Project Stint, 2014–2017

1 Summary of the proposal

Induced subgraphs play a central role in both structural and algorithmic graph theory. A graph H is an *induced subgraph* of a graph G if one can delete vertices of G to obtain H. This is the strongest notion of subgraph, hence being H*-free* (that is not containing H as an induced subgraph) is not a very restrictive requirement. Weaker notions of containment, like for instance minors, are now well understood, and the next achievement in Graph Theory should certainly be the understanding of forbidden induced structures. We focus in this proposal on the following very general question: *Given a (possibly infinite) family ψ of graphs, what properties does a ψ-free graph have?*

This is the key question of many important and longstanding problems, because many crucial graph classes are defined in terms of forbidden induced subgraphs. This field is now quickly growing, and new techniques and tools have been recently developed.

Our first goal is to establish bounds on some classical graph parameters for ψ-free graphs, such as the clique number, the stability number and the chromatic number. A second goal is to design efficient algorithms to recognize ψ-free graphs and to determine or approximate some parameters for those graphs.

For this purpose, we plan to use and develop various proof techniques, some of these being recently discovered, such as the structural description of graph classes, the regularity lemma, graph limits, flag algebras, VC-dimension, discharging method as well as computer-assisted proofs.

Members of the project

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2 Context

For more than a hundred years, the development of graph theory has been inspired and guided by some important conjectures, the most prominent one being the Four-Colour Conjecture. To attack these conjectures, graph theorists have developed powerful new techniques. Some of them aim at providing structural results on graphs; in other words, they are concerned with establishing results that characterize various properties of graphs, and use these theorems in the design of efficient algorithms and other applications. For example, one of the dramatic developments over the past thirty years has been the creation of the theory of graph minors by Robertson and Seymour. In a long series of deep papers, they revolutionized graph theory by introducing original and incisive ways of viewing graph structures. This theory was developed to tackle a famous conjecture of Wagner, and it did solve it. In the process, it has led to the design of polynomial-time algorithms for solving a variety of hitherto intractable problems, including that, called Linkage, of finding a collection of pairwise-disjoint paths between prescribed pairs of vertices. This theory gives in particular powerful tools to design algorithms for families of graphs defined in terms of excluded minors.

However, it turns out that many interesting graph classes are not defined in terms of forbidden graph minors but rather in terms of forbidden induced subgraphs.

This is for example the case of line graphs (characterized by Beineke [Bei66] and Bermond and Meyer [**BeMe73**] for multigraphs) or more generally claw-free graphs. But the best-known example of a class defined by excluding induced subgraphs is certainly that of perfect graphs. In 1961, Berge [Ber61] conjectured that a graph is perfect if and only if no induced subgraph is a cycle of odd length at least 5 or the complement of such a cycle. Today such graphs are called *Berge graphs*. This conjecture, known as the Strong Perfect Graph Conjecture, was proved after four decades by Chudnovsky et al. [CRST06]. All these results lead to the following very general meta-question.

Given a (possibly infinite) family ψ of graphs, what properties does a ψ-free graph have?

