$ of available colours, that is the list of colours in $\{1, \dots, k\}$ that are not used on a nieghbour of v in X. Clearly, $|L(v)| \geq k - |N(v) \cap X| \geq d_H(x)$.

We then check if G is Gallai tree or not. If it is, then we return $\texttt{colour}(H, L)$. If not, then we return 'Yes'. This is valid by Theorem 12. □

Remark 16. The proof of all above theorems are constructive, so there is an algorithm that given a graph G with at most p vertices of degree more than k, returns in $O(p^k)(n^2)$ time a k-colouring of G if one exists, or returns 'No' if G is not k-colourable.

Theorem 11 shows that the following problem is FPT. Instance: A graph G and an integer k
Parameter: Number of vertices of degree greater than k and k.
Instance: Is G k-colourable ?

Problem 17. Does the above problem have a polynomial kernel ?

1.5 Colouring graphs with pair of 4-connected vertices

Presented by Nicolas Trotignon.

Let $\kappa\ (x,y)$ be the *connectivity between x and y*, that the maximum number of vertex disjoint path between x and y. Let $K(G) = \max\{\kappa(x,y) \mid x,y \in V(G)\}$.

One can show that if $K(G) = 3$, then G is 4-colourable.

Problem 18. Can we decide in polynomial whether a graph with $K(G) = 3$ is 3-colourable or not.

1.6 List colouring large portion of a t-choosable graphs

Presented by Louis Esperet.

Conjecture 19 (Janssen et al. [7])**.** Let G be a graph of order n. If G is t-choosable, then for any s-list assignment, $s < t$, there is an induced subgraph of size $\frac{s}{t} \cdot n$ that is L-colourable.

As shown by Janssen et al. [7], we cannot hope for an induced subgraph of size $\frac{s}{t} \cdot n$ which is This disproves a Conjecture of Albertson et al.

This is true if s divides t. Suppose $s = p \times t$ with p an integer. Let L be an s-list assignment of G. Let L' be the t-list assignment of G defined by $L(v) = \bigcup_{c \in L(v)} \{(c,1), \cdots, (c,p)\}$. The graph G admits an L'-colouring ϕ. For any $1 \leq i \leq p$, let V_i be the set of vertices whose colour is of type (c,i). Clearly, there is some k such that $|V_k| \geq n/p = \frac{s}{t} \cdot n$. And $G[V_k]$ is L-colourable. Just forget about the second member of the pair of ϕ. It also implies that there is an L-colourable induced subgraph of size $\frac{1}{\lceil t/s \rceil} \cdot n$. Another evidence is that there is an L-colourable induced subgraph of size $\frac{6s}{7t} \cdot n$.

1.7 A bit of discrete geometry

Presented by Pierre Aboulker.

If S is a set of points in the plane, then $\mathcal{L}(\mathcal{S})$ is the set of lines containing at least two points of S.

Theorem 20 (De Bruijn and Erdős)**.** $|\mathcal{L}(S)| \geq |S|$ *or all the points of S are colinear.*

Let $I_{av}(S) = \frac{\sum_{e,f \in \mathcal{L}(S) e \neq f} e \cap_S f}{\binom{m}{2}}$, where $e \cap_S f$ is the set of points of S in e and f.

The following conjecture strengthen De Bruijn–Erdős Theorem.

Conjecture 21 (Aboulker)**.** Let S be a set of points that are not colinear. Then

$$I_{av}(S) \cdot |\mathcal{L}(S)| \geq |S|.$$

For every point v of S, its degree $d(v)$ is the number of lines of $\mathcal{L}(S)$ to which it belongs. Then $I_{av}(S) \cdot |\mathcal{L}(S)| \geq |S|$ is equivalent to $\sum_{v \in S} d(v)(d(v)-1) \geq |S|(|\mathcal{L}(S)|-1)$.

We know that $|\mathcal{L}(S)| = |S|$ if and only if there are n points, $n-1$ being colinear and the last out of their lines. In this case, we clearly have $I_{av}(S) = 1$.

1.8 Colouring wheels

Presented by Pierre Aboulker.

A k-wheel is a graph which is the union of a cycle and a vertex adjacent to k vertices of the cycle.

A graph is *k-wheel-free* if it contains no k-wheel as a subgraph.

Conjecture 22. Let $k \geq 3$. Every k-wheel-free graph is k-colourable.

