Decomposition of graphs in given factors

Martín Matamala

Departamento de Ingeniería Matemática & Centro de Modelamiento Matemático
Universidad de Chile, Santiago, Chile

22th January 2008
Collaborators.

- **José Correa**, School of Business, Universidad Adolfo Ibáñez, Santiago, Chile.
- **Flavio Guiñez**, Departement of Mathematical Engineering, Universidad de Chile, Santiago, Chile.
- **José Zamora**, Departement of Mathematical Engineering, Universidad de Chile, Santiago, Chile.
Content

1. Decomposition in factors
   - A family of games
   - Graph Setting
   - The problem

2. Computational Complexity
   - \( p = 2 \)
   - \( p \geq 3 \)

3. Scaled factors always exists
   - Motivation
   - Known results
   - Weighted version

4. Open Problems
   - \( \lambda \)-factors
   - Complete graphs and special \( A \)
Decomposition in factors
Computational Complexity
Scaled factors always exists
Open Problems

A family of games
Graph Setting
The problem

CELLST version 1.0 (1)

Martín Matamala
Factors
### CELLST version 1.0 (1)

<table>
<thead>
<tr>
<th></th>
<th>3</th>
<th>2</th>
<th>1</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>3</th>
<th>2</th>
<th>1</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>•</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
<td>•</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td></td>
<td></td>
<td>•</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td>•</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>3</th>
<th>2</th>
<th>1</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>•</td>
<td>•</td>
<td>•</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
<td>•</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td></td>
<td></td>
<td>•</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td>•</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>3</th>
<th>2</th>
<th>1</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>•</td>
<td>•</td>
<td>•</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
<td>•</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td></td>
<td></td>
<td>•</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td>•</td>
</tr>
</tbody>
</table>

---

**Martín Matamala**

**Factors**
### Decomposition in factors

- **Computational Complexity**
  - Scaled factors always exist
- **Open Problems**

#### A family of games

- **Graph Setting**

#### The problem

**CELLST version 1.0 (1)**

<table>
<thead>
<tr>
<th>3</th>
<th>2</th>
<th>1</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>3</th>
<th>2</th>
<th>1</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>3</th>
<th>2</th>
<th>1</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>•</td>
<td>•</td>
<td>•</td>
</tr>
<tr>
<td>2</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>3</th>
<th>2</th>
<th>1</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>•</td>
<td>•</td>
<td>•</td>
</tr>
<tr>
<td>2</td>
<td>•</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>3</th>
<th>2</th>
<th>1</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>•</td>
<td>•</td>
<td>•</td>
</tr>
<tr>
<td>2</td>
<td>•</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Martín Matamala**

**Factors**
<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Decomposition in factors
Computational Complexity
Scaled factors always exist
Open Problems

A family of games
Graph Setting
The problem

CELLST version 1.0 (2)

Martín Matamala
Factors
### Decomposition in factors

#### Computational Complexity

Scaled factors always exist

#### Open Problems

A family of games

**Graph Setting**

**The problem**

---

**CELLST version 1.0 (2)**

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Martín Matamala**

**Factors**
CELLST version 1.0 (2)

1  2  3  3
1
2
3
3

1  2  3  3
1
2
3
3

1  2  3  3
1
2
3
3

1  2  3  3
1
2
3
3

1  2  3  3
1
2
3
3

1  2  3  3
1
2
3
3
Decomposition in factors
Computational Complexity
Scaled factors always exists
Open Problems

A family of games
Graph Setting
The problem

CELLST version 2.0

Martín Matamala
Factors
### CELLST version 2.0

<table>
<thead>
<tr>
<th></th>
<th>3</th>
<th>2</th>
<th>1</th>
<th>1</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>1</td>
<td></td>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>2</td>
<td>3</td>
<td>1</td>
<td></td>
<td>2</td>
</tr>
<tr>
<td>1</td>
<td>3</td>
<td></td>
<td></td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>3</td>
<td></td>
<td></td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

