Structural aspects of tilings

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Plan de l'exposé

Context

- Focus : structure of discrete tilings
- Tileset : "Local rules"
- Tiling (produced by a tileset) : "Infinite object that respects local rules"

Context

- ➡ Focus : structure of discrete tilings
- ➡ Tileset : "Local rules"
- ➡ Tiling (produced by a tileset) : "Infinite object that respects local rules"
- ➡ Several equivalent definitions

Configurations and patterns

Discrete tilings of the plane ($\mathbb{Z}^2$)

Set of states Q

Definition. (Configuration)

Configuration : element of $Q^{\mathbb{Z}^2}$

Definition. (Pattern)

$V \subset \mathbb{Z}^2$, V *finite*

Pattern: $P \in Q^V$

Tileset and tilings

Definition. (Tileset)

Tileset $\tau = (Q, \mathcal{P}_\tau)$. $\mathcal{P}_\tau$*: finite set of patterns.*

w.l.o.g: $\mathcal{P}_\tau \subseteq Q^V$ (patterns have the same domain)

Definition. (Tiling)

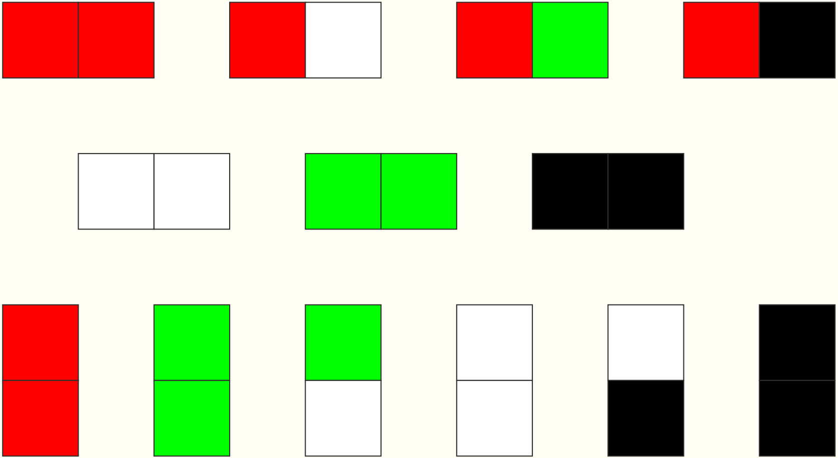
$c \in Q^{\mathbb{Z}^2}$ *is a tiling by* τ *if it contains only allowed patterns.*

i.e., $\forall x \in \mathbb{Z}^2, c|_{V+x} \in \mathcal{P}_\tau$

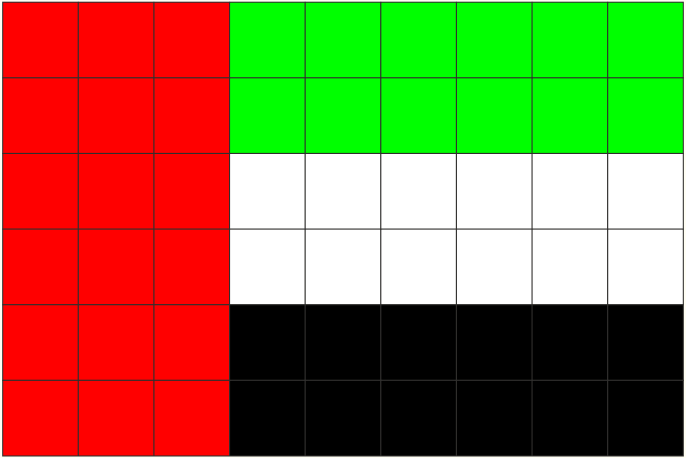
Forbidden patterns : $\mathcal{F}_\tau = Q^V \setminus \mathcal{P}_\tau$