2.1 Objectifs et caractère ambitieux et novateur du projet / Objectives, originality and novelty of the project

To understand the structure of ψ-free graphs, the best approach is to study how classical graph parameters are affected by forbidden induced structures. In particular, the clique number, the stability number and the chromatic number, which are mutually but loosely dependent, are expected to become more dependent when a given induced subgraph is forbidden. Our first aim is to establish bounds on these parameters. Let us first give some formal definitions. A *clique* in a graph is a set of pairwise adjacent vertices, and a *stable set* is a set of pairwise non-adjacent vertices graph. The size of a largest clique in G is the *clique number* $\omega(G)$; the size of a largest stable set in G is the *stability number* $\alpha(G)$. A (proper) k-*colouring* of a graph G is a mapping f from its vertex set $V(G)$ into $\{1, 2, \ldots, k\}$ such that $c(u) \neq c(v)$ for every edge uv of G. Equivalently, a k-colouring may be seen as a partition of the vertex set of G into k disjoint stable sets $S_1, \ldots S_k$, where each S_i is the set of vertices coloured i. The *chromatic number* $\chi(G)$ of G is the smallest integer k such that G admits a k-colouring. Graph colouring is a central area in graph theory and many important conjectures deal with this parameter (See [JeTo95]). It follows immediately from the definition of the chromatic number that $|V(G)|/\alpha(G) \leq \chi(G)$ and $\omega(G) \leq \chi(G)$. However, these two inequalities can be very loose. The disjoint union of a clique of size k and a stable set of size k is an example of a graph G with $|V(G)|/\alpha(G) \leq 2$ and $\chi(G) = k$. There are many families of graphs with clique number 2 (triangle-free graphs) and arbitrarily large chromatic number; the first one was given by Zykov [Zyk49]. The gap between the chromatic number and the clique number of a graph G can be seen as a measure of the (structural) complexity of a graph G. So, for some appropriate families ψ, we would like to show that the chromatic number of every ψ-free graph is bounded by a function of its clique number. The maximum size of a clique or a stable set is also strongly affected by forbidding an induced subgraph (see the Erdős-Hajnal conjecture). Furthermore, the separation of cliques and stable sets is a key-question in communication complexity and combinatorial optimization (extended formulations of polytopes). Our second aim is to design efficient algorithms for ψ-free graphs. In the first place, recognizing graphs that are ψ-free is a natural problem. In the second place, for each graph parameter, we would like to design efficient algorithms to determine the exact value of the parameter or to approximate it. For example, if we show that every graph G of some class Γ satisfies $\chi(G) \leq c \cdot \omega(G)$ for some absolute constant c, and if the proof can be turned into a polynomial-time algorithm that finds a colouring with at most $c \cdot \omega(G)$ colours, then we have obtained a c-approximation algorithm for colouring graphs in Γ.

A particular attention will be devoted to directed graphs (digraphs). Even though digraph theory has numerous important applications, for various reasons undirected graphs have been studied much more extensively than directed graphs. One of the reasons is that undirected graphs form in a sense a special class of directed graphs (symmetric digraphs), hence problems that can be formulated for both directed and undirected graphs are often easier for the latter. Another reason is that basic digraph parameters are much more complicated than their undirected counterparts. For instance, even the notion of (out)degree is not well understood. Understanding how forbidden induced subgraphs affect degrees is a major open area.

Advances in discrete structures are generally driven by difficult conjectures. For instance, the Four-Colour conjecture gave rise to the Discharging Method, Wagner's conjecture gave tree-decompositions and extremal graph problems gave the Regularity Lemma. Basically, the persistence of a hard problem generally indicates that some key tools are missing in the theory. Discovering these new tools is the real challenge hidden behind the open questions. Let us mention here some conjectures which have resisted investigations for decades.

A combinatorial algorithm for perfect graph colouring A graph G is *perfect* if every induced subgraph H of G satisfies $\chi(H) = \omega(H)$. Chudnovsky et al. [CRST06] proved that perfect graphs are exactly the Berge graphs (those where no induced subgraph is a cycle of odd length at least 5 or the complement of such a cycle), thereby confirming the Strong Perfect Graph Conjecture of Berge [Ber61]. One of the main reasons for which perfect graphs have fascinated researchers is that they are central subjects in both graph theory and linear programming. The relation lies in the fact that the facets of the stable set polytope of a perfect graph G correspond to the cliques of G [Chv75]. Together with the ellipsoid method, this provides a polynomial-time algorithm for several hard optimization problems, such as finding the chromatic number (which equals the clique number) of any perfect graph [GLS88]. The crucial point that is still lacking is that despite the characterization of perfect graphs [CRST06], no combinatorial (or even purely LP) algorithm is known for colouring perfect graphs.

χ-bounded classes – Gyárfás' conjecture on trees In perfect graphs, the chromatic number is equal to the clique number for every induced subgraph, and this property ensures that these NP-hard parameters are easy to compute for this class. Gyárfás [Gya87] proposed the following generalization of perfect graphs: a graph G is *χ-bounded* by a function f if every induced subgraph H of G satisfies $\chi(H) \leq f(\omega(H))$. A class of graphs is *$\chi$-bounded* if there exists a function f such that every graph in the class is χ-bounded by f. Thus, perfect graphs are precisely the graphs that are χ-bounded by the identity function. The aforementioned construction of Zykov [Zyk49] implies that the class of all graphs is not χ-bounded. One of the most famous question here is Gyárfás' conjecture on trees [Gya87], which reads: *for every tree T, the class of T-free graphs is χ-bounded.* Intuitively forbidding a tree (or some other simple structure) must simplify the structure of cliques and stable sets, hence allow colouring close to clique number. Expressing this idea formally is a real challenge.

