Lower Bounds for Geometric Diameter Problems

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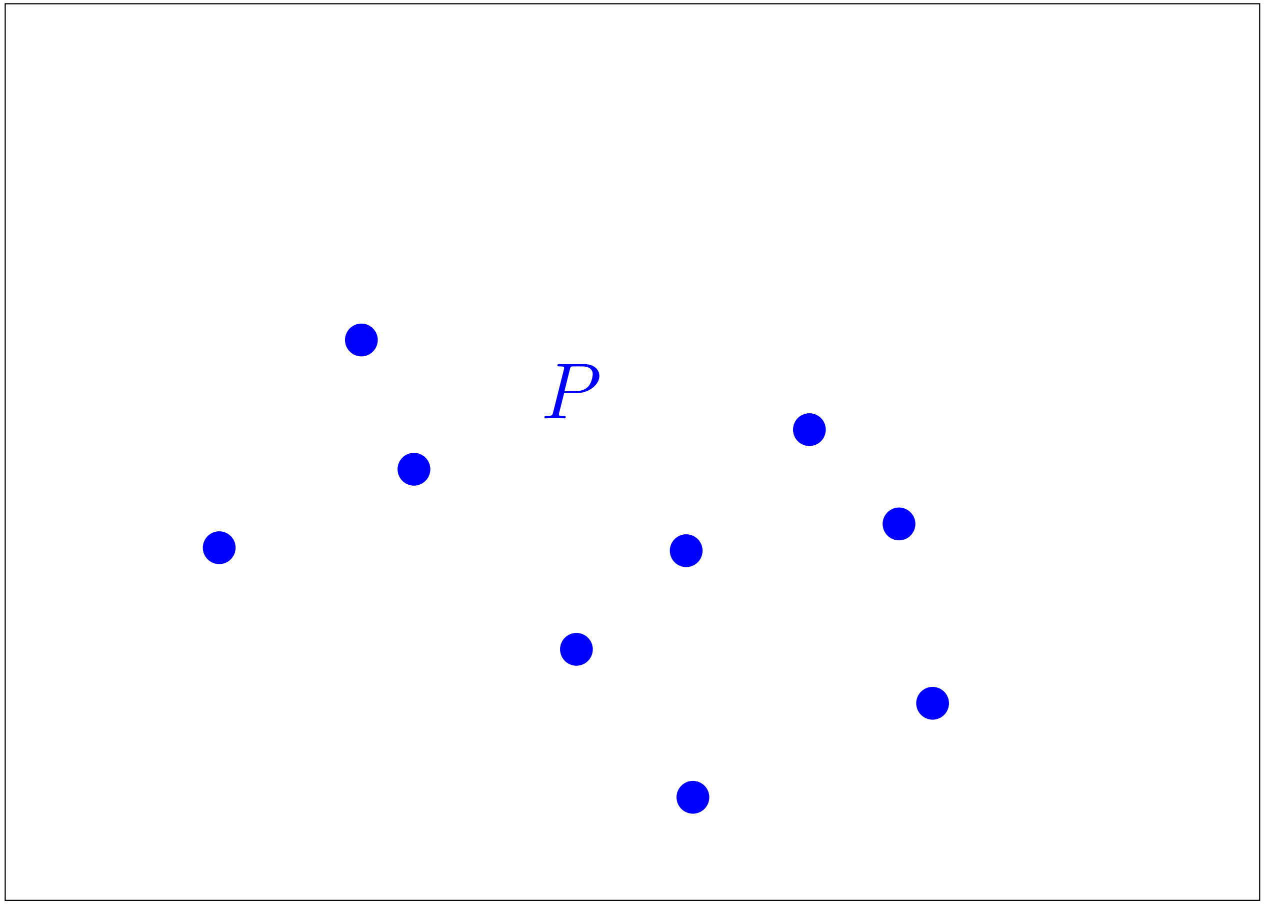
Outline

- Review of previous work on the 2D and 3D diameter problems.
- $\Omega(n \log n)$ lower bound for computing the diameter of a 3D convex polytope.
- Reduction from Hopcroft's problem to the diameter problem for point sets in $\mathbb{R}^7$.

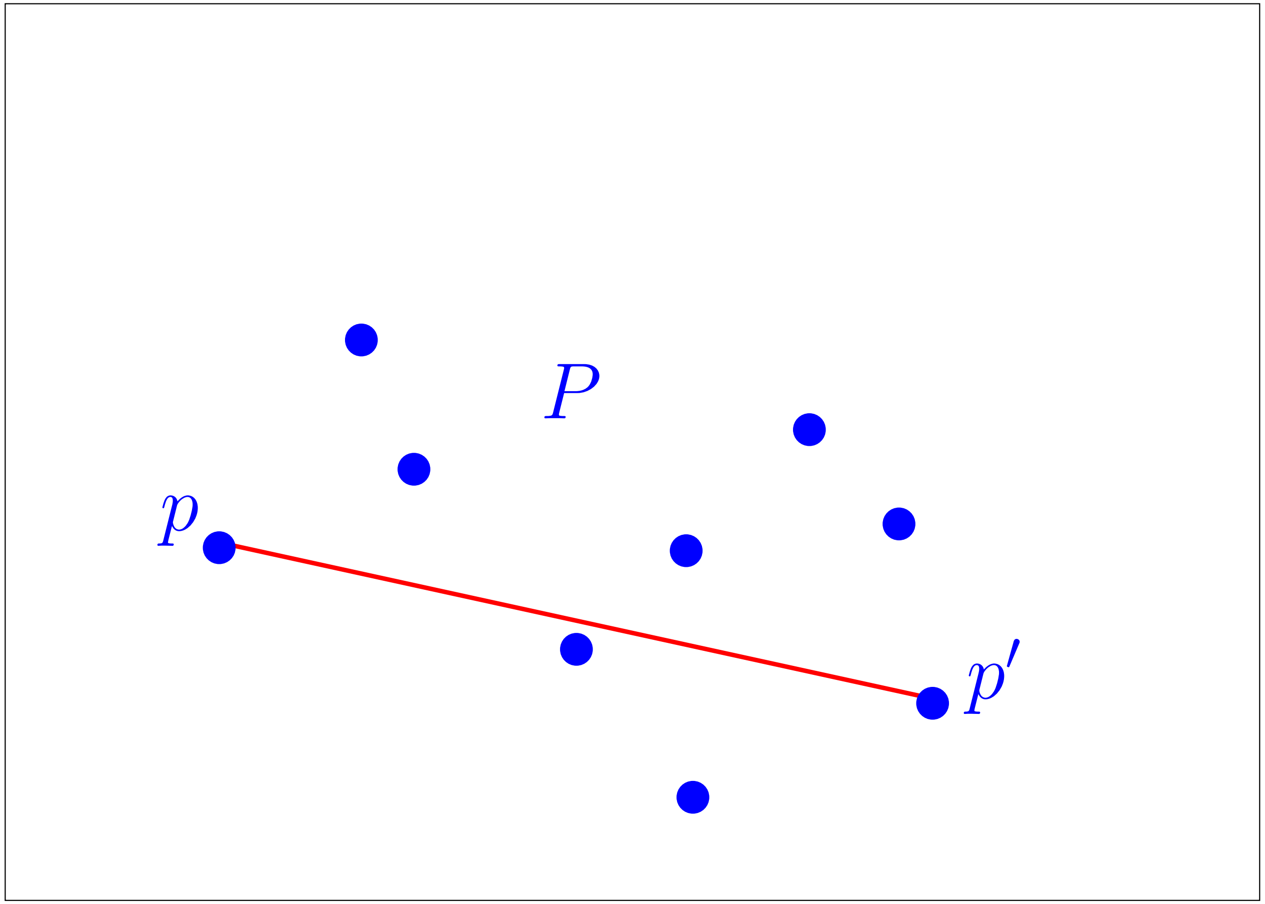
Previous work

The diameter problem



- INPUT: a set P of n points in $\mathbb{R}^d$.
- OUTPUT: $\mathrm{diam}(P) := \max\{\mathrm{d}(x,y) \mid x, y \in P\}$.

The diameter problem

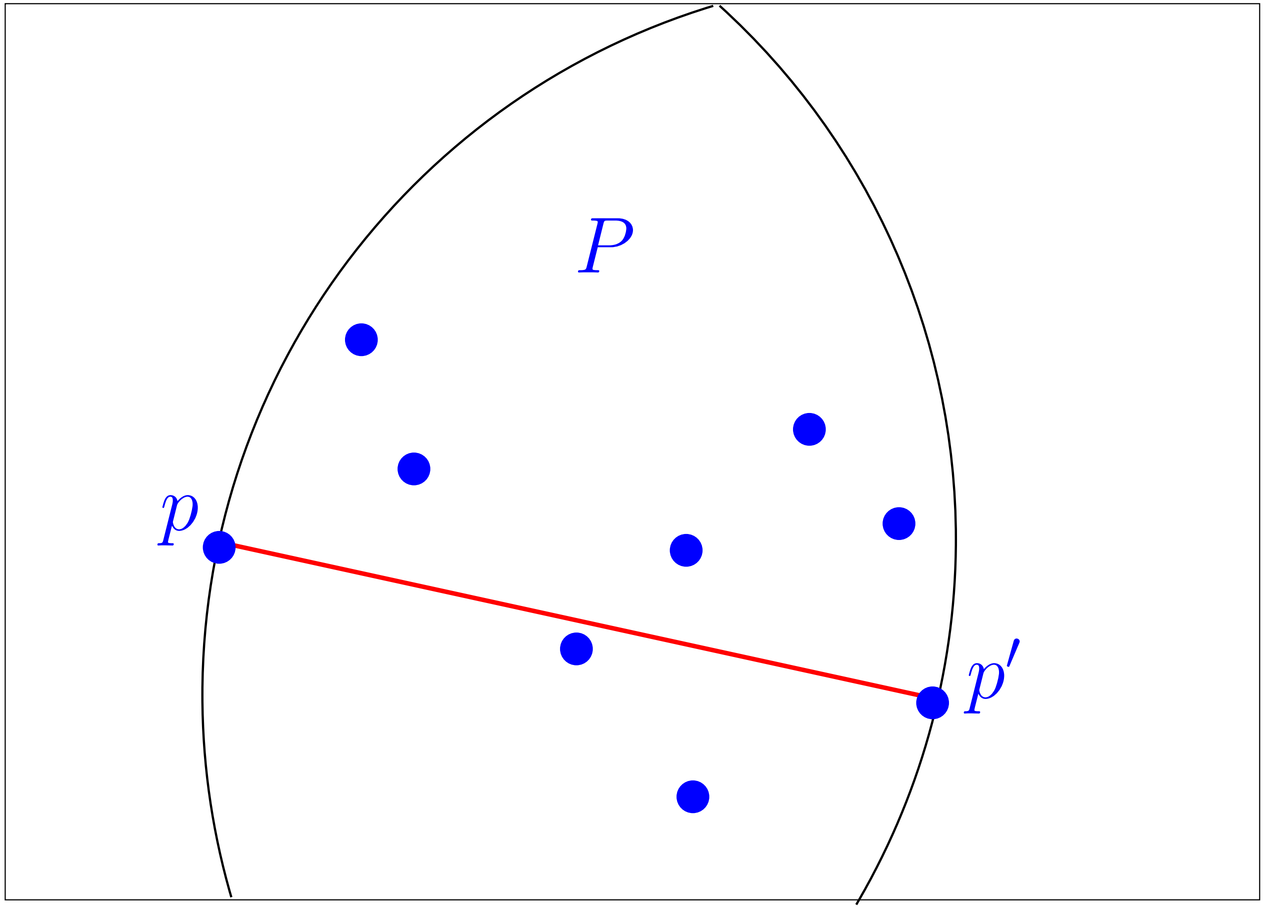


- $\mathrm{diam}(P) = \mathrm{d}(p, p')$.