Thomassen and Toft [8] proved the conjecture for $k = 3$ and Aboulker [1] proved it for $k = 4$. Turner III [9] proved that they are $(k+1)$-colourable, because it is k-degenerate.

Conjecture 23. If G is k-wheel-free ($k \geq 3$), then either it contains a vertex of degree at most $k-1$, or it contains a pair of non adjacent twins.

The conjecture holds for $k = 3, 4$.

1.9 Polynomial χ-bounding function for χ-bounded hereditary classes

Presented by Pierre Aboulker.

We denote by $\omega(G)$ (resp. $\chi(G)$) the clique number (resp. chromatic number) of the graph G. A class of graph $\mathcal{C}$ is *χ-bounded* if there exists a function f such that $\chi(G) \leq f(\omega(G))$ for all graph $G \in \mathcal{C}$. The function f is said to be *χ-bounding* for $\mathcal{C}$.

Problem 24 (L. Esperet)**.** Let $\mathcal{C}$ be an hereditary χ-bounded class of graphs. Does it have a polynomial χ-bounding function ?

We do not know if they all admits exponential (or whatever huge can be) χ-bounding function. A unique ultimately χ-bounding function might exist.

Problem 25. Does there exists a function f, such that for every hereditary χ-bounded class $\mathcal{C}$ of graphs, there is $w_{\mathcal{C}}$, such that if $G \in \mathcal{C}$ and $\omega(G) \geq w_{\mathcal{C}}$, then $\chi(G) \leq f(\omega(G))$?

1.10 Betti numbers and chromatic number

Presented by S. Thomassé.

Conjecture 26 (Kalai and Meshulam)**.** Let G be a graph. There exists a function f such that if the sum of the Betti numbers of the stable complex of every induced subgraph of G is bounded by k, then $\chi(G) \le f(k)$.

For the stable complex of a cycle C_ℓ, the sum of the Betti numbers is 1 if $\ell \equiv 1,2 \mod 3$, and 0 otherwise. Hence the case $k = 2$ for this conjecture is implied by the following theorem of Bonamy, Charbit, and Thomassé [3].

Theorem 27 (Bonamy, Charbit, and Thomassé [3])**.** *There is a constant C such that every graph with chromatic number at least C contains an induced cycle of length* $0 \mod 3$.

A natural question is to ask what the best constant C could be.

Conjecture 28. Every graph with chromatic number at least 4 contains an induced cycle of length $0 \mod 3$.

More generally,

Problem 29. What is the value of $f(2)$?

2 Notes possiblement informes sur quelques jolies preuves

2.1 L-Colouring large portion of a 3-choosable graph

Theorem 30. *Let* $\sigma = \frac{\sqrt{5}-1}{2} = 0.6180\ldots$. *Let G be a* 3*-choosable graph. For any* 2*-list assignment L, there is a set S of at least* $\sigma \cdot v(G)$ *vertices such that* $G[S]$ *is L-colourable.*

Proof. Observe that $\sigma^2 + \sigma - 1 = 0$. Let α be a colour that is not in $U = \bigcup_{v \in V(G)} L(v)$. Let L' be the 3-list assignment defined by $L'(v) = L(v) \cup \{\alpha\}$ for every $v \in V(G)$. G has an L-colouring c. Let A be the set of vertices such that $c(x) = \alpha$ and $B = V(G) \setminus A$. Observe that A is a stable set.

Now put each colour in U with probability σ in U_1 and probability $1-\sigma$ in U_2. For $i = 1,2$, let B_i be the set of vertices v such that $c(x) \in U_i$. Clearly, $\mathbf{E}(|B_1|) = \sigma|B|$. Now let A_2 be the set of vertices a of A such that $L(a) \cap U_2 \neq \emptyset$. For every $a \in A$, $\mathbf{Pr}(a \in A_2) = 1 - \sigma^2$, so $\mathbf{E}(|A_2|) = (1-\sigma^2)|A| = \sigma|A|$.