The Erdős-Hajnal conjecture One of the first results in random graph theory is that the expected clique number (or stability number) in a (uniformly chosen) random graph with n vertices is $O(\log n)$. This essentially says that in the class of all graphs, typical cliques and stable sets have logarithmic size. But random graphs are universal objects, in the sense that they contain all small graphs as induced subgraphs with high probability. So a natural question arises here: *in the class of H-free graphs, for some fixed graph H, is the typical clique (or stability) number greater than $O(\log n)$?* Erdős and Hajnal [ErHa89] conjectured that the answer is positive. Indeed they even asked for a polynomial bound: *For a fixed graph H, does there exist a positive constant c, depending on H only, such that every H-free graph has a clique or a stable set of size n^c?* This question is still wide open for most instances of the graph H, even with few vertices; for example for the cycle on 5 vertices and for the path on 5 vertices. This conjecture is related to Gyárfás' conjecture, because if a class ψ of graphs is χ-bounded by a polynomial function f, then ψ satisfies the Erdős-Hajnal conjecture. Again this question strongly suggests that forbidding any induced graph simplifies the structure of cliques and stable sets. A new tool is needed here to capture this idea.

The Caccetta-Häggkvist conjecture undirected graphs and directed graphs, which amounts to the complexity step between symmetric and general binary relations. The particular case of classes of oriented graphs defined by forbidden induced oriented graphs makes no exception to this difficulty gap. This is one of the most intriguing open questions in graph theory [CaHa78]: *does every oriented graph with no (induced) directed cycle of length 3 have a vertex with outdegree less than $n/3$?* What is surprising here is that the forbidden structure is extremely simple, and the conclusion is easy to test — a degree condition is much simpler to check than the clique number for instance. This indicates that even the notion of minimum degree is not well understood for oriented graphs and, again, that our understanding is limited by dark zones in our current theory of induced forbidden sub(di)graphs.

We could have expanded this list with dozens of other open problems. Our goal in this proposal is certainly not to specifically solve one of these but to develop tools which will be relevant for attacking these questions.

2.2 État de l'art / State of the art

We introduce in this part the main tools used so far to tackle the kind of questions mentioned above. For each of them, we highlight in which case it is particularly useful. The aim of this proposal is to use and develop these tools to prove new results and to hopefully identify some new general technique.

Structural description of classes

Key point: Completely describe a graph class using basic classes and elementary operations.

The proof of the Strong Perfect Graph Conjecture follows from a decomposition theorem for Berge graphs. More precisely, it is proved in [CRST06] that every Berge graph either belongs to one of a few well-understood families of basic graphs or admits a decomposition among several decomposition schemes known to preserve perfection. Theorems following the same general paradigm are known for ψ-free graphs for other families ψ. Some of them are easy — for example, it is almost immediate to see that if ψ consists of a single graph that is an induced two-edge path, then every ψ-free graph is either complete or disconnected. Others are difficult — e.g., if ψ is the set of all even-length cycles, or the set of all cycles of odd length at least five (the corresponding results being deep theorems of Conforti et al. [CCKV02] and of Conforti, Cornuéjols, and Vušković [CCV04], respectively). One might then ask whether a structural theorem of that kind exists for every family ψ. Certainly, a significant part of this question is to have a meaningful definition of the graphs that should be considered basic and of the kinds of decompositions that should be allowed. However, it is of great interest to understand to what extent forbidding an induced subgraph in a graph impacts the global structure of the graph. In the last few years, several researchers have studied ψ-free graphs for different families ψ, in an attempt to obtain some insight into this question. For example, Chudnovsky and Seymour, in a series of five papers [ChSe07, ChSe08a, ChSe08b, ChSe08c, ChSe08d], proved that every claw-free graph is either a basic graph or admits one of a few decompositions. Chudnovsky [Chu12a, Chu12b] did the same for bull-free graphs. One of the goals of this proposal is to find new such decomposition theorems and try and emphasize the similarities among them and the ones in the literature. An important feature of such theorems is whether they can be "reversed", that is whether they give a procedure for building all graphs in the class under consideration, starting from some basic pieces. If such is the case, then it is often a lot easier to use the theorem for establishing properties of the graphs in the class. For example, the decomposition theorem of claw-free graphs given by Chudnovsky and Seymour can be reversed. But the decomposition theorem of Berge graphs by Chudnovsky et al. [CRST06] cannot, and finding a reversible one is challenging open problem.