Decision problem

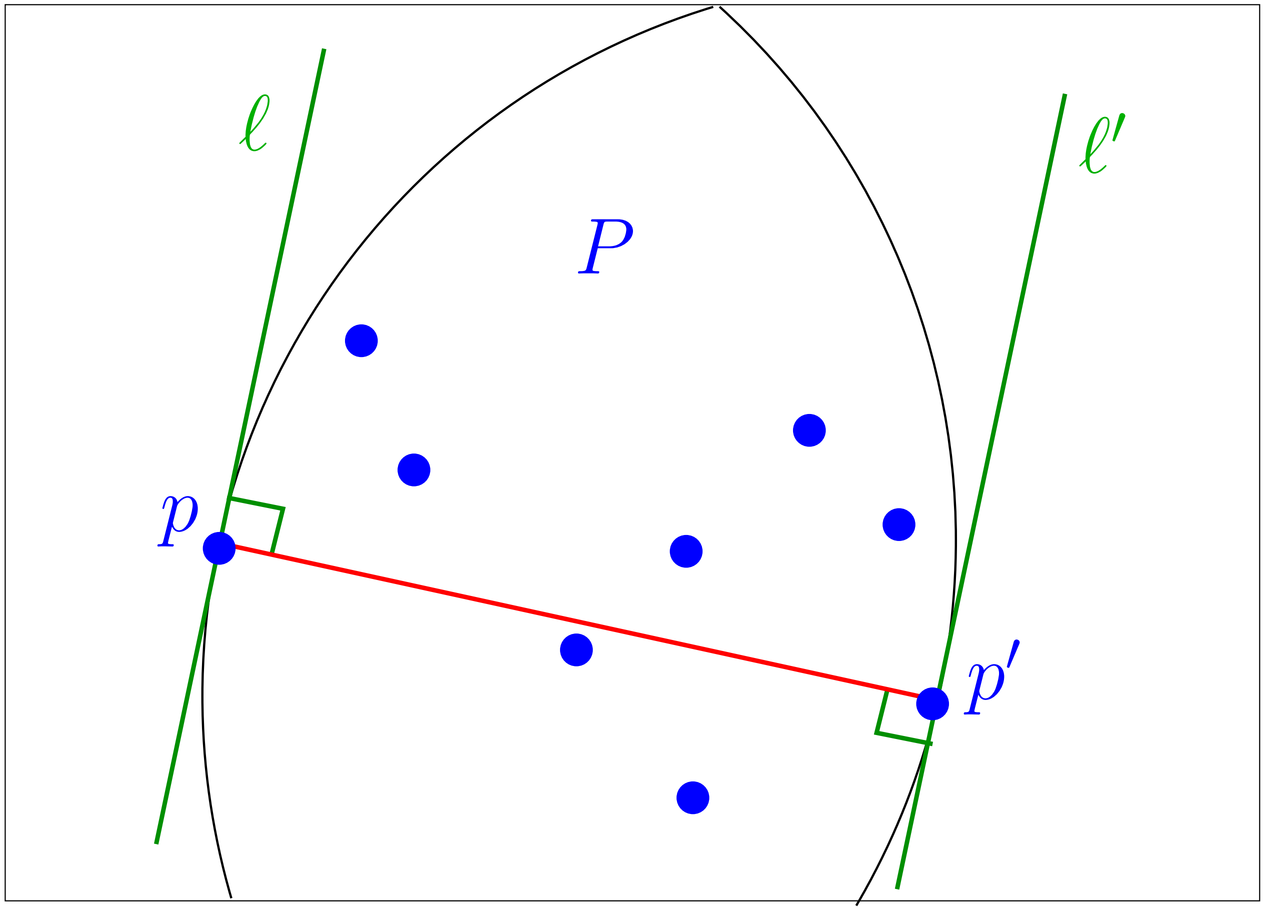
- We will give lower bounds for the *decision problem* associated with the diameter problem.
- INPUT: a set P of n points in $\mathbb{R}^d$.
- OUTPUT:
  - YES if $\mathrm{diam}(P) < 1$
  - NO if $\mathrm{diam}(P) \geqslant 1$

Observation



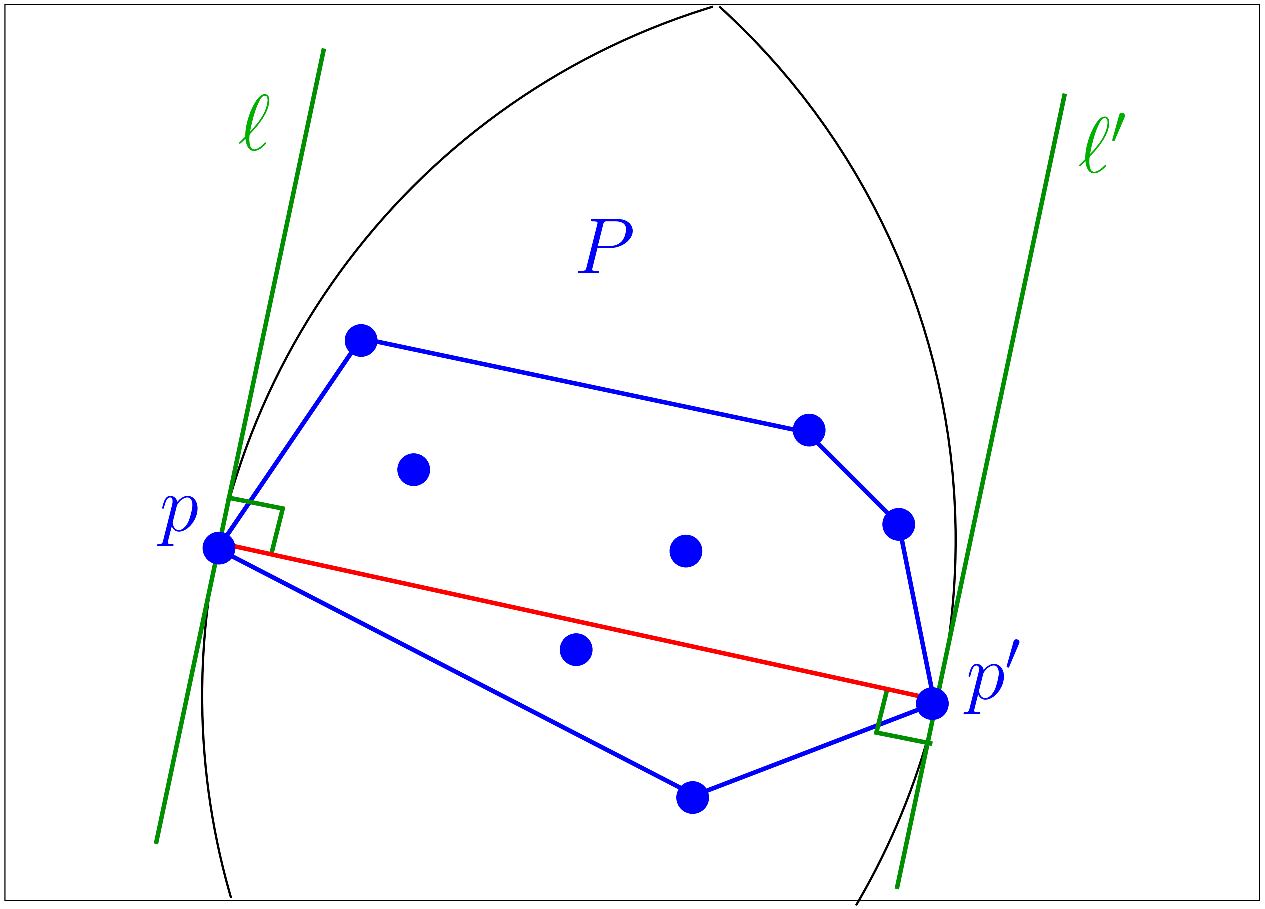
- P lies in the intersection of the two balls with radius $\mathrm{d}(p, p')$ centered at p and p'.

The diameter problem



- P lies between two parallel hyperplanes through p and p'. We say that (p, p') is an *antipodal pair*.

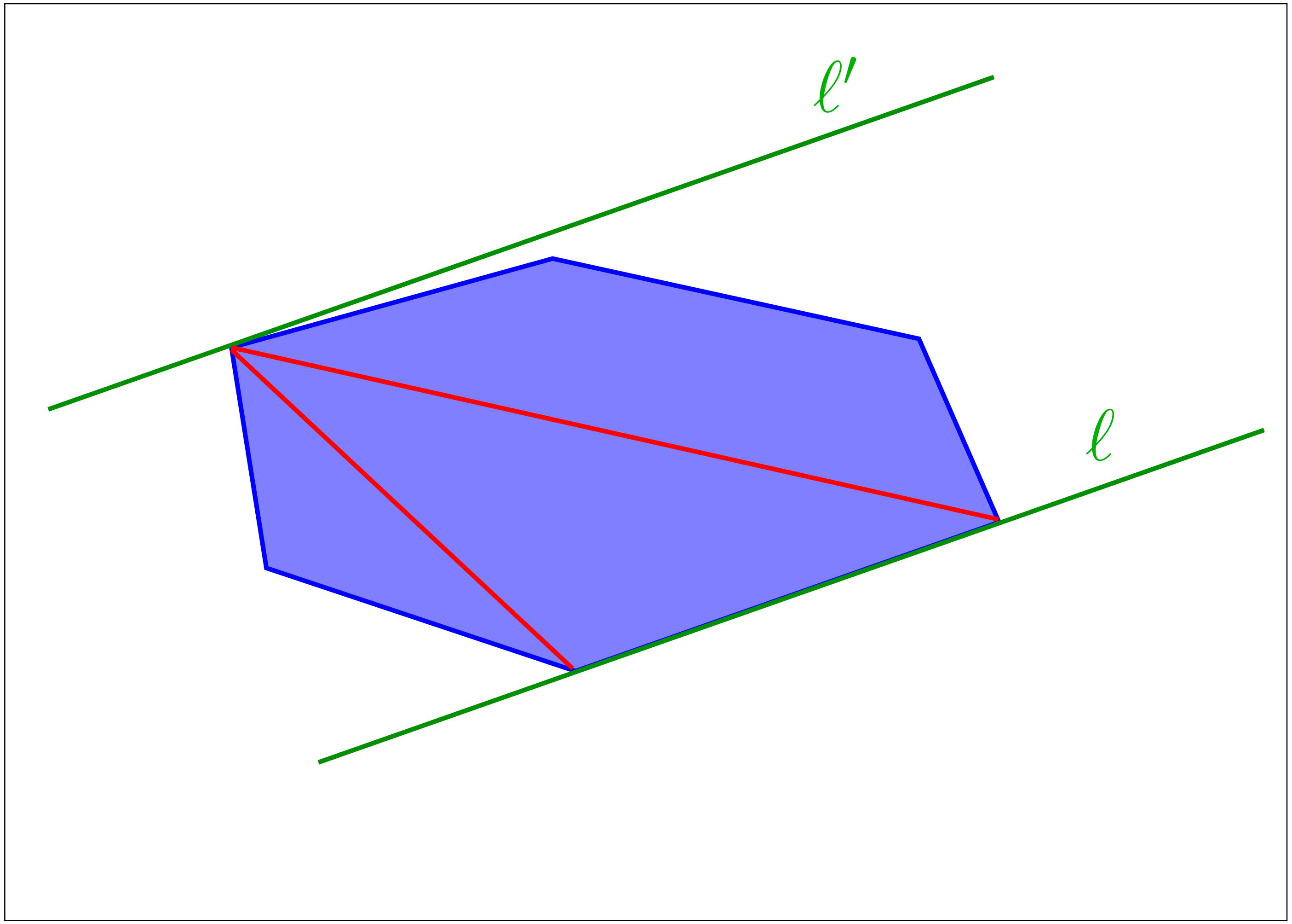
The diameter problem



- Any antipodal pair (and therefore any diametral pair) lies on the convex hull $\mathrm{CH}(P)$ of P.