Set oF Tilings (SFT) by τ : $\mathcal{T}_\tau$

Allowed patterns : $\mathcal{P}_\tau$



Produced tilings : $\mathcal{T}_\tau$



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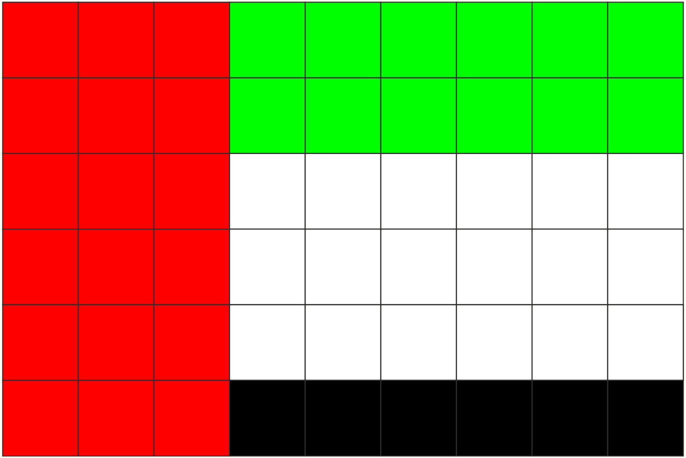

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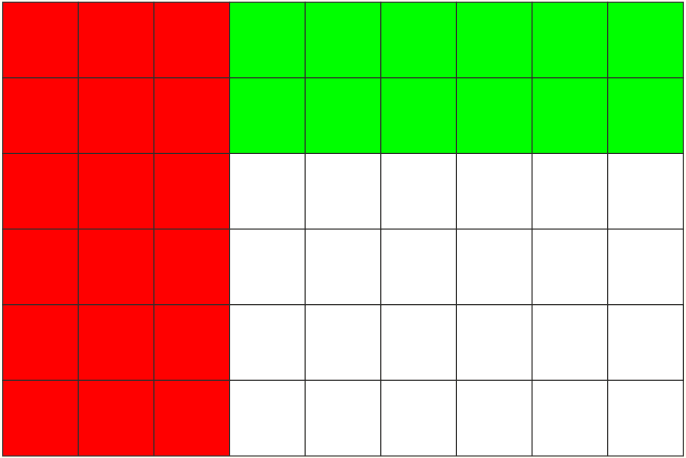

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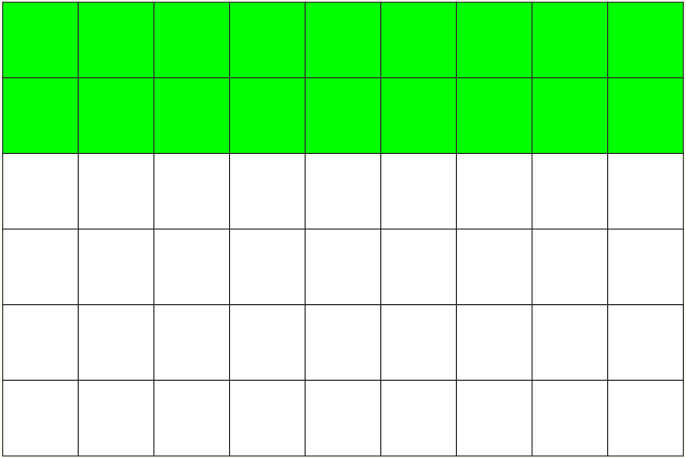


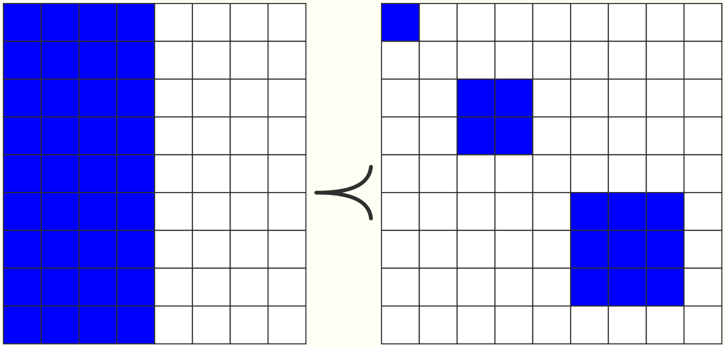
Plan de l'exposé

Pre-order

$x, y \in Q^{\mathbb{Z}^2}$

Definition.
$x \preceq y$ iff any pattern that appears in x also appears in y.