To sum up, it seems today that the structural description of classes is one of the main tools to tackle hard questions, and we plan to use it. The tool is very powerful, but a problem is that for every particular class, all the work must be restarted from scratch. We do not consider this feature as unavoidable; on the contrary, developing generic tools is one of our goals. Also, we find it strange that this technique is almost never used in the realm of digraphs, so this is something we would like to change and better comprehend.

Regularity lemma, graph limits, flag algebras

Key point: partite graphs. This allows statistical counting. Theories and tools to deal with huge graphs and comprehend patterns arising in extremal graph theory.

We take a bird's view at the domain of graph limits and graph regularity, which provides theories and tools both for theoretical topics (extremal graph theory, property testing) and more real-world problems (social networks).

A particularly timely challenge concerns huge graphs. Typically, we mean here graphs that are too large to be stored in a computer (and which, also, may dynamically change). Such graphs come from many real-world situations (e.g., protein-protein interactions networks, web graphs and social networks). How to answer questions about a huge graph?

A natural approach is to use sampling techniques: this supposes a way to pick "randomly" (usually, uniformly) a certain number of nodes and answer questions using this "sample", that is, the subgraph induced by the selected nodes. The question is then to know to which extent the sample is representative of the whole graph. More precisely, if we fix a number k, choose randomly k vertices of the huge graph and consider the subgraph induced by the k chosen vertices, this yields a probability distribution on the set of all non-isomorphic graphs on k vertices. These distributions capture a number of essential properties of the huge graph. In the last six years, a framework for studying such distributions has been developed. It is known as graph limits. From a theoretical point of view, designing a concept of limit for graphs is very natural if we consider the setting of extremal graph theory.

Extremal graph theory is a deep field, the difficulty of which partly comes from combinatorial explosion (e.g., there are more than one million non-isomorphic graphs on ten vertices). This transpires, for instance, in Ramsey theory: the Ramsey number $R(5,5)$ is unknown, and $R(6,6)$ may stay unknown forever, as Erdős pointed out. On the other hand, many extremal problems can be solved for graphs with a large number n of vertices. This is because small graphs often exhibit peculiar behaviours (boundary effects), while patterns start to appear for larger values of n. Consequently, it is natural to try and give a meaning to "n goes to infinity" and, if possible, drop the parameter n by defining and studying a limit-object.

To develop such a framework, one needs to introduce a notion of distance between graphs, which measures their "similarity". Limit objects are then obtained by a completion process (in which a limit point is added for each Cauchy sequence of (hyper)graphs). Such a framework for dense graphs, initiated mainly by Lovász and Szegedy (see [BCL+08, BCL+12] and the recent book by Lovász [Lov12]), comprises a number of beautiful results and is still widely open. Many fundamental questions about graph limits are unanswered. There is not, also, one theory of graph limits, but several different ones depending on the density of the considered graphs. For instance, the aforementioned framework is of no use for sequences of graphs (G_n) such that $e(G_n) = o(n^2)$; on the contrary, there is a specific notion of convergence for graphs with bounded maximum degree (the local-global convergence [HLS], which extends the Benjamini-Schramm convergence [BeSc01]). Aiming to unify Lovász-Szegedy and Benjamini-Schramm notions of graph limits, Nešetřil and Ossona de Mendez [**NeOs12a**] proposed a notion of "structural limits". In this setting, a sequence of graphs is convergent if the probability that any fixed first-order formula is satisfied by a random assignment of vertices to the free variables converges. The key idea is to consider a functional analysis point of view: finite graphs define continuous linear forms on some Banach space and are represented, in a usual way, by a probability measure. Then, convergence of graphs correspond to weak convergence of measures, which is well understood. Also, in some cases, a representation of the limit as a measurable graph exists, which extends the notion of "graphing" used to represent limits of bounded degree graphs.