Finding the antipodal pairs

- The rotating calipers technique.



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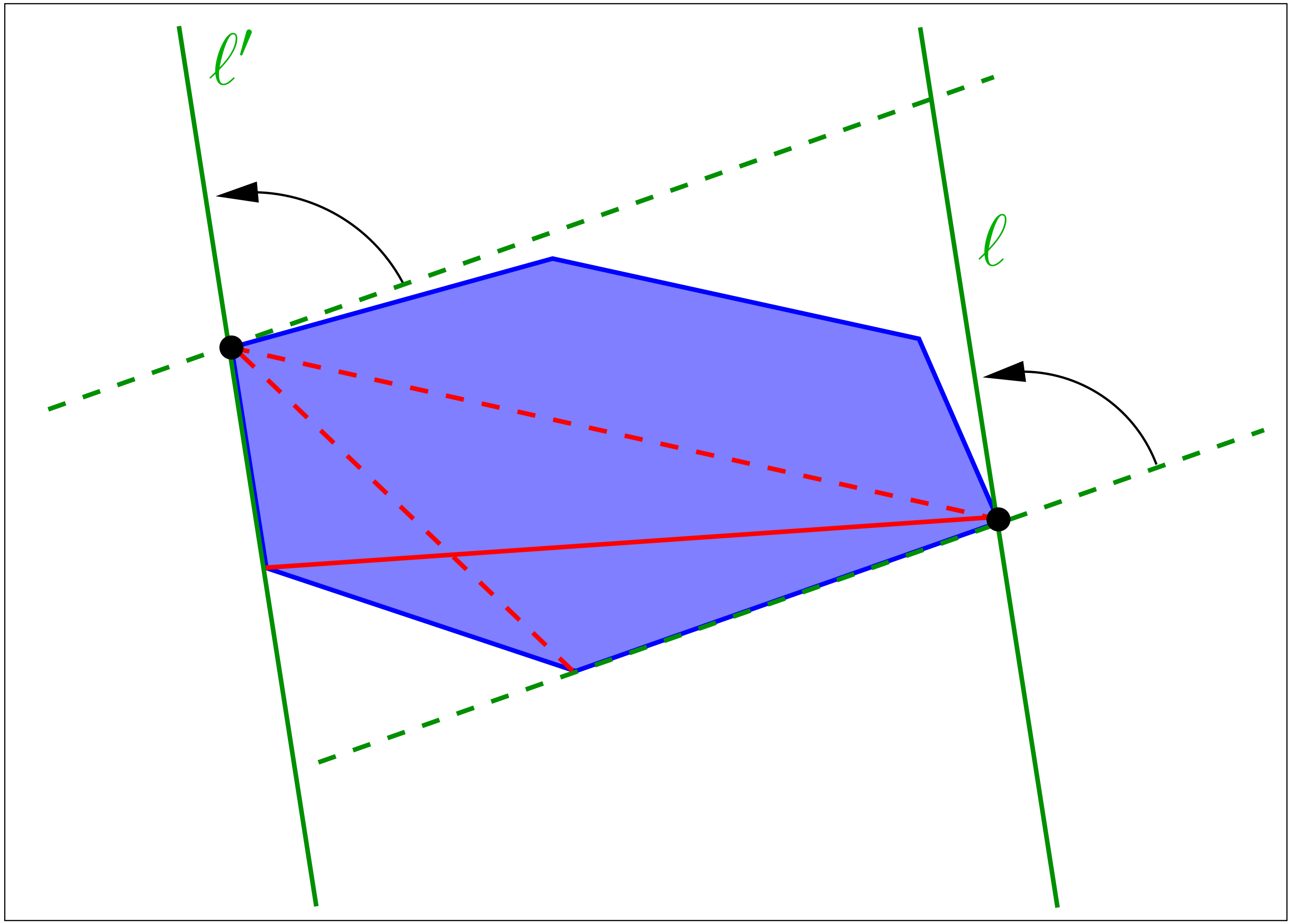

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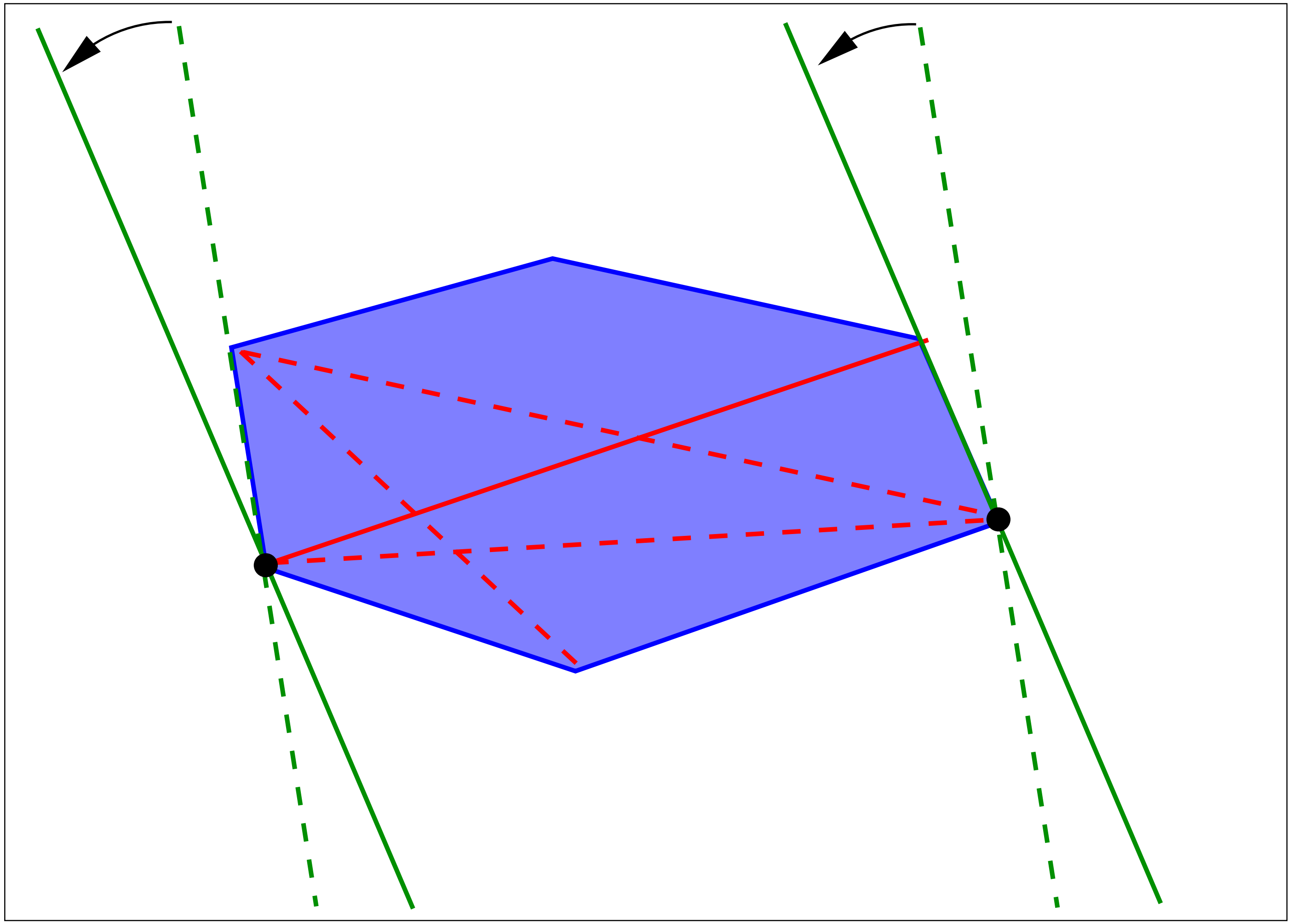

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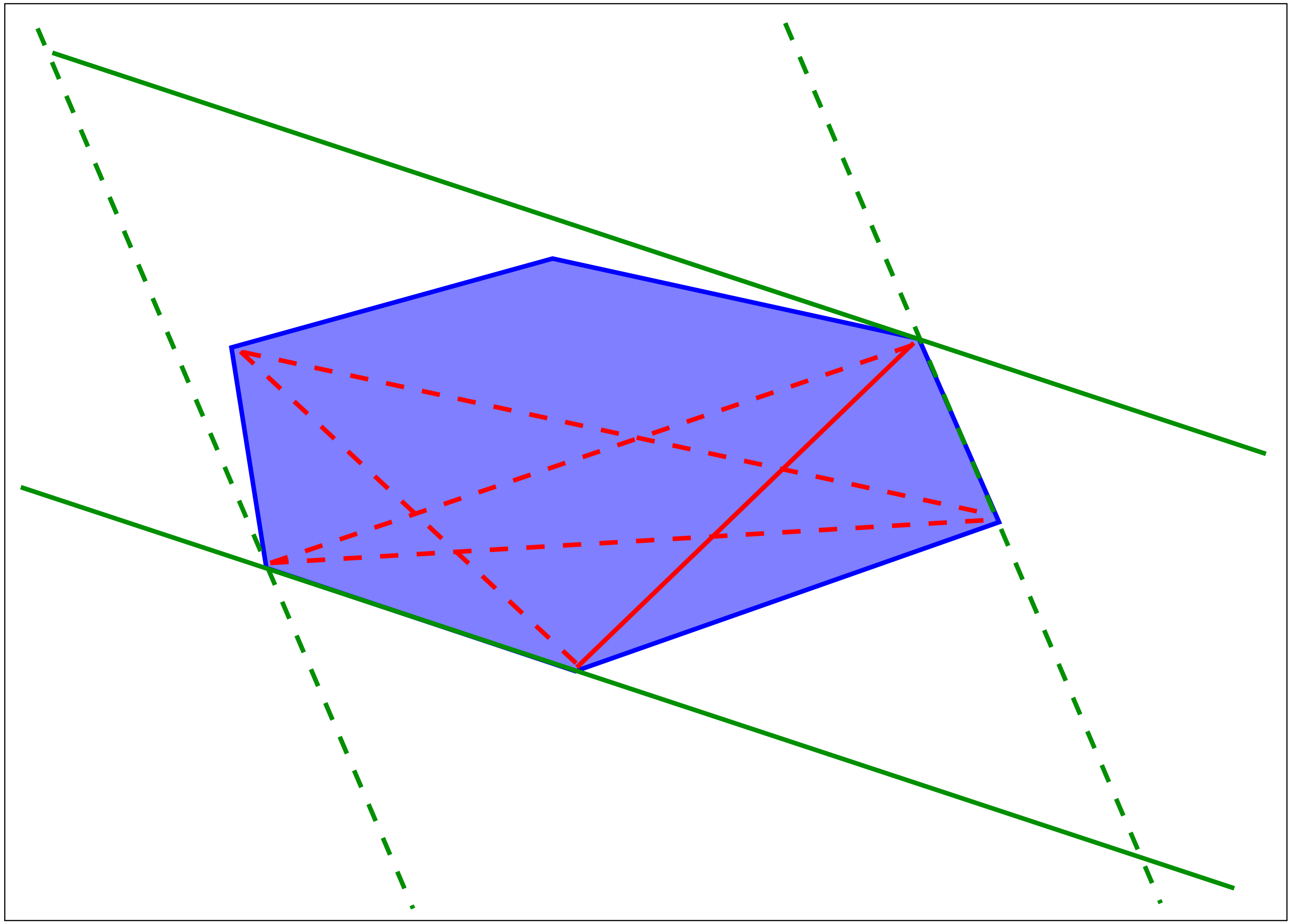

