Problem Session – STINT Meeting – July 2015

July 2, 2015

Partitioning a planar graph into 3 forests of equitable size

Communicated by L. Esperet.

A partition $(V_1, \dots, V_k)$ is *equitable* if $|V_i - V_j| \leq 1$ for all $1 \leq i, j \leq k$. A k-*colouring* of a graph is a partition into k stable sets.

Theorem 1 (Hajnal, Szemerédi [7]). *If $k \geq \Delta(G) + 1$, then G admits an equitable k-colouring.*

A natural question is whether for some graph class $\mathcal{G}$, there exists a constant c such that every graph $G \in \mathcal{G}$ has an equitable k-colouring for all $k \geq c$. However, this does not work for most graph classes. It is already false for stars, because every k-colouring of a star of size at least $2k + 2$ is not equitable (the centre forms a colour class of size 1).

However it turns out to be true for *forest partitions* and planar graphs. An *forest k-partition* is a partition $(V_1, \dots, V_k)$ of G such that $G\langle V_i \rangle$ is a forest for all $1 \leq i \leq k$.

Theorem 2 (Esperet, Lemoine, and Maffray [5]). *For all $k \geq 4$, and every planar graph G, there is an equitable forest k-partition of G.*

The proof starts by considering an acyclic 5-colouring of a planar graph, which exists by a celebrated theorem of Borodin [4]. Recall that an acyclic colouring of a graph is a partition into stable sets such that the union of any two parts induces a forest. In fact, we just need a partition in 5 sets such that the union of any two induces a forest. The parts being stable sets is useless.

Problem 3. Does every planar graph admit a forest 3-partition ?

Another natural question is the following ?

Problem 4. Does there exists a constant c such that, for all $k \geq c$, every planar graph admits an equitable acyclic k-colouring ?

Does there exists a constant c such that, for all $k \geq c$, every planar graph admits an equitable forest k-partition ?

Clique cover of claw-free graph

Communicated by P. Charbit.

A *claw* in a graph is an induced subgraph isomorphic to $K_{1,3}$. A graph is *claw-free* if it contains no claw.

$\mathrm{cc}(G)$ is the minimum number of cliques to cover all edges of a given graph.

Problem 5. Is it true that for every claw-free graph $\mathrm{cc}(G) \leq |V(G)|$?

It is not even known for graphs with stability 2, which are peculiar claw-free graphs.

Problem 6. Is it true that if $\alpha(G) \leq 2$, then $\mathrm{cc}(G) \leq |V(G)|$?

It is not difficult to prove that $\mathrm{cc}(G) \leq 2|V(G)|$ when $\alpha(G) \leq 2$. One can then derive $\mathrm{cc}(G) \leq 3|V(G)|$ when G is claw-free.

A *quasi-line graph* is a graph such that the neighborhood of every vertex can be covered by two cliques. Quasi-line graph are claw-free. For such graphs the answer is known.

Theorem 7. *If G is a quasi-line graph, then* $\mathrm{cc}(G) \leq |V(G)|$.

Disjoint directed cycles of different lengths

Communicated by A. Harutyunyan.

Conjecture 8 (Bermond and Thomassen [2])**.** If $\delta^+(D) \geq 2k-1$, then D has at least k disjoint directed cycles.

Alon [1] proved that it holds if $\delta^+(D) \geq 64k$. Bessy, Lichiardopol and Sereni [3] also proved it for tournaments.

Conjecture 9 (Lichiardopol [9])**.** There is a function g such that if $\delta^+(D) \geq g(k)$, then D contains k disjoint cycles of different lengths.

We know that if g exists, then $g(k) \geq \frac{k^2}{2}$. Lichiardopol [9] proved that $g(2) = 4$. The conjecture also holds for tournaments. Consider a multipartite digraphs obtained by blowing up a directed 3-cycle. Bensmail, Harutyunyan and Le proved the conjecture for $k^2/2$-diregular digraphs (i.e. $d^+(v) = d^-(v) = k^2/2$ for all vertex v).

Covering regular graphs by few disjoints paths

Communicated by S. Thomassé.

Conjecture 10 (Magnant and Martin [10])**.** If G is d-regular, then $V(G)$ can be covered by at most $\frac{n}{d+1}$ disjoint paths.

The number $\frac{n}{d+1}$ is sharp because of the union of disjoint K_{d+1}.

As shown by Feige, Ravi and Singh [6] is intersecting for TSP. They proved that if G is d-regular, then $V(G)$ can be covered by at most $\frac{n}{\sqrt{d}}$ disjoint paths.

Seymour posed the following question.

Problem 11. What happens when $d = \epsilon n$? The answer should be : if G is ϵn-regular, then $V(G)$ can be covered by at most $\lceil \frac{1}{\epsilon} \rceil$ disjoint paths.

Using the following result of Jackson, we can prove that it holds for $\epsilon \geq 13$.

Theorem 12 (Jackson [8]). *If G is d-regular, with $d \geq n/3$, and 2-connected, then G has a hamiltonian cycle.*

Corollary 13. *If G is d-regular, with $d \geq n/3$, then $V(G)$ can be covered by at most 2 disjoint paths.*

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