Graph limits is a concrete field in which new methods and tools have arisen. Of particular interest is the concept of flag algebras, developed by Razborov [1] in the context of limits for sequences of dense (hyper)graphs. Though it is an abstract and general framework, it can be described as a systematic approach to counting arguments. In our scope, this mainly boils down to exploiting correlations between the densities of fixed (small) sized induced subgraphs to gain knowledge about the original graph. When the size of the considered subgraphs is small, then the method is amenable to computers, using in particular semi-definite program solvers. We point out that, unlike some computer-aided proofs, the flag algebra computations can be presented in a form that can be verified by hand, although very tediously. The number of applications of flag algebras in extremal graph theory is already impressive and it keeps growing with time. be solved. For example, the most recent advances on Caccetta-Häggkvist Conjecture use flag algebras [HKS09]. We found [**KLS+**] new and original uses of the flag algebras formalism and the scope of applications goes beyond purely extremal graph theory, as the method brought new results, for instance, in discrete geometry [**KMS12**].

Another natural approach to the study of huge graphs is to design random models. Indeed, maybe sampling k nodes is an expansive process or there is no clear way how to set (or approach) a uniform distribution on the vertex set of the huge graph, for instance. While the historical random graph model of Erdős and Rényi provides a deep and useful theory, it does not capture essential properties commonly observed in real-life networks (e.g. clustering and scale invariance). This is why other models of random graphs were developed (e.g. the preferential attachment model initiated by Barabási and Albert and formally defined by Bollobás and Riordan). These models are randomly growing random graphs. If the model is accurate, then one can predict how the network may evolve or use the model to test algorithms.

An often-used notion is that of pseudo-randomness. The idea is to find a set of (relevant) properties that are all fulfilled by random graphs (according to the model in consideration) and such that, whenever a graph satisfies one of these properties, then it satisfies all of them. The graphs satisfying these properties are then called *quasi-random*. A well-known notion of quasi-randomness for sequences of graphs is as follows. Consider a sequence (G_n) of graphs such that $V(G_n) = \{1, ..., n\}$. (In particular, $V(G_i)$ contains $V(G_j)$ if $j \leq i$.) The sequence (G_n) is *p-quasi-random* if for every subset U of $V(G_n)$ the sequence defined by the number of edges in the subgraph of G_n induced by U is $p \times |U|(|U|-1)/2 + o(n^2)$.

For large dense graphs — that is, with a quadratic number of edges — an awesome approximation was found by Szemerédi [Sze78]. It allowed him to generalize Roth's theorem to arithmetic progressions of arbitrary lengths, as conjectured by Erdős. The approximation, now known as Szemerédi's Regularity Lemma, had a tremendous impact in a number of areas of mathematics. Several new proofs were given — some are purely combinatorial while other made use of ergodic theory — and extensions to hypergraphs and to digraphs were found. Roughly, Szemerédi's Regularity Lemma states that every large graph can be well approximated by the union of a bounded number of quasi-random bipartite graphs. This approximation is particularly useful to obtain counting lemmas and removal lemmas, enabling us to comprehend embeddings of sparse graphs into large dense graphs. There are countless applications of regularity, counting and removal lemmas in graph theory and in other fields, such as algebraic number theory.

We now point out an application of the directed Regularity Lemma, established by Alon and Shapira [AlSh03], to the aforementioned Erdős-Hajnal conjecture. To this end, we first reformulate this conjecture in terms of digraphs (using the notion of induced subdigraph). A *tournament* is an orientation of a complete graph. A tournament is *transitive* if it contains no directed cycles (equivalently, if it contains no directed triangles). We denote by $\alpha(T)$ the largest integer k such that a tournament T contains an induced transitive subdigraph on k vertices. Alon, Pach and Solymosi [APS01] showed that the Erdős-Hajnal conjecture is equivalent to the following statement. *For every tournament S, there exists a positive constant $\delta(S)$ such that every S-free tournament T satisfies $\alpha(T) \geq |V(T)|^{\delta(S)}$.* The directed Regularity Lemma allowed Berger, Choromanski and Chudnovsky [BCC] to construct an infinite family of prime tournaments with the Erdős-Hajnal property. A tournament S has *the Erdős-Hajnal property* if every S-free tournament T satisfies $\alpha(T) \geq |V(T)|^{\delta(S)}$. A tournament is *prime* if it cannot be obtained from another tournament by the substitution operation, which preserves the Erdős-Hajnal property. In contrast, no such result is known for graphs: the largest prime graph known to satisfy the Erdős-Hajnal property has five vertices.