- **Left Matrix:**

  - Top Left: 3
  - Top Middle: 1
  - Top Right: 2
  - Middle Left: 2
  - Middle Right: 3
  - Bottom Left: 1
  - Bottom Right: 3

- **Right Matrix:**

  - Top Left: 3
  - Top Middle: 1
  - Top Right: 2
  - Middle Left: 3
  - Middle Right: 1
  - Bottom Left: 2
  - Bottom Right: 1

- **Cells:**

  - Left Matrix: 1, 1, 3, 3
  - Right Matrix: 1, 1, 3, 3
Some cells are forbidden

Martín Matamala
Factors
Some cells are forbidden

<table>
<thead>
<tr>
<th></th>
<th>3</th>
<th>2</th>
<th>1</th>
<th>1</th>
<th>X</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>3</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>3</th>
<th>2</th>
<th>1</th>
<th>1</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>3</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td></td>
<td></td>
<td></td>
<td>X</td>
</tr>
</tbody>
</table>

Factors

Martín Matamala
### Three or more colors

<table>
<thead>
<tr>
<th></th>
<th>2</th>
<th>1</th>
<th>1</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>1</td>
<td>3</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>2</th>
<th>1</th>
<th>1</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>2</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>1</td>
<td>3</td>
<td></td>
</tr>
</tbody>
</table>

- Blue
- Red
- Green

### A family of games

Graph Setting

The problem

<table>
<thead>
<tr>
<th></th>
<th>2</th>
<th>1</th>
<th>1</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>2</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>1</td>
<td>3</td>
<td></td>
</tr>
</tbody>
</table>

- Blue
- Red
- Orange

### Open Problems

- Martín Matamala

Factors
### Three or more colors

<table>
<thead>
<tr>
<th></th>
<th>2</th>
<th>1</th>
<th>1</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>3</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>2</th>
<th>1</th>
<th>1</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>2</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>1</td>
<td>3</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>2</th>
<th>1</th>
<th>1</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>1</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>1</td>
<td>3</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>0</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>3</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>2</th>
<th>1</th>
<th>1</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>2</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>0</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>3</td>
<td></td>
</tr>
</tbody>
</table>

Martín Matamala  Factors
Tableaux → complete bipartite graph \((X \cup Y, E)\).
- Rows → \(X\); Columns → \(Y\).
- Cells → \(E\):
  - cell \((i, j)\) is the edge between vertex \(i \in X\) and vertex \(j \in Y\).

Input: A function \(a : X \cup Y \rightarrow \mathbb{N}\).
Output: A set \(F \subseteq E\) such that:

\[
\forall v \in X \cup Y, |\{e \in F : v \in e\}| = a(v)
\]

Number of edges of \(F\) incident with \(v\) are exactly \(a(v)\).
Tableaux → complete bipartite graph \((X \cup Y, E)\).
- Rows → \(X\); Columns → \(Y\).
- Cells → \(E\):
  cell \((i,j)\) is the edge between vertex \(i \in X\) and vertex \(j \in Y\).

Input: A function \(a : X \cup Y \to \mathbb{N}\).
Output: A set \(F \subseteq E\) such that:

\[
\forall v \in X \cup Y, |\{e \in F : v \in e\}| = a(v)
\]

Number of edges of \(F\) incident with \(v\) are exactly \(a(v)\).
Tableaux → complete bipartite graph \((X \cup Y, E)\).
- Rows → \(X\); Columns → \(Y\).
- Cells → \(E\):
  - cell \((i, j)\) is the edge between vertex \(i \in X\) and vertex \(j \in Y\).

Input: A function \(a : X \cup Y \to \mathbb{N}\).
Output: A set \(F \subseteq E\) such that:

\[\forall v \in X \cup Y, |\{e \in F : v \in e\}| = a(v)\]

Number of edges of \(F\) incident with \(v\) are exactly \(a(v)\).
Graphs setting

Definitions

Definition (Degree function)

Let $G = (V, E)$ be a graph. The function $a : V \to \mathbb{N}$ defined by

$$a(v) := |\{e \in E : v \in e\}|$$

is the degree function of $G$

- The degree function is denoted by $d_G$. 
The degree function of the complete graph $K_n = (V, E)$ is $n - 1$, for each vertex in $V$.