The order



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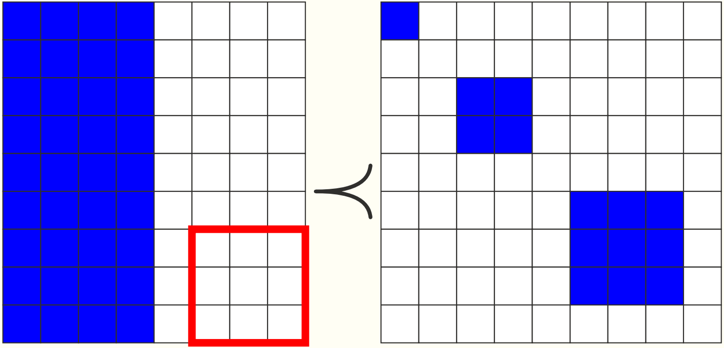

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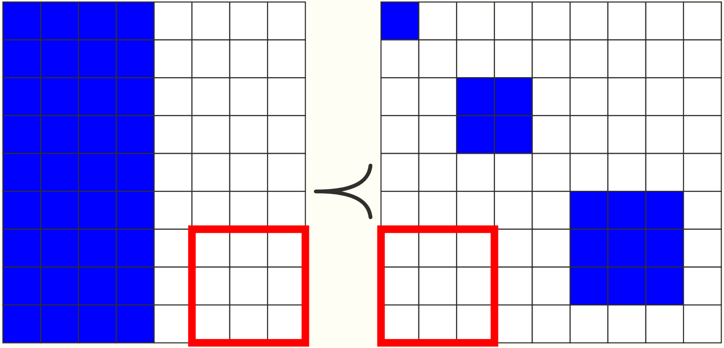

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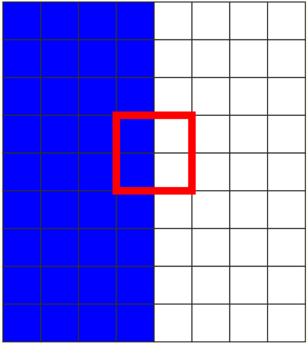
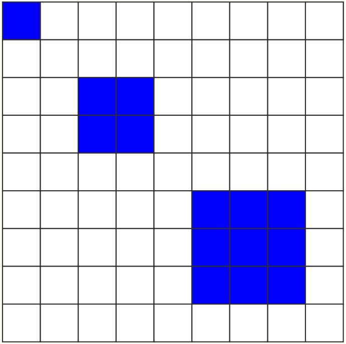

$\prec$


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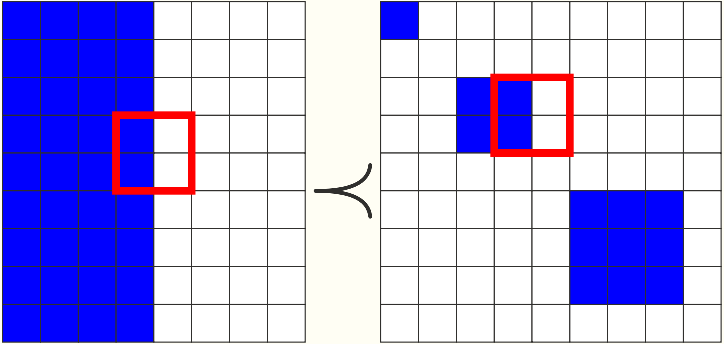

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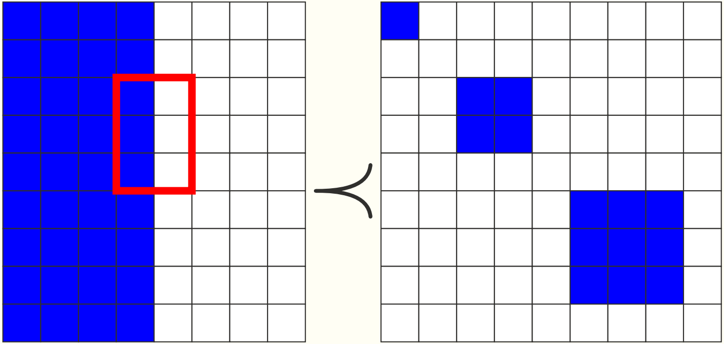