Vapnik-Cervonenkis dimension

Key point: Forbidding a bipartite structure is equivalent to bounding the VC-dimension.

A natural structure to consider when dealing with a graph G is its *neighbourhood hypergraph*, whose vertices are those of G and whose hyperedges are the neighbourhoods of the vertices in G. This hypergraph reflects some of the properties of G. In particular, one may consider a classical complexity parameter of hypergraphs: the *Vapnik-Cervonenkis dimension*, a.k.a. *VC-dimension*. One can easiliy checks that the neighborhood hypergraph of G has bounded VC-dimension if and only if G does not contain any induced subgraph in the closure of some fixed bipartite graph H. Here, the *closure* of a bipartite graph H is the set of all graphs that can be obtained from H by adding edges that join two vertices in the same part of the bipartition of H. Having bounded VC-dimension is very restrictive, the major fact being that usual graph parameters can generally be bounded in terms of their fractional relaxation. This partly explains why forbidding the closure of bipartite graph H is considerably stronger than only excluding H as an induced subgraph. To illustrate this, observe that forbidding the closure of a fixed complete bipartite graph $K_{t,t}$ results in a class of sparse graphs — that is, the number of edges is $o(n^2)$ — whereas a $K_{t,t}$-free graph can have up to $\Omega(n^2)$ edges. Let us mention some results that are nearly direct consequences of bounded VC-dimension: we showed [**LuTh**] that triangle-free graphs with minimum degree $n/3$ have bounded chromatic number. unbounded chromatic number and minimum degree $c \cdot n$ for any constant c less than 1/3. We also showed [**BoTh**] that maximum triangle-free graphs avoiding any induced subdivision of a fixed graph H have bounded chromatic number. More recently, we proved [**BLT**] that the class of H-free graphs has the polynomial clique/stable set separation if H is a *split graph* — i.e., the vertex set of H can be partitioned into a clique and a stable set. This property, introduced by Yannakakis [Yan91], means that there exists a polynomial number of bipartitions such that every pair composed of disjoint clique and stable set is separated by one of these bipartitions.

The VC-dimension approach is both appealing — since we obtain sharp results — and mysterious — since no structure of the graph is produced. For instance even if we forbid a fixed randomly generated bipartite graph H on 10^{10} vertices, we will keep some control on the class of H-free graphs, even though no structural description of the class can be expected.

Nowhere-dense classes *Key point*: taxonomy of graph classes based on structural and model theoretical properties; algorithmically efficient decomposition and approximation of sparse structures.

Nešetřil and Ossona de Mendez introduced a taxonomy of graph classes, which is based on the study of shallow minor densities [**NeOs08b**, **NeOs10a**, **NeOs10c**, **NeOs10d**], which is the subject of the monograph [**NeOs12b**]. Our approach is very general, and applies to classical sparse classes defined through their geometric representations (like sphere packings [BeCu11, BeSc01], meshes [MTTV98], or intersection graphs [**NOW12**]).

Although the three standard minor-like constructions (minors, topological minors, immersions) behave very differently, their "shallow" versions are deeply related, and the main results obtained by the theory are independent of the considered type of shallow minor. The *topological resolution* of a class C of graphs is the monotone sequence of graph classes $C_0, \dots, C_t, \dots$ such that C_t contains all those graphs H such that a subdivision of H with at most $2t$ subdivision vertices per edges appears as a subgraph of a graph of C. In other words, C_t is the class of all shallow topological minors of depth t of graphs in C. A class C of graph is *somewhere dense* if there exists a threshold t such that C_t contains all graphs (or, equivalently, such that the clique number is unbounded on C_t). The class C is *nowhere dense* otherwise. A nowhere dense class C has bounded expansion if, for every integer t, not only the clique number is bounded on C_t, but so is the average degree [**NeOs06a**] (or, equivalently, for every t the chromatic number is bounded on C_t [**NeOs10d**]). The stability of this classification is not only witnessed by the independence to the considered type of shallow minors, but also by the diversity of the (non-obvious) equivalent characterizations, which can be given in terms of subgraphs, minors, partitions, game theory, stability, chromatic number, density, and homomorphism statistics. Nowhere dense classes are also exactly those classes of graphs that are quasi-wide (notion introduced by Dawar in the context of first-order model checking), as proved in [**NeOs10b**]. Examples of nowhere dense classes are abundant. They include classes of graphs omitting a fixed graph as a minor (proper minor closed classes), and classes of graphs omitting a fixed graph as a topological minor (these classes include classes of bounded degree graphs). As (rather non-trivial) examples consider:

