Problem Session – STINT Meeting – January 2017

January 26, 2017

Problems on theta-free graphs

Communicated by N. Trotignon

A *theta*-free graph is a graph with no induced subdivision of $K_{2,3}$.

Take any problem which has a positive answer for claw-free graphs and prove or disprove that the same holds for theta-free graphs. For example, recognizing claw-free graphs is polynomial-time solvable (trivially) and it is also the case for theta-free graphs (proved by Chudnovsky and Seymour). It is still open for χ-boundedness by a polynomial.

Problem 1. Is there a polynomial f such that every theta-free graph G satisfies $\chi(G) \leq f(\omega(G))$?

Another example would be to prove Erdős-Hajnal Conjecture.

Problem 2. Is there a positive constant c such that every theta-free graph on n vertices has either a clique of size n^c or a stable set of size n^c?

If the above fails one may still try it for even-hole-free graphs (which form a subclass of theta-free graphs).

Recall that $\alpha(G)$ denotes the maximum size of a stable set in the graph G.

Problem 3. Can $\alpha(G)$ be computed in polynomial time for all theta-free graph G?

Radovanović, Trotignon and Vušković proved that recognizing (theta, wheel)-free graphs can be done in polynomial time. Their proof uses a decomposition theorem where the basic graphs are graphs whose vertex-set can be partitioned into a complete graph and a line-graph with additional properties. The following natural question arises.

Problem 4. Can we decide in polynomial time whether a graph can be partitioned into a complete graph and a line-graph ?

A *jellyfish* is the union of a cycle C and three disjoints paths P_1, P_2, and P_3 that attach on the cycle (i.e. their intersection with the cycle is one of their ends). The *extremities* of the jellyfish are the ends of the P_i not in C if P_1 has length at least 1, and the vertex of P_i otherwise.

The problem 3-IN-A-JELLYFISH is: given a graph G and three vertices a, b, c of G, is there an induced jellyfish with extremities a, b, c in G?

Problem 5. Can 3-IN-A-JELLYFISH be solved in polynomial time ?

If yes, this would prove that deciding whether a graph has an induced subdivision of K_4 is polynomial-time solvable. This latter fact was recently proved by Le.

Dichromatic number of digraphs with no induced subdigraphs

Communicated by Pierre Aboulker

A k-*dicolouring* of a digraph is a partition of its vertex set into k acyclic subdigraphs. The *dichromatic number* of a digraph D, denoted by $\vec{\chi}(D)$, is the minimum k such that D has a k-dicolouring.

Harutyunyan and Mohar proved the existence of digraphs with arbitrarily large girth and dichromatic number, by means of probabilistic arguments. What about explicit constructions? [Note: the *girth* is that of the underlying undirected graph.]

Conjecture 6 (Neumann-Lara & Erdős, 1980)**.** There exists an integer L such that every graph G with chromatic number greater than L admits an orientation D such that $\vec{\chi}(D) \geq 3$.

Given a sequence S of digraphs, we define $\mathrm{Forb}(S)$ to be the class of all digraphs that do not contain any member of S as an induce subdigraph. We say that $\mathrm{Forb}(S)$ has bounded dichromatic number if every digraph in $\mathrm{Forb}(S)$ has bounded dichromatic number.

We denote by $\vec{C}_{\geq k}$ the set of directed cycles of length at least k. We let TT_k be the transitive tournament on k vertices.

Conjecture 7. $\mathrm{Forb}(TT_3, \vec{C}_{\geq 4})$ has bounded dichromatic number.

A tournament H is a *hero* if every tournament with no subtournament isomorphic to H has bounded dichromatic number. Heroes have been characterized by Berger et al. We denote by $\overleftrightarrow{K}_k$ the symmetric complete digraph on k vertices.

Conjecture 8 (Aboulker, Charbit, and Naserasr)**.** For every hero H, $\mathrm{Forb}(\overleftrightarrow{K}_2, H, K_1 + TT_2)$ has bounded dichromatic number.

Aboulker, Charbit, and Naserasr proved it for $H = TT_k$ and $H = \vec{C}_3$.

We denote by S_k the digraph with k vertices and no arcs.

Conjecture 9 (Aboulker, Charbit, and Naserasr)**.** For every hero H, $\mathrm{Forb}(\overleftrightarrow{K}_2, H, \vec{C}_3)$ has bounded dichromatic number.

Harutyunyan, Le, Newman, and Thomassé proved it for $H = \vec{C}_3$.

Conjecture 10. For every hero H and every oriented star S, $\mathrm{Forb}(\overleftrightarrow{K}_2, H, \vec{S})$ has bounded dichromatic number.

This Conjecture has been verify for $H = \vec{C}_3$ by Kierstead and Rödl [2], and for $H = TT_3$ by Aboulker, Bang-Jensen, Bousquet, Charbit, Havet, Maffray and Zamora. In both case, it was actually proved that the chromatic number was bounded (recall that the chromatic number of a digraph is the chromatic number of its underlying graph) and the later authors proposed the following conjecture:

Conjecture 11 (Aboulker, Bang-Jensen, Bousquet, Charbit, Havet, Maffray, Zamora)**.** If k is a positive integer and $\vec{S}$ is an orientation of a star, then $\mathrm{Forb}(TT_k, \vec{S})$ has bounded chromatic number.