The degree function of the complete bipartite graph $K_{n,m} = (X \cup Y, E)$ is $m$ for vertex in $X$ and $n$ for vertex in $Y$. 
The degree function of the complete graph $K_n = (V, E)$ is $n - 1$, for each vertex in $V$.

The degree function of the complete bipartite graph $K_{n,m} = (X \cup Y, E)$ is $m$ for vertex in $X$ and $n$ for vertex in $Y$. 

Martín Matamala
Graphs setting
Definitions

Definition (Factor)
Let \( G = (V, E) \) be a graph and \( a : V \rightarrow \mathbb{N} \).

A \( a \)-factor of \( G \) is a subgraph \( (V, F) \) of \( G \) such that \( a \) is the degree function of \( (V, F) \).

Remark
A subgraph \( (V, F) \) is a \( a \)-factor of \( (V, E) \) if and only if \( (V, E \setminus F) \) is a \((d_G - a)\)-factor of \( (V, E) \).
Graphs setting

Definitions

**Definition (Factor)**

Let $G = (V, E)$ be a graph and $a : V \rightarrow \mathbb{N}$.

- A $a$-factor of $G$ is a subgraph $(V, F)$ of $G$ such that $a$ is the degree function of $(V, F)$.

**Remark**

A subgraph $(V, F)$ is a $a$–factor of $(V, E)$ if and only if $(V, E \setminus F)$ is a $(d_G - a)$–factor of $(V, E)$. 

Martín Matamala

Factors
Graph setting
Some cells are forbidden

- Tableaux → bipartite graph $G = (X \cup Y, E)$.
  - Rows → $X$; Columns → $Y$.
  - Allowed Cells → $E$:
    - Input: A function $a : X \cup Y \rightarrow \mathbb{N}$.
    - Output: A $a$–factor of $G$. 
Graph setting
Some cells are forbidden

Tableaux $\rightarrow$ bipartite graph $G = (X \cup Y, E)$.
- Rows $\rightarrow X$; Columns $\rightarrow Y$.
- Allowed Cells $\rightarrow E$:

Input: A function $a : X \cup Y \rightarrow \mathbb{N}$.
Output: A $a$–factor of $G$. 
Graph setting
Three or more colors

- **Tableaux** → **bipartite graph** \( G = (X \cup Y, E) \).
  - Rows → \( X \); Columns → \( Y \).
  - Allowed Cells → \( E \)

- Input: A set \( A \) of three or more functions.
- Output: A partition \( \mathcal{P} = \{F_a : a \in A\} \) of \( E \).
  \((X \cup Y, F_a)\) is a \( a \)--factor of \( G \), for each \( a \in A \).
Graph setting

Three or more colors

- Tableaux → bipartite graph $G = (X \cup Y, E)$.  
  - Rows → $X$; Columns → $Y$.  
  - Allowed Cells → $E$

- Input: A set $A$ of three or more functions.
- Output: A partition $\mathcal{P} = \{F_a : a \in A\}$ of $E$.  
  $(X \cup Y, F_a)$ is an $a$–factor of $G$, for each $a \in A$.  

Martín Matamala  Factors
Graph setting
Three or more colors

- Tableaux → **bipartite graph** $G = (X \cup Y, E)$.
  - Rows → $X$; Columns → $Y$.
  - Allowed Cells → $E$

- Input: A set $A$ of three or more functions.
- Output: A partition $\mathcal{P} = \{F_a : a \in A\}$ of $E$.
  
  $(X \cup Y, F_a)$ is an $a$–factor of $G$, for each $a \in A$. 
Decomposition of graphs in factors

- **Input:** A graph \( G = (V, E) \) and a set \( A \) of functions.