- A d-dimensional mesh of aspect ratio bounded by a is the 1-skeleton of a complex in which every d-simplex has aspect ratio at most a. (This means that there are no "very flat" simplices.) Form the bounds obtained in [PRS94] on the maximum order of a complete shallow minor of depth r of a d-dimensional mesh of given aspect ratio, it follows that every class of d-dimensional meshes with bounded aspect ratio is nowhere dense.
- The class of the graphs whose girth is greater than the maximum degree is nowhere dense (but has unbounded chromatic number).

From an algorithmic point of view, the following problem has been considered by several researchers, including Dawar-Kreutzer and Dvořák-Král'-Thomas: *Is it true that for every monotone class of graphs C first-order model checking is fixed parameter tractable?* Existence of a fixed parameter linear-time algorithm for bounded expansion classes has been proved by

Dvořák, Král' and Thomas [DKT10]. They also obtained a nearly linear-time algorithm for classes with local bounded expansion, which still form a proper subset of nowhere dense classes.

In this context, it is natural to seek for a generalization of nowhere denses classes that would be closed under model theoretical interpretations. Obviously, such a characterization could not rely on a parameter like the clique number. However, it is plausible that the nowhere dense/somewhere dense dichotomy may be related to the complexity of the admissible subgraphs in terms of VC-dimension.

Classes C that are nowhere dense without having bounded expansion have the property that for some integer t the graphs in C_t have both unbounded chromatic number and bounded clique number. This is why we call such classes sparse Erdős classes. Structure and properties of sparse Erdős classes are difficult and lead to some core problems of finite combinatorics (extremal problems and restricted Ramsey theorems); see e.g. [NeRo77, CoGo10, KLR97, RoSc07]. The logical characterization of sparse Erdős classes puts some of the old problems in new light. For example the following problem was isolated recently by Nešetřil and Ossona de Mendez in the context of characterization of bounded expansion classes by means of their First-Order definable CSPs [**NeOs**]: *Is it true that for every monotone sparse Erdős class there exists an integer s such that the class includes s-subdivisions of graphs with arbitrarily large chromatic number and odd-girth?* A positive answer to this question would follow from a positive answer to any of the two following conjectures.

- The first conjecture has been proposed by Erdős and Hajnal [Erd69]: *Is it true that for all integers c, g there exists an integer $f(c, g)$ such that every graph G of chromatic number at least $f(c, g)$ contains a subgraph of chromatic number at least c and girth at least g?* The case $g = 4$ has been settled by Rödl [Rod77] and $g > 4$ remains open.

- The second conjecture has been proposed by Thomassen [Tho83]: *Is it true that for all integers c, g there exists an integer $f(c, g)$ such that every graph G of average degree at least $f(c, g)$ contains a subgraph of average degree at least c and girth at least g?* This problem is linked to the previous one by the fact that every graph with large minimum degree contains the 1-subdivision of a graph with large chromatic number as a subgraph [Dvo08]. The case $g = 6$ has been settled by Kühn and Osthus [KuOs04] and $g > 6$ remains open.

Unavoidable sets of induced subgraphs

Key point: Finding an appropriate unavoidable set of induced subgraphs yields inductive proofs.

A common way to prove that every graph G in some class ψ has some property P is to proceed by induction or, equivalently, by considering a minimum counterexample. The aim is to find an unavoidable set of reducible configurations for ψ. The concept of *configuration* is very close to the concept of induced subgraph. It is indeed an induced subgraph H of G together with the edges between H and $G-H$. A configuration is *reducible* if it cannot be a configuration of a minimum counterexample. A set U of configurations is *unavoidable* (for ψ), if every graph G of ψ contains a configuration of U.