Existence of subgraphs with large girth keeping some parameters large

Communicated by Patrice Ossona de Mendez

Recall that the *girth* of a graph is the length of a smallest cycle (or $+\infty$ if the graph is acyclic).

Conjecture 12 (Erdős and Hajnal)**.** For every positive integers k and g, there exists a constant C such that every graph G with $\chi(G) \geq C$ has a subgraph H with girth g and chromatic number at least k.

This conjecture is true for $g = 4$, i.e. there exists $f(k)$ s.t. every graph with chromatic number at least $f(k)$ has a triangle-free subgraph with chromatic number at least k (Rödl).

Conjecture 13 (Thomassen)**.** For every positive integers k and g, there exists a constant C such that every graph G with $\delta(G) \geq C$ has a subgraph H with girth g and minimum degree at least k.

Observe that taking a maximum bipartite subgraph, we only have to care about odd-length cycles. True for $g \leq 6$ (Kühn and Osthus).

An $(\leq s)$-*subdivision* of a graph G is a graph that can be obtained from G by replacing each edge by a path of length at most $s + 1$.

Conjecture 14 (Nešetřil and Ossona de Mendez)**.** There exists s such that for every k and g, there exists a constant C such that every graph G with $\delta(G) \geq k$ has a subgraph H which is an $(\leq s)$-subdivision of a graph H' with $\delta(H') = c$ and girth at least g.

There exists f such that if $\delta(G) \geq f(k)$, then G contains a (≤ 1)-subdivision of a graph with chromatic number k.

χ-boundedness of graph with no hole of length at least 5

Communicated by Frédéric Maffray

Vaidy Sivaraman showed that if a graph G has no hole of length at least 5 then $\chi(G) \leq 2^{2^{\omega(G)}}$, and claimed that Paul Seymour can prove $\chi(G) \leq 2^{\omega(G)^2}$.

Problem 15 (Sivaraman, [5])**.** Is the following true: if a graph G has no hole of length at least 5 then $\chi(G) \leq \omega(G)^2$?

Taking products of complements of C_7, Scott and Seymour [4] proved that there exists such graph with $\chi(G) \geq \omega(G)^c$ for some $1 < c < 2$.

Problem 16 (Sivaraman [5])**.** Prove that every planar graph can be partitioned into two perfect graphs without using the Four-Colour Theorem.

Splitting into subgraphs with positive genus

Communicated by Lucas Pastor

A cycle is *splitting* if its separates a graph into two subgraphs of positive genus.

Conjecture 17 (Barnette, 1982, [3] p. 166)**.** Every triangulation of a surface of genus at least 2 has a splitting cycle.

Conjecture 18. If a graph has sufficiently large genus, there is a partition of its edge set into two subgraphs with positive genus.

Conjecture 19 (Balanced version, would imply the previous one)**.** If a graph has sufficiently large genus g, there is a partition of its edge set into two subgraphs with genus at least $\lfloor g/2 \rfloor$.

Hamiltonian subgraphs with large chromatic number

Communicated by Stéphan Thomassé

A graph is *Hamiltonian* (resp. *traceable*) if it has a Hamiltonian cycle (resp. path).

Problem 20 (Thomassé)**.** Does there exist $f(k)$ such that every graph with chromatic number at least $f(k)$ has a Hamiltonian (or traceable) subdigraph with chromatic number at least k?

Colouring Jordan curves

Communicated by Louis Esperet

A family of Jordan curves is *simple* if any two of its members intersect at most once. The family is *k-touching* if no point of the plane belongs to more than k of these curves.

Conjecture 21 (Esperet, Gonçalves, Labourel [1])**.** There exists a constant c such that any simple and k-touching family of Jordan curves can be coloured with at most $k + c$ colours.

A proof, a conjecture and a decision

Collegial work

Recall that the following question was raised at the STINT meeting in Vercors.

Question 1. Is the recipe of chocolate mousse by Aurélie's mother really delicious ?

During the meeting, Aurélie made the demonstration that it is indeed delicious. All the participants tasted it and everyone was convinced. The following decision was then unanimously made.

Decision 1. A chocolate mousse must be prepared (by Aurélie) at every STINT meeting.

However, Pierre Aboulker claimed that his recipe is even more delicious. He made the following conjecture.

Conjecture 22 (Aboulker)**.** There is a better chocolate mousse than the one of Aurélie's mother. In particular, Aboulker's recipe is a lot better.

Unfortunately, he gave no evidence for this unbelievable conjecture. We hope that he could provide some in the near future.

References

[1] L. Esperet, D. Gonalves and A. Labourel, Coloring non-crossing strings, *Electronic Journal of Combinatorics* 23(4) (2016), #P4.4

[2] H. A. Kierstead and V. Rödl. Applications of hypergraph colouring to colouring graphs that do not induce certain trees. *Discrete Math.*, 150:187–193, 1996.

[3] B. Mohar and C. Thomassen. *Graphs on Surfaces*. Johns Hopkins Studies in the Mathematical Sciences. Johns Hopkins University Press, 2001.

[4] A. Scott, P. Seymour, Induced subgraphs of graphs with large chromatic number. I. Odd holes, *J. Combin. Theory Ser. B* 121 (2016), 68–84.

[5] V. Sivaraman, Some problems on induced subgraphs, arXiv:1612.09349 [math.CO], 2016.