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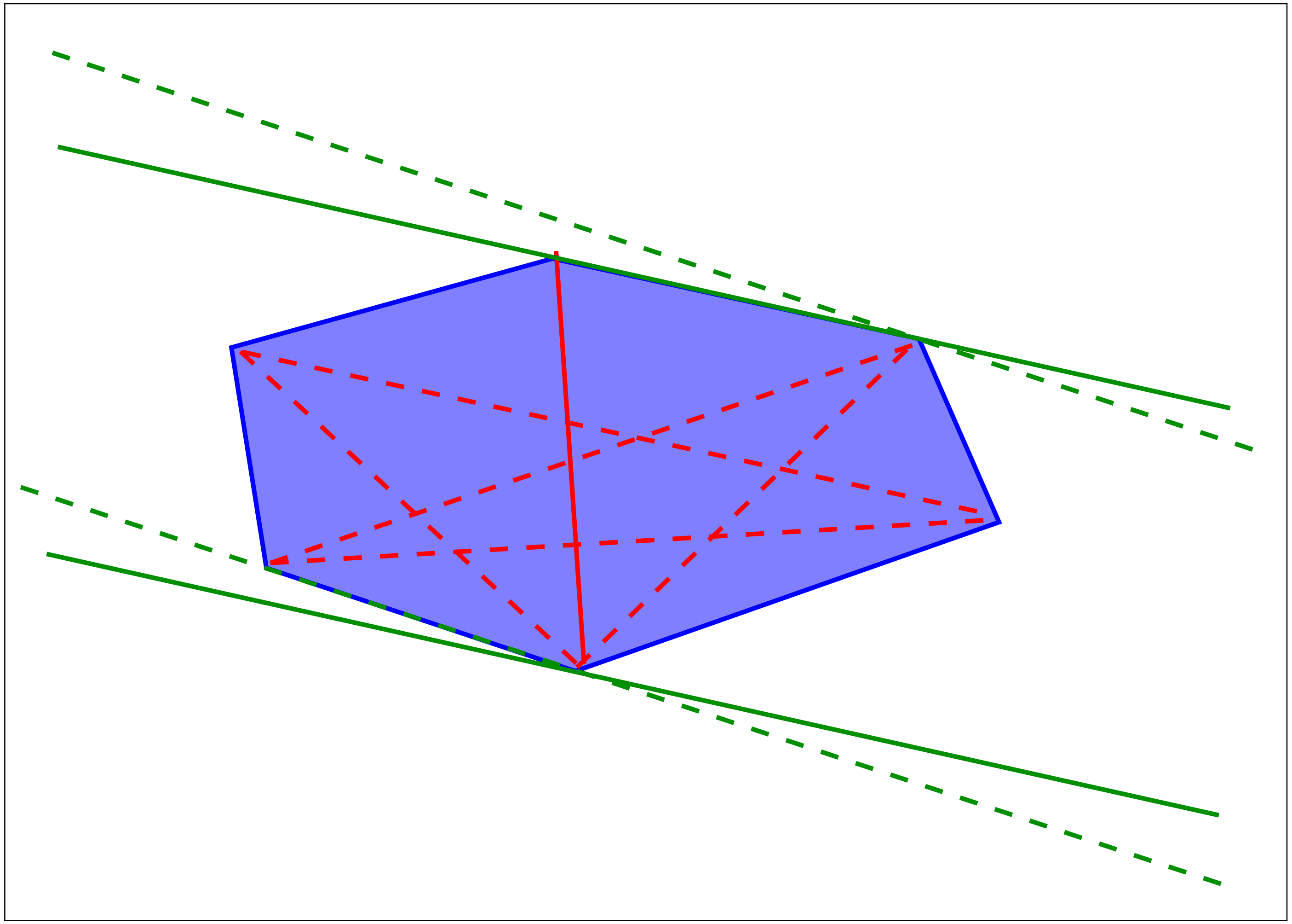


Computing the diameter of a 2D-point set

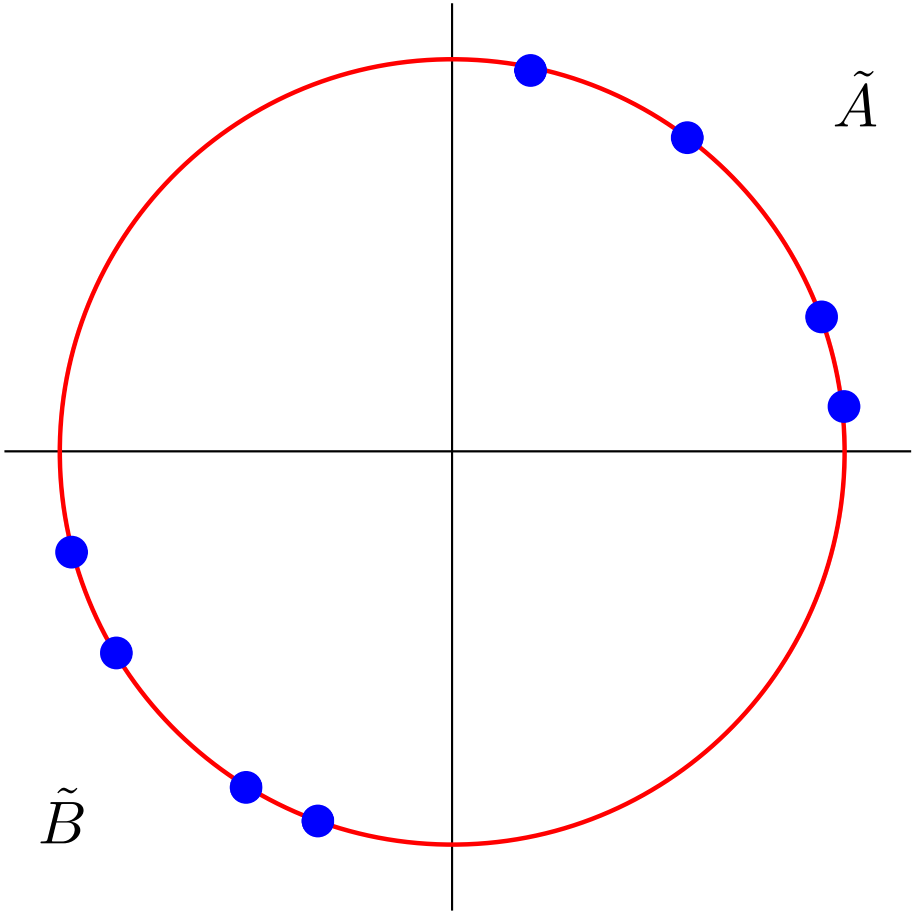
- Compute the convex hull $\mathrm{CH}(P)$ of P.
  - $O(n \log n)$ time.
- Find all the antipodal pairs on $\mathrm{CH}(P)$.
  - There are at most n such pairs in non–degenerate cases.
  - $O(n)$ time using the rotating calipers technique.
- Find the diametral pairs among the antipodal pairs.
  - $O(n)$ time by brute force.
- Conclusion:
  - The diameter of a 2D-point set can be found in $O(n \log n)$ time
  - The diameter of a convex polygon can be found in $O(n)$ time.

Diameter in $\mathbb{R}^3$ and higher dimensions

- Randomized $O(n \log n)$ time algorithm in $\mathbb{R}^3$ (Clarkson and Shor, 1988).
  - Randomized incremental construction of an intersection of balls and decimation.
- Deterministic $O(n \log n)$ time algorithm in $\mathbb{R}^3$ (E. Ramos, 2000).
- In $\mathbb{R}^d$, algorithm in $n^{2-2/(\lceil d/2 \rceil+1)} \log^{O(1)} n$ (Matoušek and Schwartzkopf, 1995).

Lower bound on the diameter

- $\Omega(n \log n)$ lower bound in $\mathbb{R}^2$.
  - Reduction from Set Disjointness.
    Given $A, B \subset \mathbb{R}$, decide if $A \cap B = \emptyset$.



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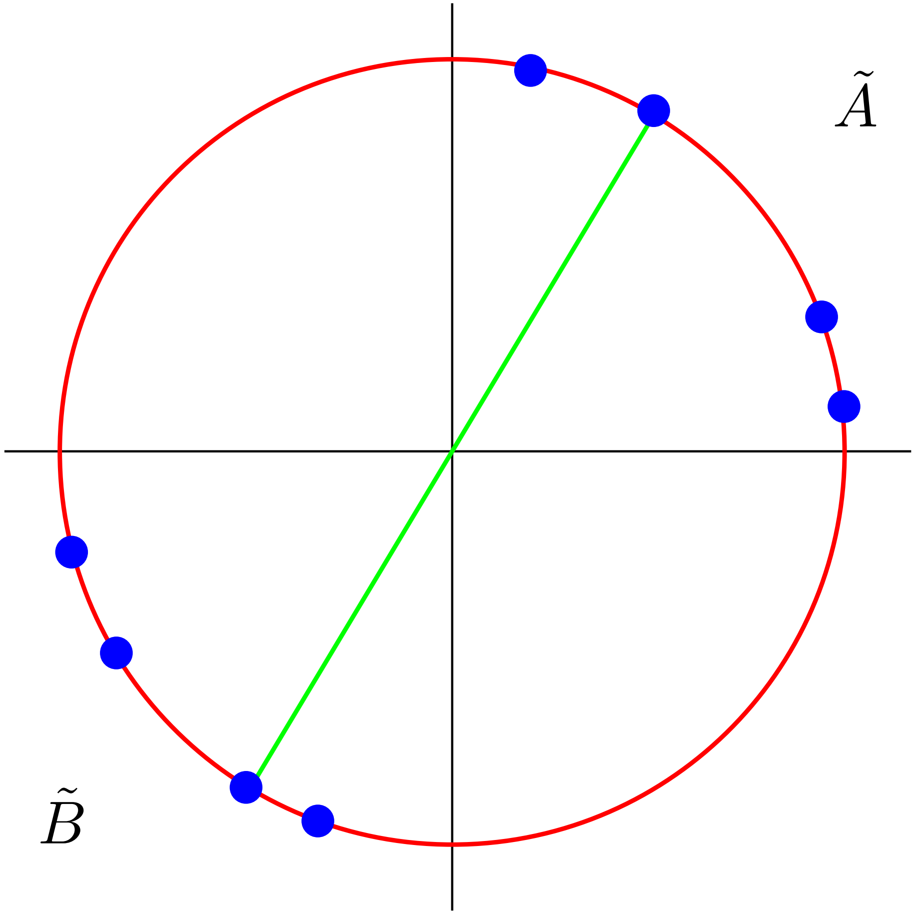


Diameter of a polytope

- The diameter of a convex polygon in $\mathbb{R}^2$ can be found in $O(n)$ time.
- Can we compute the diameter of a convex $3D$-polytope in linear time?
  - No, we give an $\Omega(n \log n)$ lower bound.