Unavoidable sets can be found by a process called the Discharging Method. Its principle is as follows. We pick a graph G in ψ and assign charges to elements (vertices, edges, faces ...) of G. Using properties of ψ, we show that the total is a constant (generally negative). We then redistribute the charges according to a certain number of discharging rules that we define, so that the total charge remains unchanged. After this discharging phase, we show that either the total charge is different (generally by proving that every element has non-negative charge), which is impossible, or that G contains an element of U. This process is nothing more than an ingenious and highly effective way of averaging the charge of the elements.

For example, the Four-Colour Theorem was proved via the Discharging Method. In the original proof, Appel and Haken [ApHa77] had a set of 1936 reducible configurations. Robertson et al. [RSST97], using more refined techniques, constructed a smaller set, consisting of 633 reducible configurations.

So far, in graph theory, the Discharging Method has mainly been used to prove theorems on planar graphs. This is mainly due to the existence of Euler's Formula, which is useful to show that the initial charge is constant for many charge functions: specifically, it allows us to translate a local property (the charge of every vertex and face) into a global one (the total sum is negative). The Discharging Method has also been successfully applied to the more general case of graphs with bounded density. But ψ-free graph classes do not have bounded density: in general the clique number, and hence the density, can be arbitrarily large in such a family. However, we believe that the Discharging Method could be applied on such classes, either directly using ingenious charge functions — possibly using an infinite set of values — or through an auxiliary graph. Let us point out here that the Discharging Method has also been successfully used outside of graph theory, for instance to the checker problem. Moreover, the Discharging Method might well be useful to establish theorems on some 'basic' graph classes needed in a decomposition theorem, thereby providing the starting point of a generic inductive proof.

We also note that recent techniques used to prove that certain classes are χ-bounded (for instance, the class of bull-free graphs), consist in assigning charges to the vertices of the graphs in order to satisfy some local/global property (i.e. if locally some condition is satisfied, then globally another condition is satisfied). Even if there is no discharging here (charges are static), the spirit is very close to what has been mentioned above.

Computer-assisted proofs

Key point: Try to unify the discharging proofs and the methods of flag algebras.

The use of computers has enabled new types of proof in combinatorics: it is now possible to automatically verify a set of cases that is impossible to handle manually. The increase in computing power is constantly pushing the limits of the possible in the field. Half-theoretical, half-computational methods helped answer questions, sometimes very old. Famous examples are, once again, the proof of the Four-Colour Conjecture by Appel and Haken [ApHa77], and the non-existence of a finite projective plane of order 10 by Lam [Lam91].

Although this approach is common in some areas of combinatorics (as in combinatorics on words), it remains fairly unused in graph theory, despite the interest it presents. This comes firstly from the scepticism of some combinatorialists, especially in graph theory. For example, many combinatorialists doubted the original proof of Appel and Haken [ApHa77], and another proof was later given by Robertson et al. in [RSST97]. This scepticism may come from the fact that (general) graphs are somewhat complex objects to handle by a computer program. A few lines of easily understandable code may suffice to disprove the existence of an infinite word avoiding some factors (e.g., Abelian-squares in ternary words), but checking every graph of a class requires more complex functions, such as isomorphism or subgraph checks. Nevertheless, this way of thinking is changing. The second proof the Four-Colour Theorem is now accepted by almost all the community, even though it uses the computer more intensively. Paradoxically, this technique clarified some points of the proof, especially the discharging procedure, concentrating the effort on the method and postponing the tedious case analysis to the computer part.

We think that these computer-aided proofs will become more important in the future. On the one hand, nowadays proofs tend to become more complicated and sometimes require a long case analysis. Even if we can handle them "by hand", the probability that an error will slip through the attention of the authors and reviewers increases with the complexity of the proof. On the other hand, computers become more and more powerful, and most of the researchers have access to a computing center. We can use a variety of computer tools, such as high level languages which can handle complex structures and software suites which bring together several combinatorial libraries, like Sage. We have also now powerful proof assistants (e.g., the Four-Colour Theorem has been proved in Coq by Gonthier in 2004).

We believe that the computerization of parts of the proofs via the Discharging Method is possible. Currently, the discharging rules are found "by hand", and become increasingly complex and technical. Usually, when developing a discharging proof, "problematic" configurations are given by the tight inequalities of a linear program. Hence, a first idea is to use a kind of algebras, similar to some extent to flag algebras, in order to obtain a contradiction to a linear system.

The names of the participants to this project, as well as the keys of their publications, are in bold.

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