- **Output:** A partition \( \mathcal{P} = \{F_a : a \in A\} \) of \( E \),

  \[ \forall a \in A, (V, F_a) \text{ is a } a-\text{factor of } G. \]

  When it is possible \( G \) is called \( A-\text{decomposable}. \)
Decomposition of graphs in factors

- **Input:** A graph $G = (V, E)$ and a set $A$ of functions.

- **Output:** A partition $\mathcal{P} = \{F_a : a \in A\}$ of $E$, $\forall a \in A, (V, F_a)$ is an $a$–factor of $G$.

When it is possible $G$ is called $A$–decomposable.
Decomposition of graphs in factors

- Input: A graph $G = (V, E)$ and a set $A$ of functions.

- Output: A partition $\mathcal{P} = \{F_a : a \in A\}$ of $E$,
  $$\forall a \in A, (V, F_a) \text{ is a } a-\text{factor of } G.$$ 

When it is possible $G$ is called $A$–decomposable.
$p$–decomposition problem.

Input: A graph $G = (V, E)$ and a set of functions $A$ with $|A| = p$.

Question: Does $G$ have a $A$–decomposition?
Decomposition
Two parts

Definition (Factors)
Let $G = (V, E)$ be a graph and $a, b : V \to \mathbb{N}$.

A $(a, b)$-factor of $G$ is a subgraph $(V, F)$ of $G$ such that $a \leq d_G \leq b$, where $d_G$ is the degree function of $(V, F)$. 
Two parts: \((a, b)\)-factors

Known results

- **Tutte** (1970): NSC for perfect matching and for \(a\)–factors.
- **Lovász** (1970): NSC for \((a, b)\)–factors.
- **Anstee** (1985, 1990, 1994): S conditions and a \(O(n^3)\) algorithm for \((a, b)\)–factors.
- **Heinrich, Hell, Kirkpatrick, Liu.** (1990). *Simpler* conditions when \(a < b\) or \(G_{a=b}\) bipartite.
- **Correa, M.** (2006) new NSC for \((a, b)\)–factors.
Three or more parts

Polynomial

- **Guiñez, M.** (2007), Any $G$ and $|A| \geq 3$ such that at most two function assign to $v$ a nonzero value.

- **Kundu** (1970), **Kleitman, Koren and Li** (1973), **Guiñez, M.** (2007), $G$ a complete graph, $|A| = 3$ and at least one function $a$ has span $1$.

- **Guiñez, M.** (2007), $G$ acyclic (forest) and $|A| \geq 3$. 
Three or more colors
NP-completeness

- Dürr, Marek (2001), Bipartite graphs and $|A| = 3$.
- Dürr, Marek (2001), $|A| = 4$ and $G$ a complete or a complete bipartite graph.
- Open for $G$ complete or complete bipartite and $|A| = 3$. 
Scaled factors
Known scaled factors

- **Lovász (1970):** If $G = (V, E)$ has maximum degree $k$ and $k_1 + k_2 = k + 1$, then $E$ can be partitioned into a $(0, k_1)$–factor and a $(0, k_2)$–factor.

- **Tutte (1978):** If $G = (V, E)$ is $k$-regular graph and $0 \leq r < k$, then $G$ has a $(r, r + 1)$–factor.

- **Gupta (1978):** If $G = (V, E)$ has minimum degree $k$ and $k_1 + k_2 = k - 1 \geq 1$, then $E$ can be partitioned into a $(k_1, d_G)$–factor and a $(k_2, d_G)$–factor.

- **Thomassen (1980):** If $G = (V, E)$ is such that $\deg_E(v) \in \{k, k + 1\}$ and $0 \leq r < k$, then $G$ has a $(r, r + 1)$–factor.
**Scaled factors**

**Known scaled factors**

- **Lovász (1970):** If $G = (V, E)$ has maximum degree $k$ and $k_1 + k_2 = k + 1$, then $E$ can be partitioned into a $(0, k_1)$–factor and a $(0, k_2)$–factor.