Now consider $A_2 \cup B_1$. Colouring every vertex $b \in B_1$ with $c(b)$ and every vertex $a \in A_2$ with a colour in $L(a_2) \cap U_2$, we obtain an L-colouring of $G[A_2 \cup B_1]$. Moreover $|A_2 \cup B_1| \ge \sigma|A| + \sigma|B| = \sigma \cdot v(G)$. □

2.2 Strengthening the 4-colour theorem

Theorem 31. *Every (2-connected) planar graph has a 4-colouring such that every vertex of degree at least* 2 *has two neighbours of different colours.*

Proof. Give distinct names to the faces. Assign to each vertex v the list $L(v)$ of names of its incident faces. Doing so $|L(v)| \geq d(v)$. If G is a clique or an odd cycle, then find a dedicated procedure. Since G is 2-connected, not a clique nor an odd cycle, G is a Gallai tree, so it is L-colourable. Now if a vertex is assign the face f add the chord jumping over v in f. One checks that the obtained graph H is planar, two added chord may not cross because the L-colouring was proper. Hence it is 4-colourable by the 4-Colour Theorem. Now every vertex of G is in a triangle in H, and so the 4-colouring of H is the desired colouring of G. □

2.3 Potential technique

In this subsection, critical means vertex-critical.

Solving a conjecture of Ore, Kostochka and Yancey [6] showed the following.

Theorem 32 (Kostochka and Yancey [6])**.** *If G is 4-critical, then* $|E(G)| \geq \frac{5v(G)-2}{3}$.

This bound is tight has can been shown by using Hajos construction. In fact, there a more general theorem for k-critical, for any k.

Proof. For any $R \subset V(G)$, let $p(G) = 5|R| - 3e(G\langle R\rangle)$.

We need to show that if G is 4-critical, then $p(V(G)) \leq 2$.

Let G be a minimum counterexample to the theorem. G is 4-critical and $p(V(G) > 2$.

Observe that if $|R| \leq 4$, then $p(R) \geq 2$.

Claim 32.1. *There is no set* $R \subset V(G)$, $R \neq V(G)$ *with* $|R| \geq 2$ *and* $p(R) \leq 5$.

Subproof. Take R, such that $|R| \geq 2$ that minimizes p. $p(R) \leq 5$ so R has at least 4 vertices.

Take a 3-colouring of $G\langle R\rangle$, and replace R by a triangle T by identifying the vertices of same colour. The resulting graph G' contains a 4-critical subgraph, which has potential 2 because G was the minimum counterexample. Let X' be its vertex set. Let $X = X' \setminus V(T) \cup R$.

One checks that $p(X) \leq p(R) - 3 \leq 2$. But X was minimizing p, so the only possibility is $X = V(G)$. But it is impossible because $p(V(G)) > 2$. ◊

Using the same technique one can prove the following.

Claim 32.2. *The only R,* $R \neq V(G)$, *with* $p(R) = 6$ *are* 3*-cliques.*

As a consequence there is no $K_4 \setminus e$.

Claim 32.3. *Every triangle contains at most one vertex of degree* 3.

Subproof. Assume that there is a triangle $u_1u_2u_3$with two 3-vertices, say u_1 and u_2. Let v_i be the neighbour of u_i not in the triangle. Remove u_1, u_2 and add the edge v_1v_2. One checks that the resulting graph H is not 3-colourable. So H contains a 4-critical subgraph induced by Y. By minimality of G, $p(Y) \leq 2$. Y must contain a and b, for otherwise $G\langle X\rangle = H\langle X\rangle$, and X contradicts Claim 32.1. Considering $Y = X \cup \{u_1, u_2\}$, we have $p(Y) \leq p(X)+4$, so $p(Y) \leq 6$. As it is not a triangle it has to be the whole set. But then$u_3 \in X$ and $p(Y) \leq p(X)+4-6$, a contradiction. ♢

Using the same technique, one can show

Claim 32.4. *If two* 3*-vertices are adjacent, then each of them is in a triangle.*

The above claim imply that every 3-vertex has at most one 3-neighbour.

It just remain to do a small discharging procedure.

Initial charge: $d(v)$ for every vertex v.

Discharging rule: Every (≥ 4)-vertex gives 1/6 to each of its 3-neighours.

The new charge is at least 10/3 for every vertex. Hence $e(G) \geq 5v(G)/3$. Thus $p(V(G)) \leq 0$, a contradiction. □

One can deduce Grötzsch's Theorem from Theorem 32

Proof of Grötzsch's Theorem. Consider a minimum counter-example. It has to be 4-critical.

Assume fisrt that G contain a 4-face $a_1b_1a_1b_2$. One shows that one of the two pairs $\{a_1, a_2\}$ or $\{b_1, b_2\}$ can be contracted so that the remaining graph is still triangle-free. But a 3-colouring of this graph would give a 3-colouring of G, a contradiction.

If G has girth at least 5, then $e(G) \leq \frac{5m-10}{3}$ by Euler's Formula. It contradicts Theorem 32. □

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