Model of computation

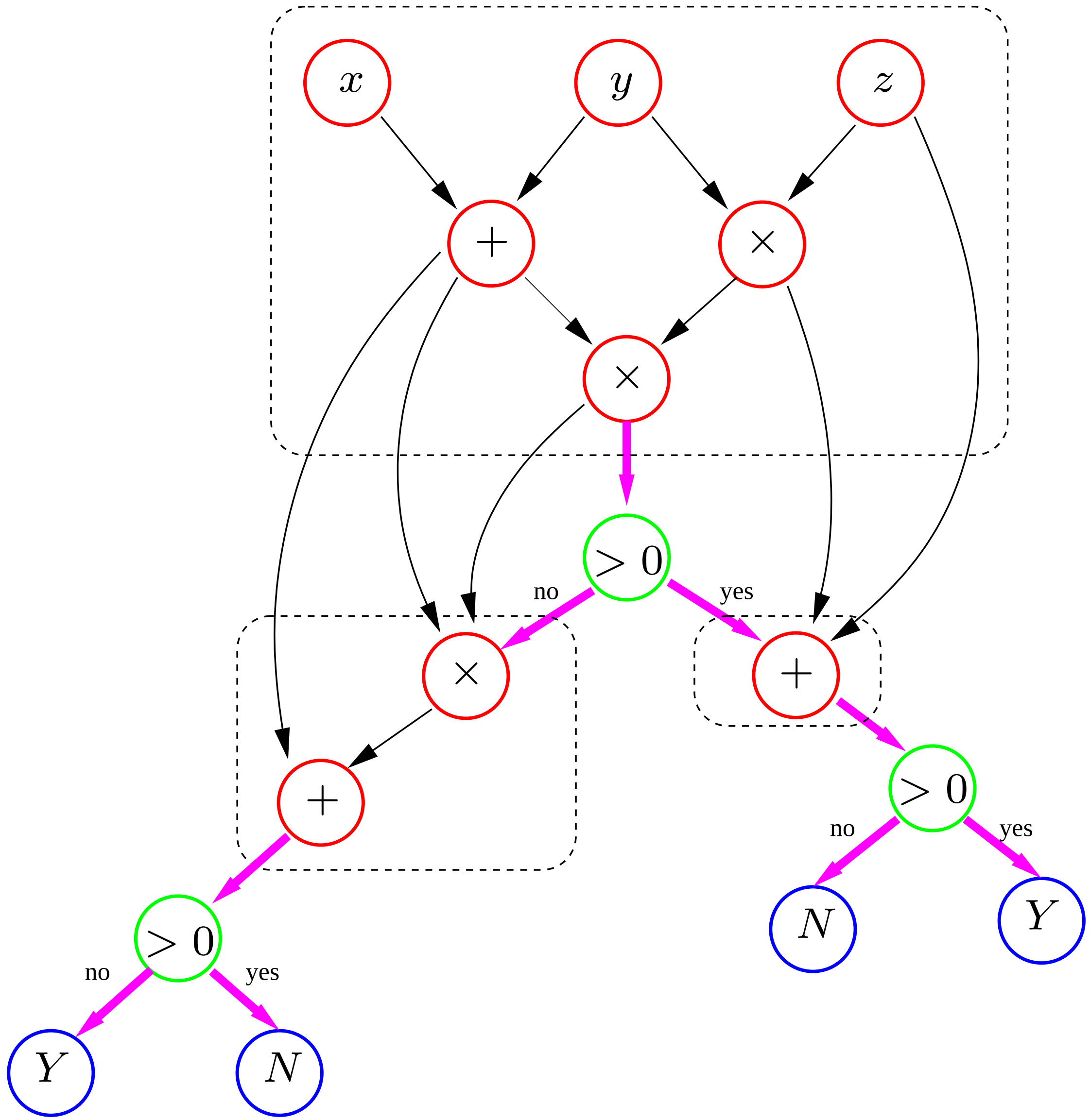
Real-RAM

- Real Random Access Machine.
- Each registers stores a *real* number.
- Access to registers in unit time.
- Arithmetic operation $(+, -, \times, /)$ in unit time.

Algebraic computation tree

- Input: $\bar{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$.
- Output: YES or NO
- It is a dag with 3 types of nodes
  - Computation nodes:
    - a real constant,
    - some input number x_i, or
    - an operation $\{+, -, \times, /, \sqrt{\cdot}\}$ performed on ancestors of the current node.
  - Branching nodes arranged in a tree: compares with 0 the value obtained at a computation node that is an ancestor of the current node.
  - Leaves: YES or NO

Algebraic computation tree: example



Algebraic computation tree (ACT)

- We say that an ACT *decides* $S \subset \mathbb{R}^n$ if
  - $\forall (x_1, \ldots, x_n) \in S$, it reaches a leaf labeled YES, and
  - $\forall (x_1, \ldots, x_n) \notin S$, it reaches a leaf labeled NO.
- The ACT model is stronger than the real–RAM model.
- To get a lower bound on the worst-case running time of a real-RAM that decides S, it suffices to have a lower bound on the *depth* of all the ACTs that decide S

Theorem (Ben-Or)**.** *Any ACT that decides S has depth*

$$\Omega(\log(\textit{number of connected components of } S)).$$

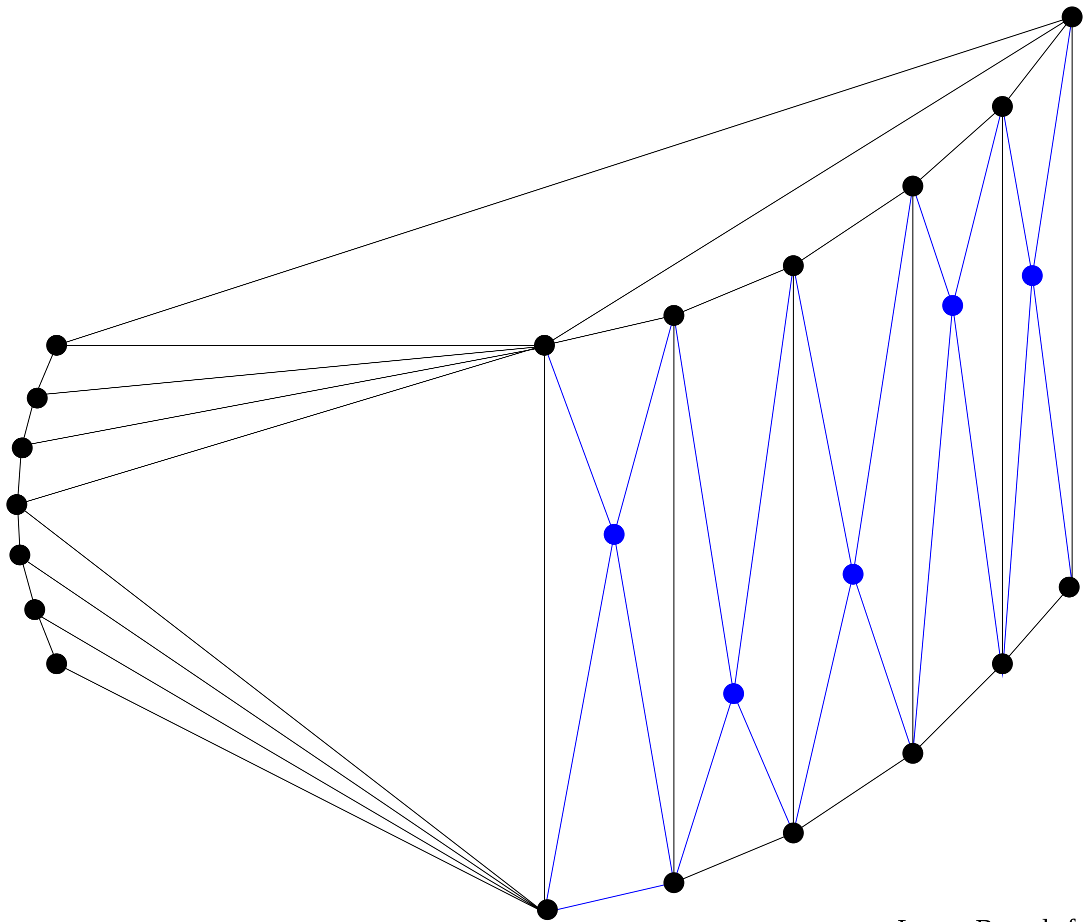
Lower bound for 3D convex polytopes

Problem statement

- We are given a convex 3-polytope P with n vertices.
- P is given by the coordinates of its vertices and its *combinatorial structure*:
  - All the inclusion relations between its vertices, edges and faces.
  - The cyclic ordering of the edges of each face.
- Remark: the combinatorial structure has size $O(n)$.
- Problem: we want to decide whether $\mathrm{diam}(P) < 1$.
- We show an $\Omega(n \log n)$ lower bound. Our approach:
  - We define a family of convex polytopes.
  - We show that the sub-family with diameter < 1 has $n^{\Omega(n)}$ connected components.
  - We apply Ben-Or's bound.

Polytopes $P(\bar{\beta})$

- The family of polytopes is parametrized by $\bar{\beta} \in \mathbb{R}^{2n-1}$.
- When n is fixed, only the $2n - 1$ blue points change with $\bar{\beta}$.