- **Tutte (1978):** If $G = (V, E)$ is $k$-regular graph and $0 \leq r < k$, then $G$ has a $(r, r + 1)$–factor.

- **Gupta (1978):** If $G = (V, E)$ has minimum degree $k$ and $k_1 + k_2 = k - 1 \geq 1$, then $E$ can be partitioned into a $(k_1, d_G)$–factor and a $(k_2, d_G)$–factor.

- **Thomassen (1980):** If $G = (V, E)$ is such that $\deg_E(v) \in \{k, k + 1\}$ and $0 \leq r < k$, then $G$ has a $(r, r + 1)$–factor.
Scaled factors
Known scaled factors

- **Lovász (1970):** If $G = (V, E)$ has maximum degree $k$ and $k_1 + k_2 = k + 1$, then $E$ can be partitioned into a $(0, k_1)$-factor and a $(0, k_2)$-factor.

- **Tutte (1978):** If $G = (V, E)$ is $k$-regular graph and $0 \leq r < k$, then $G$ has a $(r, r + 1)$-factor.

- **Gupta (1978):** If $G = (V, E)$ has minimum degree $k$ and $k_1 + k_2 = k - 1 \geq 1$, then $E$ can be partitioned into a $(k_1, d_G)$-factor and a $(k_2, d_G)$-factor.

- **Thomassen (1980):** If $G = (V, E)$ is such that $\deg_E(v) \in \{k, k + 1\}$ and $0 \leq r < k$, then $G$ has a $(r, r + 1)$-factor.
Scaled factors

Known scaled factors

- **Lovász (1970):** If $G = (V, E)$ has maximum degree $k$ and $k_1 + k_2 = k + 1$, then $E$ can be partitioned into a $(0, k_1)$–factor and a $(0, k_2)$–factor.

- **Tutte (1978):** If $G = (V, E)$ is a $k$-regular graph and $0 \leq r < k$, then $G$ has a $(r, r + 1)$–factor.

- **Gupta (1978):** If $G = (V, E)$ has minimum degree $k$ and $k_1 + k_2 = k - 1 \geq 1$, then $E$ can be partitioned into a $(k_1, d_G)$–factor and a $(k_2, d_G)$–factor.

- **Thomassen (1980):** If $G = (V, E)$ is such that $\deg_E(v) \in \{k, k + 1\}$ and $0 \leq r < k$, then $G$ has a $(r, r + 1)$–factor.
Let $\lambda \in (0, 1)$ be. A $\lambda$-factor of $G = (V, E)$ is a subgraph $(V, F)$ of $G$ such that

$$\lceil \lambda \deg_E(v) \rceil - 1 \leq \deg_F(v) \leq \lfloor \lambda \deg_E(v) \rfloor + 1$$

that is a $(\lceil \lambda \deg_E(v) \rceil - 1, \lfloor \lambda \deg_E(v) \rfloor + 1)$-factor.

A $\lambda$-factor is equivalent to scaling the degrees of $G$ by $\lambda$.

The results of Lovász (1970), Tutte (1978), Gupta (1978), Thomassen (1980), can be formulated as the existence of a $\lambda$-factor.
**λ-factors**

**Definition (Correa, M)**

- Let $\lambda \in (0, 1)$ be. A $\lambda$-factor of $G = (V, E)$ is a subgraph $(V, F)$ of $G$ such that

$$\lceil \lambda \deg_E(v) \rceil - 1 \leq \deg_F(v) \leq \lfloor \lambda \deg_E(v) \rfloor + 1$$

- that is a $(\lceil \lambda \deg_E(v) \rceil - 1, \lfloor \lambda \deg_E(v) \rfloor + 1)$-factor.