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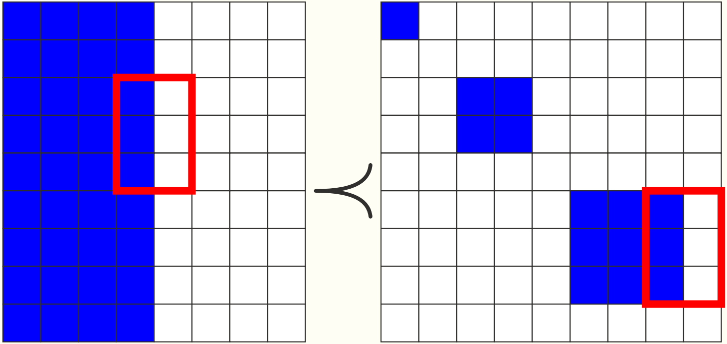

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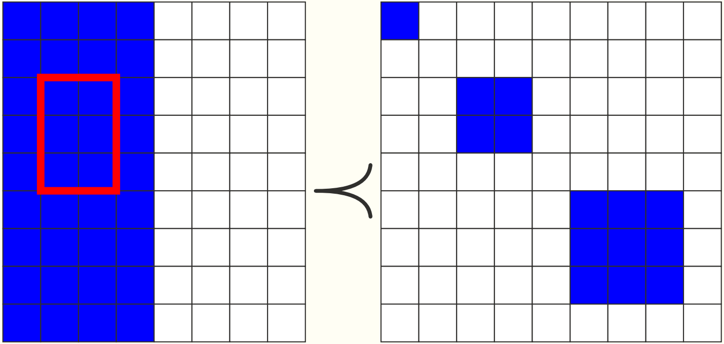

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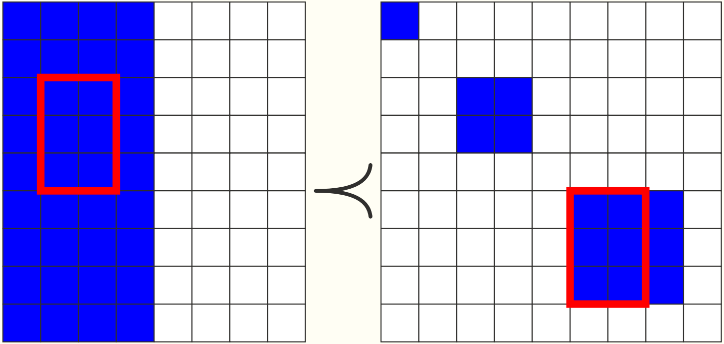

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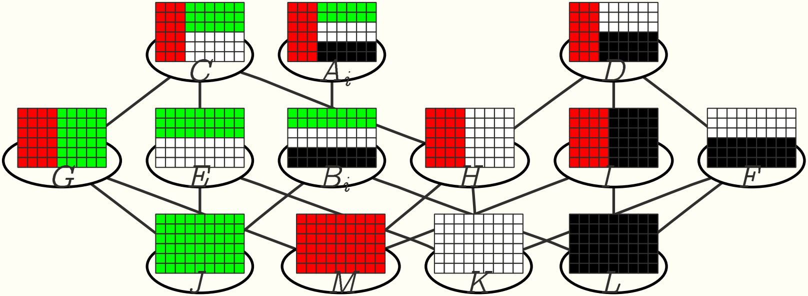

Minimal element?

Theorem (minimal elements)

For a given tileset, the corresponding set of tilings contains a minimal element for $\prec$.

Proof.

B. Durand (or Birkhoff in a topological context) ∎

Such a minimal class contains only quasiperiodic tilings.