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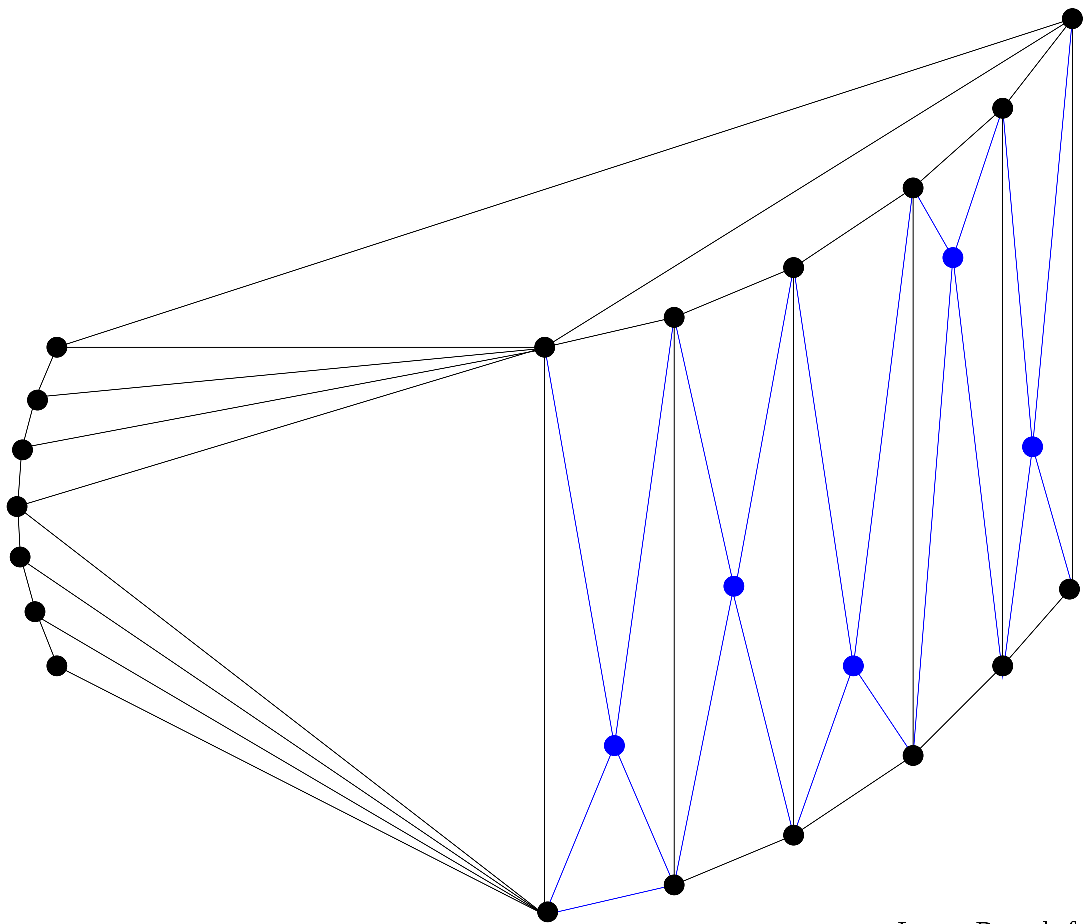
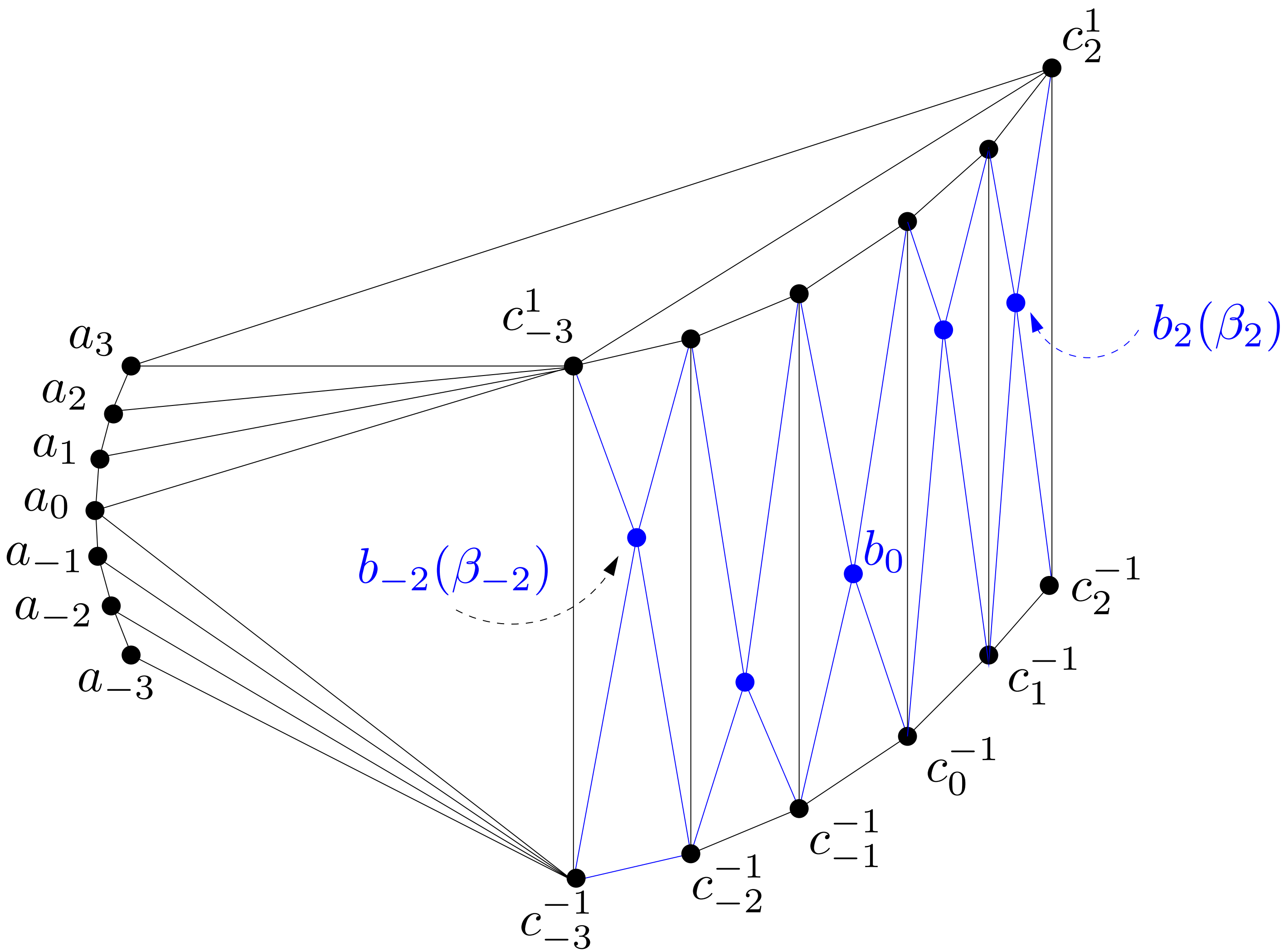


Notation

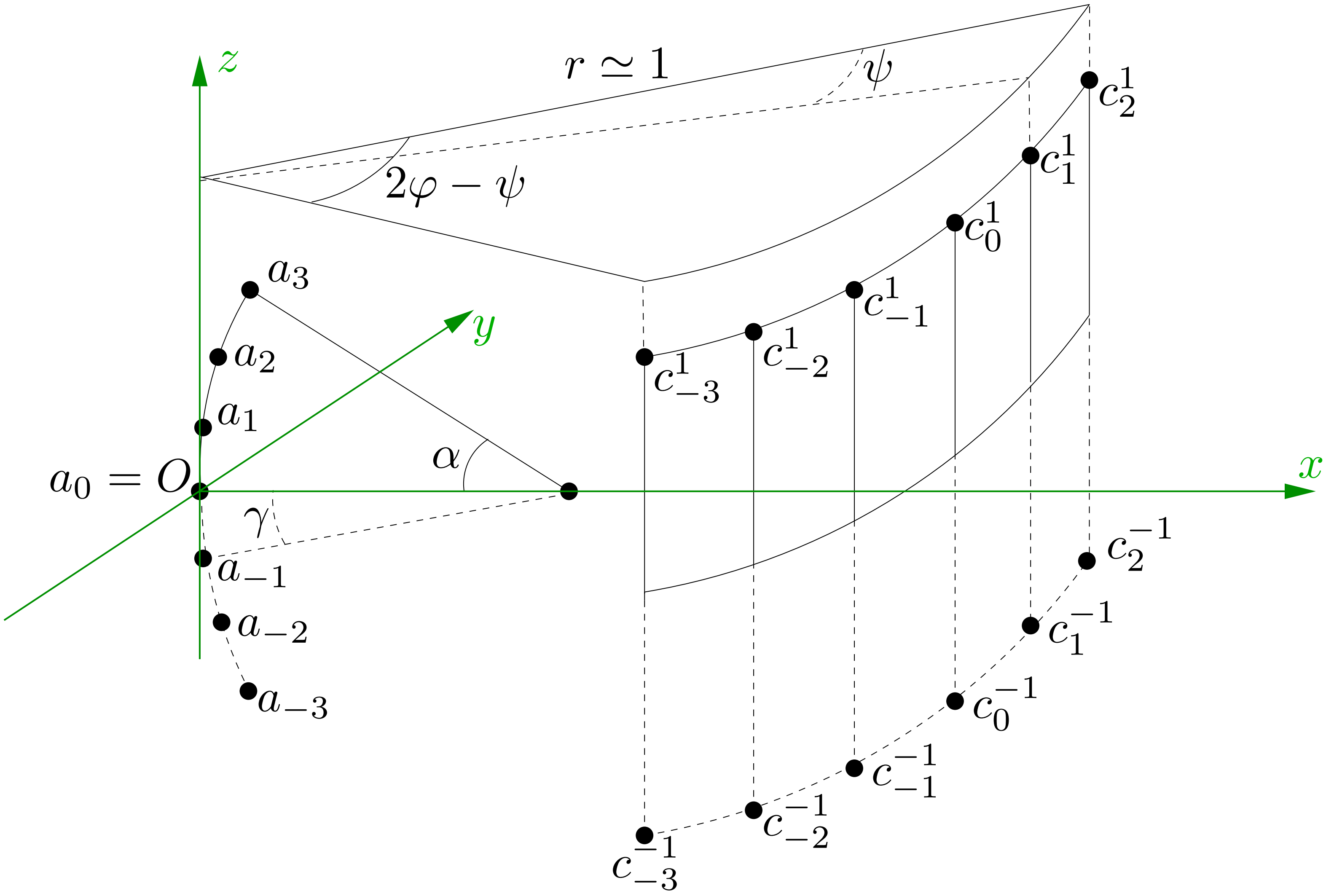
- Example where $n = 3$.