Maximal element

Theorem

For a given tileset, the corresponding set of tilings contains a maximal element.

Proof: Prove that each increasing chain C has an upper bound.

- ➡ Let $P_1, P_2, P_3, \ldots$ be the patterns that appear in some C_i.
- ➡ Build an increasing chain of patterns Q_k such that Q_k contains all patterns $P_1 \ldots P_k$
- ➡ Q_k appears in some C_i
- ➡ The "limit" $Q = \lim Q_k$ contains all patterns.

Note

- This is still valid if Q or $\mathcal{F}_\tau$ are countably infinite...
- We do not know if a minimal always exists if Q is (countably) infinite

Plan de l'exposé

Basic definitions

- Q : discrete topology
- $Q^{\mathbb{Z}^2}$: Product topology
- Topology basis : $\mathcal{O}_P$, P a pattern

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- $Q^{\mathbb{Z}^2}$: Product topology
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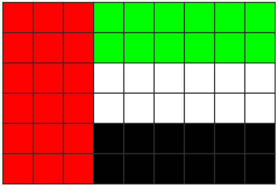
Properties : It is a Cantor space

- Compact
- Metrizable : $d(c,c') = 2^{-\min\{|i|,\ c(i)\neq c'(i)\}}$
- $0-$dimensional ($\mathcal{O}_P$ clopens)

Topological derivation

$S \subseteq Q^{\mathbb{Z}^2}$, $x \in S$

x isolated in $S \Leftrightarrow \exists P$ pattern, $\mathcal{O}_P \cap S = \{x\}$



Definition.

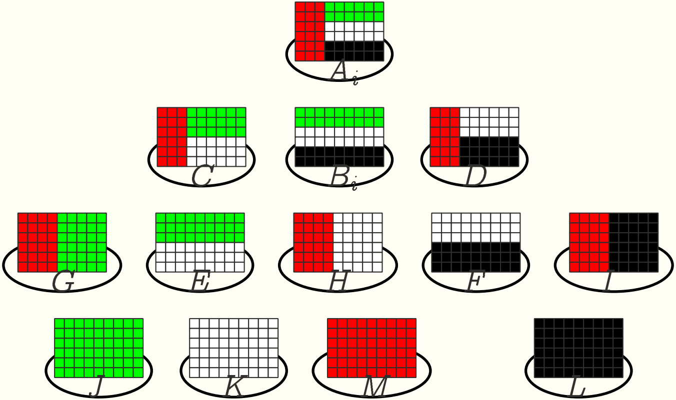
$S \rightarrow S' =$ *Set of non isolated points in* S

S subshift ($S = \mathcal{T}_\tau, \mathcal{F}_\tau$ not necessary finite) $\Rightarrow S'$ subshift

Cantor-Bendixson rank

- $S^{(0)} = S$
- $S^{(\alpha+1)} = \left(S^{(\alpha)}\right)'$
- $S^{(\lambda)} = \bigcap_{\alpha<\lambda} S^{(\alpha)}$

Example



Basic properties of C-B rank

- $\exists\lambda$ countable, $S^{(\lambda)} = S^{(\lambda+1)}$ (At most countably many finite patterns)
- Least such ordinal : *Cantor-Bendixson rank of S*
- Least ordinal λ s.t. $c \notin S^{(\lambda)} = \rho(c)$

Interesting property

Lemma

$\mathcal{T}_\tau$ *countable* $\Leftrightarrow \forall x \in \mathcal{T}_\tau, \exists \lambda, \rho(x) = \lambda.$

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$\mathcal{T}_\tau$ *countable* $\Leftrightarrow \forall x \in \mathcal{T}_\tau, \exists \lambda, \rho(x) = \lambda.$

Proof.