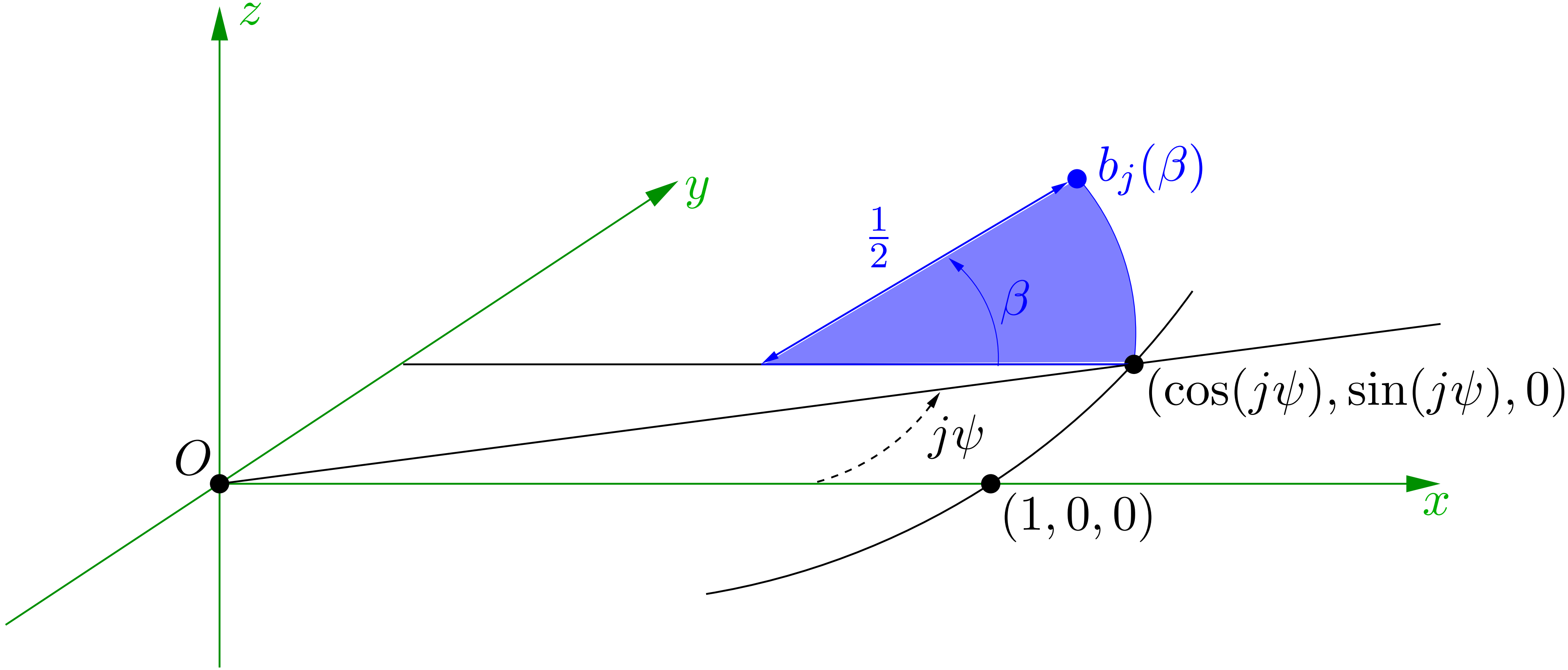
Notation

- $\bar{a} := (a_{-n}, a_{-n+1} \ldots, a_n)$.
- $A := \{a_{-n}, a_{-n+1}, \ldots, a_n\}$.
- $\bar{\beta} := (\beta_{-n+1}, \ldots, \beta_{n-1})$.
- $\bar{b}(\bar{\beta}) := (b_{-n+1}(\beta_{-n+1}), \ldots, b_{n-1}(\beta_{n-1}))$.
- $B(\bar{\beta}) := \{b_{-n+1}(\beta_{-n+1}), \ldots, b_{n-1}(\beta_{n-1})\}$.
- $\bar{c} := (c^{-1}_{-n}, c^{-1}_{-n+1}, \ldots, c^{-1}_{n-1}, c^{1}_{-n}, c^{1}_{-n+1}, \ldots, c^{1}_{n-1})$.
- $C := \{c^{-1}_{-n}, c^{-1}_{-n+1}, \ldots, c^{-1}_{n-1}, c^{1}_{-n}, c^{1}_{-n+1}, \ldots, c^{1}_{n-1}\}$.
- $P(\bar{\beta}) := \mathrm{CH}(A \cup B(\bar{\beta}) \cup C)$.

Point sets A and C



Point $b_j(\beta_j)$



- The blue region is parallel to Oxz.
- $\beta \in [-\alpha, \alpha]$

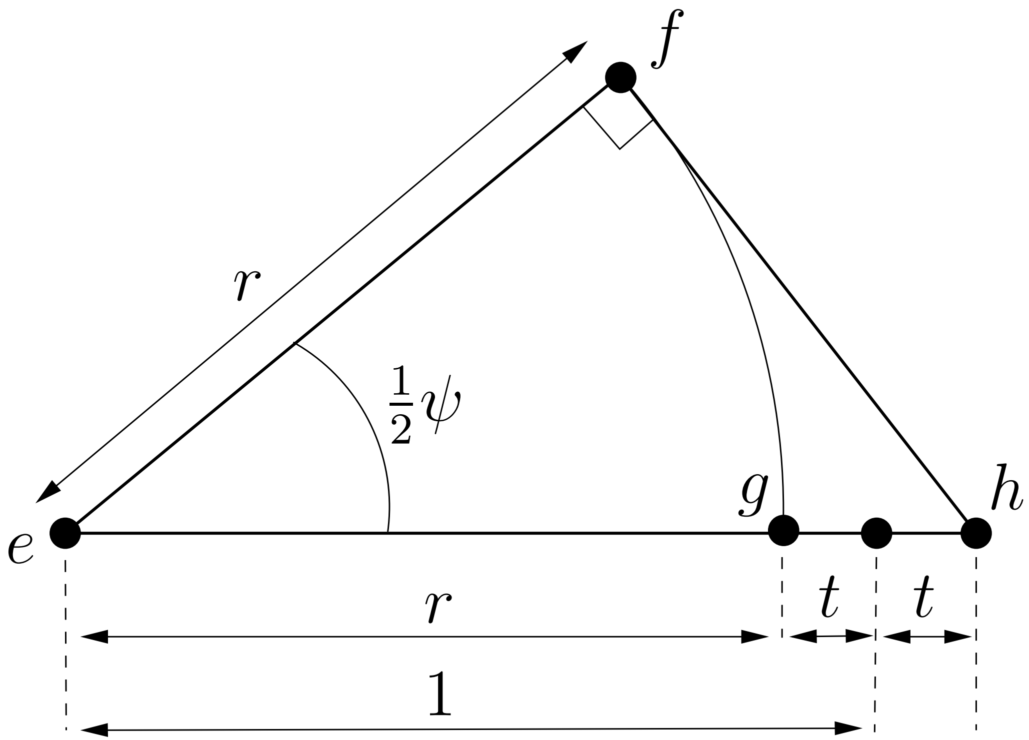
Coordinates of points in A, $B(\bar{\beta})$ and C

- $$a_i := \begin{pmatrix} \frac{1}{2}(1-\cos(i\gamma)) \\ 0 \\ \frac{1}{2}\sin(i\gamma) \end{pmatrix}$$

- $$c_i^s := \begin{pmatrix} r\cos\left(\left(i+\frac{1}{2}\right)\psi\right) \\ r\sin\left(\left(i+\frac{1}{2}\right)\psi\right) \\ \frac{1}{2}s\alpha \end{pmatrix}$$

- $$b_j(\beta) := \begin{pmatrix} \cos(j\psi) - \frac{1}{2}(1-\cos\beta) \\ \sin(j\psi) \\ \frac{1}{2}\sin(\beta) \end{pmatrix}$$

Parameters



- $\varphi = 1/4n$
- α is small.
- $\psi = \varphi/n$
- $\gamma = \alpha/n$

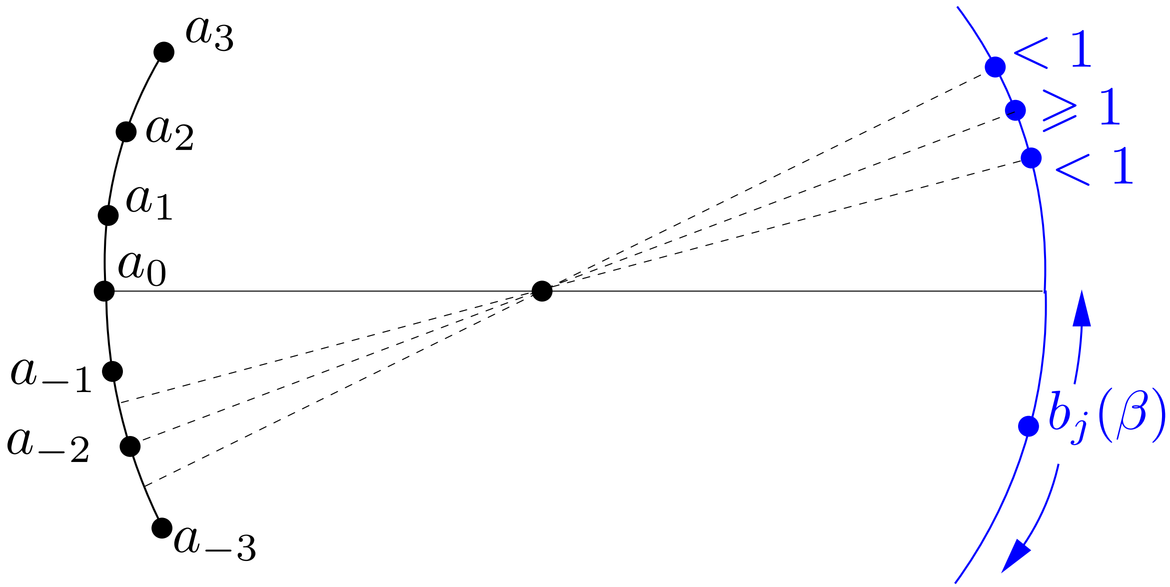
Proof

- Notation: $\mathrm{diam}(E, F) := \max\{\mathrm{d}(e, f) \mid (e, f) \in E \times F\}$.

Lemma 1. *The set*

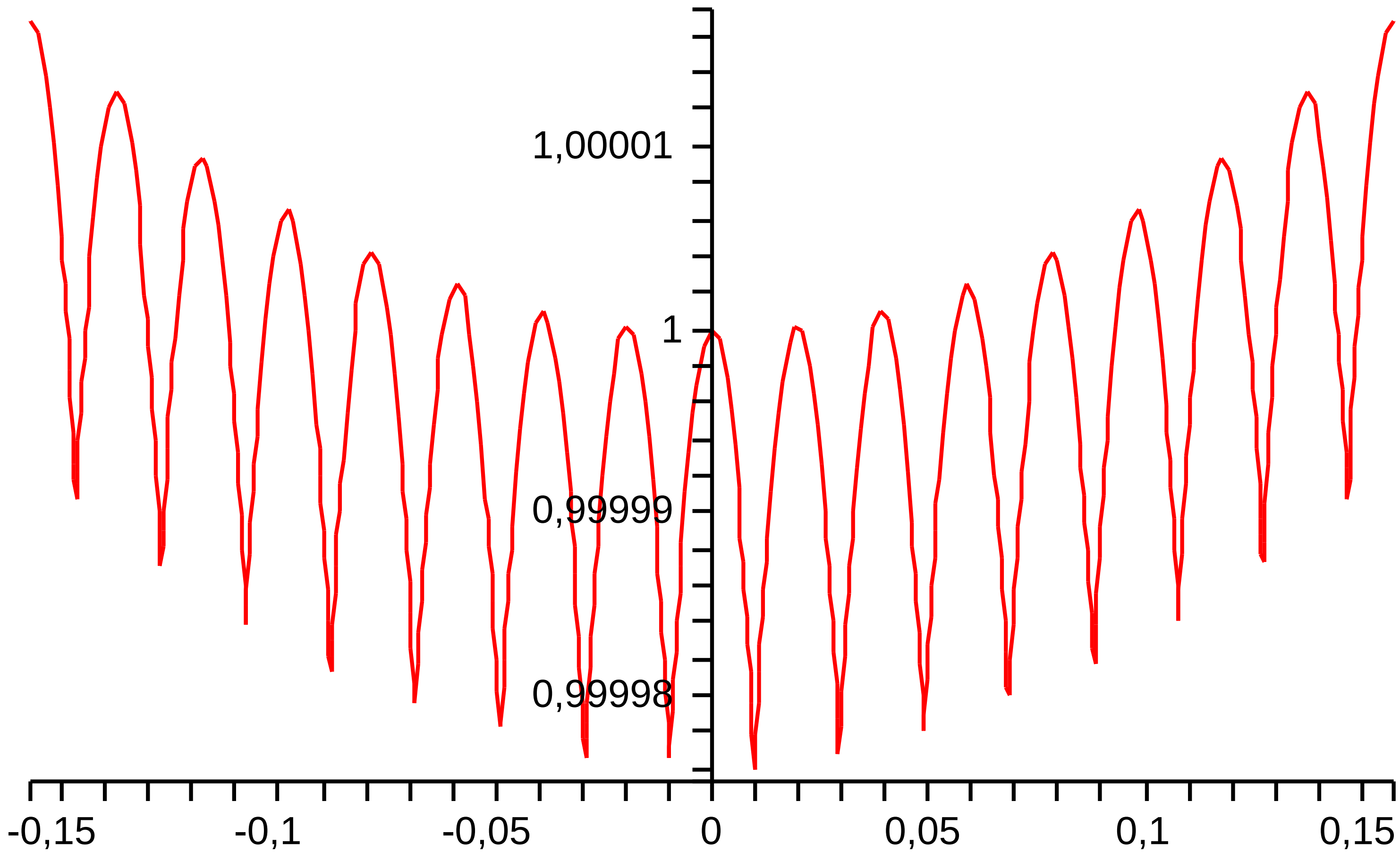
$$\{b_j(\beta) \mid \beta \in [-\alpha, \alpha] \textit{ and } \mathrm{diam}(A, \{b_j(\beta)\}) < 1\}$$

has at least $2n$ connected components.