$\Leftarrow$: Cantor-Bendixson rank of $\mathcal{T}_\tau$ countable

$\Rightarrow$: ${\mathcal{T}_\tau}^{(\lambda)} = {\mathcal{T}_\tau}^{(\lambda+1)}$, ${\mathcal{T}_\tau}^{(\lambda)}$ perfect thus uncountable if non empty (Baire) ∎

Cardinality of $\mathcal{T}_\tau$

Theorem

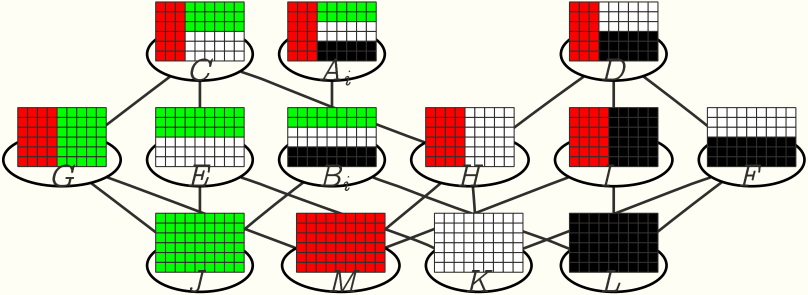
$\mathcal{T}_\tau$ is either finite, countable or has the cardinality of continuum.

Proof.

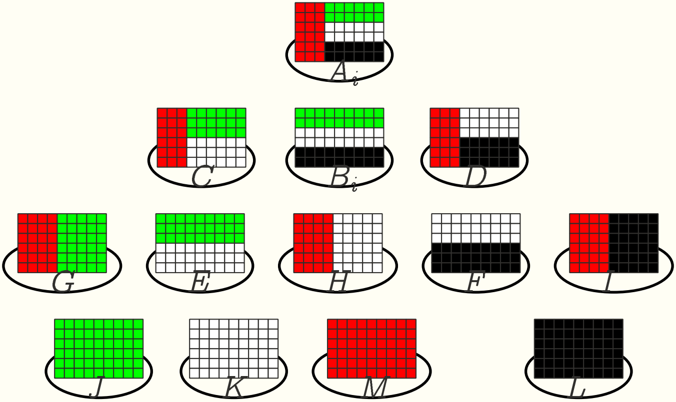
Compact, 0−dimensional. ∎

Plan de l'exposé

Pre-order



C-B ranks



First remark

x, y ranked by ρ

Lemma

$x \prec y \Rightarrow \rho(x) > \rho(y)$

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Theorem

If $\mathcal{T}_\tau$ is countable, there exists no infinite increasing chain for $\prec$

Proof.

This would give an infinite decreasing chain of ordinals. ■

Preliminary result

Theorem

S subshift that contains only periodic configurations $\Rightarrow$ S finite

Proof.

S infinite then we'll construct a sequence M_i of patterns s.t. :

- M_i square pattern centered at 0
- M_i subpattern of M_{i+1}
- $\forall i, \{x \in \mathcal{T}_\tau, M_i \in x\}$ is infinite
- $M_i \in x \Rightarrow x$ has a period greater than i

∎

Construction

- $M_0 = \emptyset$
- M_i : size $a \times a$
- C : patterns of size $(a+2(i+1)) \times (a+2(i+1))$ with M_i at their center and that are not $i+1$ periodic.
- Infinitely many $x \in S$ that contains a pattern of C (if a configuration does not contain an element of C, it is at most $i+1$ periodic)
- $M_{i+1} \in C$ s.t. there are infinetely many elements of S that contains it.

Plan de l'exposé

Non minimal tiling

Corollary.

When $\mathcal{T}_\tau$ is countable, there exists a non minimal tiling.

And now ?

Question.

What are the other tilings of $\mathcal{T}_\tau$ *?*

And now ?

Question.

What are the other tilings of $\mathcal{T}_\tau$ *?*

Theorem

There exists a tiling c with exactly one direction of periodicity.

Sketch of the proof

- ➡ There exists a tiling which is not minimal.
- ➡ There exists a tiling c which is at level 1, that is such that all tilings less than c are minimal.
- ➡ Such a tiling has exactly one direction of periodicity.

Rank of $\mathcal{T}_\tau$

Lemma

$\rho(\mathcal{T}_\tau)$ *cannot be the successor of a limit ordinal.*

Proof.

Cannot be a limit ordinal : compactness ∎

Continuation of the proof...

Proof.

- $\rho(\mathcal{T}_\tau) = \beta + 1$, $\beta = \bigcup_{i<\omega} \beta_i$
- $\mathcal{T}_\tau^{(\beta)}$ finite thus only contains periodic tilings (period p)
- $x_i \in \mathcal{T}_\tau^{(\beta_i)} \setminus \mathcal{T}_\tau^{(\beta_{i+1})}$
- w.l.o.g. x_i is not p periodic "at its center"
- $\lim x_i \in \mathcal{T}_\tau^{(\beta)}$...

∎

Tiling at level 1

Corollary.

There exists a tiling c at level 1.

Proof.

$\mathcal{T}_\tau^{(\beta-1)}$ infinite $\Rightarrow c \in \mathcal{T}_\tau^{(\beta-1)}$, c non periodic thus non minimal

But $x \prec y \Rightarrow \rho(x) > \rho(y)$, thus c is at level 1.

∎

Stucture of c

Lemma

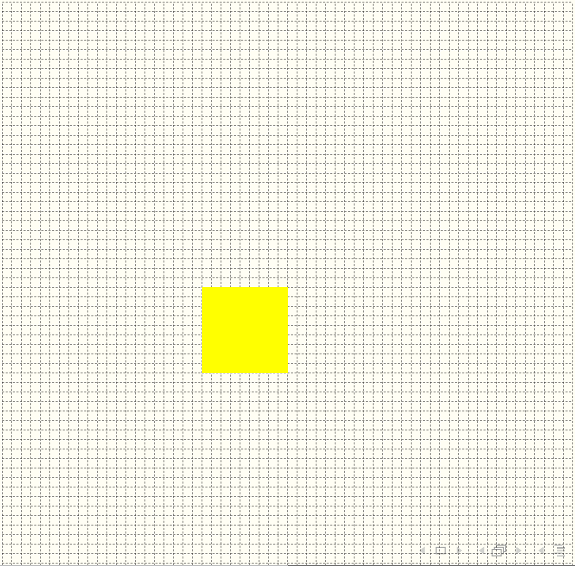
Any pattern that appears in c appears infinitely many times.

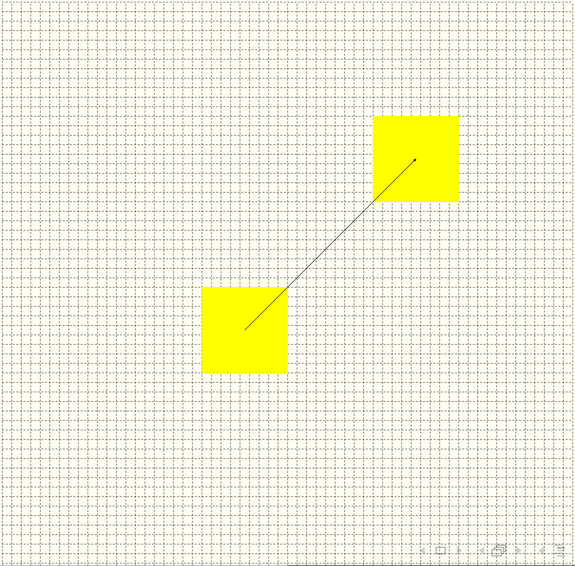
Proof.