Proof

- Proof of Lemma 1: Calculations, until the second-order terms.



Proof

Lemma 2. *The combinatorial structure of* $\mathrm{CH}(A \cup B(\bar{\beta}) \cup C)$ *is independent of* $\bar{\beta}$.

- We denote $P(\bar{\beta}) = \mathrm{CH}(A \cup B(\bar{\beta}) \cup C)$.

Lemma 3. $\mathrm{diam}(A \cup B(\bar{\beta}) \cup C) = \mathrm{diam}(A, B(\bar{\beta}))$.

Proof

- Definitions:

$$\mathcal{S}_n = \{(\bar{a}, \bar{b}(\bar{\beta}), \bar{c}) \mid \bar{\beta} \in [-\alpha, \alpha]^{2n-1}\}$$
$$\mathcal{E}_n = \{(\bar{a}, \bar{b}(\bar{\beta}), \bar{c}) \mid \bar{\beta} \in [-\alpha, \alpha]^{2n-1} \text{ and } \operatorname{diam}(P(\bar{\beta})) < 1\}$$

- Notice that $\mathcal{E}_n \subset \mathcal{S}_n \subset \mathbb{R}^{24n}$.
- Restriction to $\mathcal{S}_n$ is easy.

  Lemma 4. *The set $\mathcal{S}_n$ can be decided by an ACT with depth $O(n)$.*

- Decinding $\mathcal{E}_n$ over $\mathcal{S}_n$ is hard.

  Lemma 5. *Any ACT that decides $\mathcal{E}_n$ has depth $\Omega(n \log n)$.*

  Proof: By lemmas 1 and 3, $\mathcal{E}_n$ has at least $(2n)^{2n-1}$ connected components. Apply Ben-Or's bound.

End of the proof

Theorem. *Assume that an algebraic computation tree* T_n *decides whether the diameter of a 3-polytope is smaller than 1. Then* T_n *has depth* $\Omega(n \log n)$.

- Let d_n be the depth of T_n. The computation tree T_n can be transformed into $\tilde{T}_n$ of depth $d_n + O(n)$ which decides $\mathcal{E}_n$ over $\mathcal{S}_n$. Thus $\tilde{T}_n$ has depth $\Omega(n \log n)$ and the same holds for T_n.

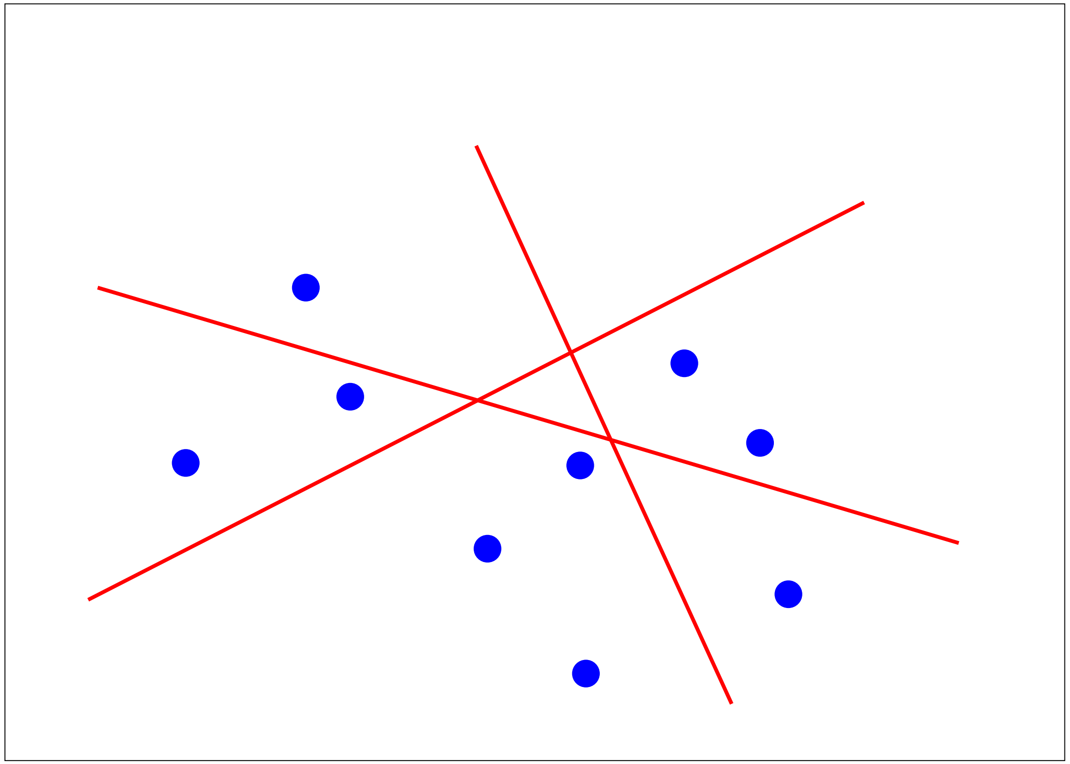
Related work

- (Chazelle) The convex hull of two 3-polytopes can be computed in linear time.
- (Chazelle et al.) It is not known whether the convex hull of a subset of the vertices of a 3-polytope can be computed in linear time.
- (Chazelle et al.) However, we can compute in linear time the Delaunay triangulation of a subset of the vertices of a Delaunay triangulation.

Diameter is harder than Hopcroft's problem

Hopcroft's problem

- P is a set of n points in $\mathbb{R}^2$.
- L is a set of n lines in $\mathbb{R}^2$.
- Problem: decide whether $\exists(p, \ell) \in P \times L : p \in \ell$.



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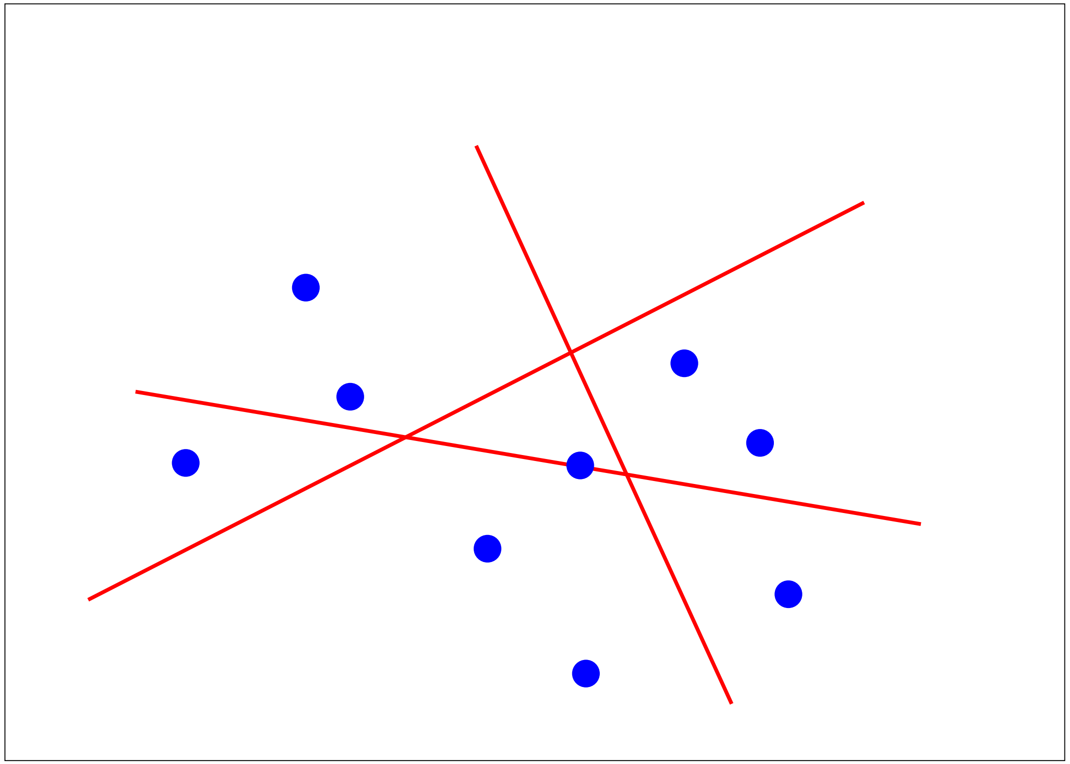

Complexity of Hopcroft's problem

- An $o(n^{4/3} \log n)$ algorithm is known. (Matoušek).
- No $o(n^{4/3})$ algorithm is known.
- Erickson gave an $\Omega(n^{4/3})$ lower bound in a *weaker model*.
    - Partitioning algorithms, based on a divide-and-conquer approach.

From Hopcroft's problem to Diameter

- We give a linear-time reduction from Hopcroft's problem to the diameter problem in $\mathbb{R}^7$.
  - Known upper bound: $n^{1.6} \log^{O(1)} n$.
- We first give a reduction to the *red-blue diameter* problem in $\mathbb{R}^6$: compute $\mathrm{diam}(E, F)$ when E and F are n-point sets in $\mathbb{R}^6$.

Proof

- $\theta(x,y,z) := \frac{1}{x^2+y^2+z^2}(x^2, y^2, z^2, \sqrt{2}xy, \sqrt{2}yz, \sqrt{2}zx)$.
- Note that $\|\theta(x,y,z)\| = 1$.
- For $1 \leqslant i \leqslant n$
  - $p_i = (x_i, y_i, 1)$
  - $\ell_i = (u_i, v_i, w_i)$ is the line $\ell_i : u_i x + v_i y + w_i = 0$.
- Let $p'_i := \theta(p_i)$ and $\ell'_j = \theta(\ell_j)$.
- We get

$$\begin{aligned}\|p'_i - \ell'_j\|^2 &= \|p'_i\|^2 + \|\ell'_j\|^2 - 2 < p'_i, \ell'_j > \\ &= 2 - 2\frac{< p_i, \ell_j >^2}{\|p_i\|^2 \|\ell_j\|^2}\end{aligned}$$

Proof

- Note that $p_i \in \ell_j$ iff $< p_i, \ell_j >= 0$.
- Thus, there exists i, j such that $p_i \in \ell_j$ if and only if $\mathrm{diam}(\theta(P), \theta(L)) = 2$.
- $\theta(P)$ and $\theta(L)$ are n-point sets in $\mathbb{R}^6$.
- Similarly, we can get a reduction from Hopcroft's problem to the diameter problem in $\mathbb{R}^7$, using this linearization:

$$\tilde{\theta}(x, y, z) := \left(\frac{1}{x^2 + y^2 + z^2} (x^2, y^2, z^2, \sqrt{2}xy, \sqrt{2}yz, \sqrt{2}zx), \pm 1 \right)$$

Related work

- The red-blue diameter in $\mathbb{R}^4$ can be computed in $O(n^{4/3}\text{polylog } n)$ (Matoušek and Scharzkopf). It would be interesting to get a reduction from Hopcroft's problem.
- Erickson gave reduction from Hopfcroft problem to other computational geometry problems.
  - Ray shooting in polyhedral terrains,
  - Halfspace emptyness in $\mathbb{R}^5$

  are harder than Hopcroft's problem.