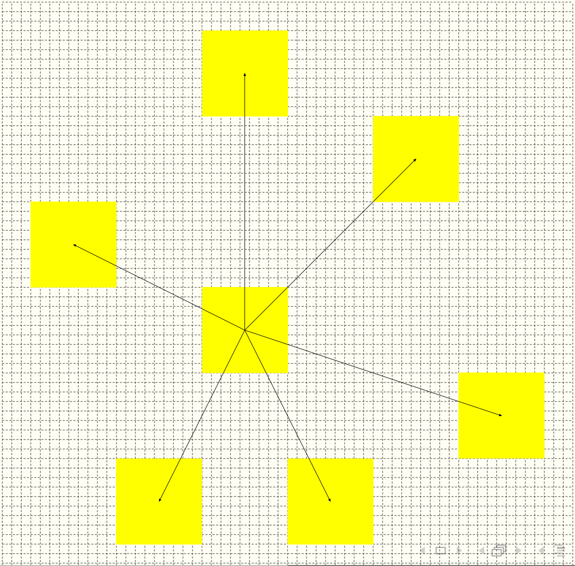
P pattern that appears only once in c

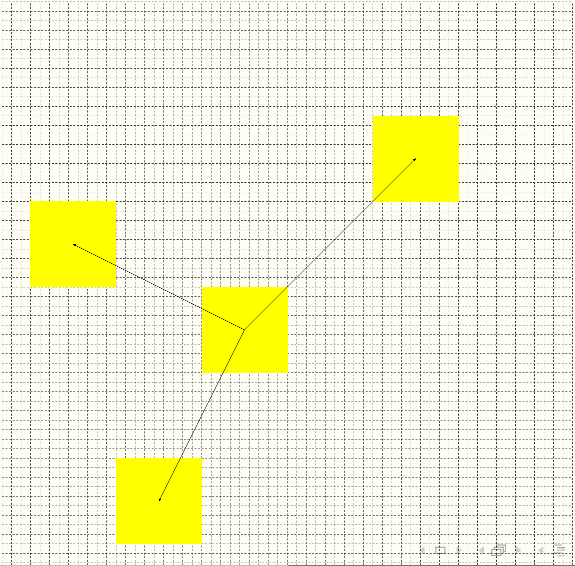
$\forall x, x \prec c, P \notin x$

Patterns of c of size $2p \times 2p$ not p periodic appears arbitrary far from P ? No : extraction ∎









To finish our proof

Theorem

There exists a tiling c with exactly one direction of periodicity.

Proof.

P isolates c in $\mathcal{T}_{\tau}^{(\beta-1)}$, appears twice, $c = \sigma(c)$. ∎

Plan de l'exposé

Conclusion

- Different points of view : Combinatorial vs. topological
- Interesting links

Questions remain :

- Characterize τ s.t. $\mathcal{T}_\tau$ is countable
- $\mathcal{T}_\tau$ countable $\Rightarrow \rho(\mathcal{T}_\tau)$ finite ?

Proof of cardinality result

Theorem

A perfect, compact, 0*−dimensional space P has cardinality of continuum.*

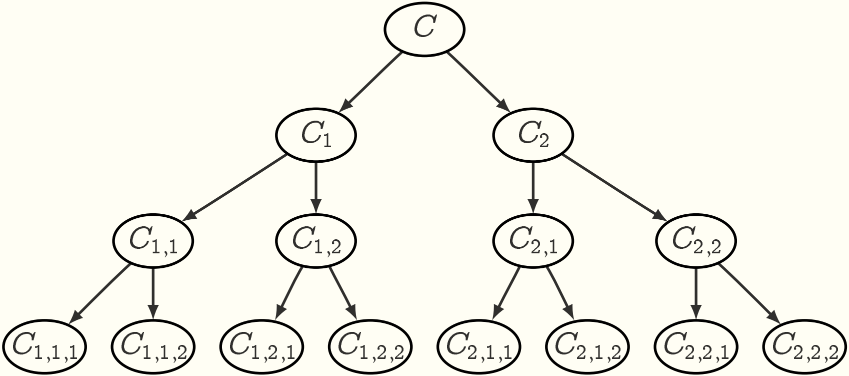
Proof.

Any non empty clopen can be split in two non empty clopen :

C clopen, $x \neq y \in C \Rightarrow P \in x, y \notin \mathcal{O}_P$

$C_1 = C \cap \mathcal{O}_P, C_2 = C \setminus C_1.$ ■

Proof of cardinality result



Proof of cardinality result

Proof.

u_n "increasing" sequence of words of length n (u_n is prefix of u_{n+1}).

$C_{u_{n+1}} \subseteq C_{u_n}$, $\bigcap_{n\in\mathbb{N}} C_{u_n} \neq \emptyset$ (compactness, $\forall n, C_{u_n} \neq \emptyset$).

∎