$\lambda$-factor $\equiv$ scaling the degrees of $G$ by $\lambda$

The results of Lovász (1970), Tutte (1978), Gupta (1978), Thomassen (1980), can be formulated as the existence of a $\lambda$-factor.
λ-factors
Definition (Correa, M)

- Let $\lambda \in (0, 1)$ be. A $\lambda$-factor of $G = (V, E)$ is a subgraph $(V, F)$ of $G$ such that

$$\left\lceil \lambda \deg_E(v) \right\rceil - 1 \leq \deg_F(v) \leq \left\lfloor \lambda \deg_E(v) \right\rfloor + 1$$

- that is a $\left(\left\lceil \lambda \deg_E(v) \right\rceil - 1, \left\lfloor \lambda \deg_E(v) \right\rfloor + 1\right)$-factor.

$\lambda$-factor $\equiv$ scaling the degrees of $G$ by $\lambda$

The results of Lovász (1970), Tutte (1978), Gupta (1978), Thomassen (1980), can be formulated as the existence of a $\lambda$-factor.
Let \( \lambda \in (0, 1) \) be. A \( \lambda \)-factor of \( G = (V, E) \) is a subgraph \((V, F)\) of \( G \) such that

\[
\lceil \lambda \deg_E(v) \rceil - 1 \leq \deg_F(v) \leq \lfloor \lambda \deg_E(v) \rfloor + 1
\]

that is a \((\lceil \lambda \deg_E(v) \rceil - 1, \lfloor \lambda \deg_E(v) \rfloor + 1)\)-factor.

\( \lambda \)-factor \equiv \text{scaling the degrees of } G \text{ by } \lambda

The results of Lovász (1970), Tutte (1978), Gupta (1978), Thomassen (1980), can be formulated as the existence of a \( \lambda \)-factor.
λ-factors

Results

- **Theorem A:** If $G$ bipartite and $\lambda \in [0, 1]$, then there exists $F$ such that
  \[
  \lfloor \lambda \deg_E(v) \rfloor \leq \deg_F(v) \leq \lceil \lambda \deg_E(v) \rceil
  \]
  Hoffman (1956), using techniques of network flows.

- **Theorem B:** λ-factors always exists.
  Kano, Saito (1983), using the NSC of Lovasz.

**Theorem A:** If $G$ bipartite and $\lambda \in [0, 1]$, then there exists $F$ such that

$$\lfloor \lambda \deg_E(v) \rfloor \leq \deg_F(v) \leq \lceil \lambda \deg_E(v) \rceil$$

Hoffman (1956), using techniques of network flows.

**Theorem B:** $\lambda$–factors always exists.

Kano, Saito (1983), using the NSC of Lovasz.

Theorem A: If $G$ bipartite and $\lambda \in [0, 1]$, then there exists $F$ such that

$$\lfloor \lambda \deg_E(v) \rfloor \leq \deg_F(v) \leq \lceil \lambda \deg_E(v) \rceil$$

Hoffman (1956), using techniques of network flows.

Theorem B: $\lambda$–factors always exists.

Kano, Saito (1983), using the NSC of Lovasz.

Definition (weighted factors)

Let \( G = (V, E) \) be a graph, let \( w : E \to [0, \infty) \) and let \( a, b : V \to \mathbb{N} \). A \( w \)-weighted \( (a, b) \)-factor of \( G \) is a subset \( F \subseteq E \) such that

\[
a(v) \leq w(\delta_v \cap F) \leq b(v), \forall v \in V,
\]

where \( x(A) = \sum_{e \in A} x_e \) and \( \delta_v = \{uv : uv \in E\} \).
Weighted $\lambda$-factors.

Definition

Let $G = (V, E)$ be a graph, let $w : E \to [0, \infty)$ and let $\lambda \in (0, 1)$. A $w$–weighted $\lambda$–factor of $G$ is a subset $F \subseteq E$ such that

$$\left| w(\delta_v \cap F) - \lambda w(\delta_v) \right| \leq \max\{w_e : e \in \delta_v\} \quad \forall v \in V.$$
A $w$–weighted $\lambda$–factor always exists.


Theorem C: Let $G = (V, E)$ be a graph and $l \leq 0 \leq u : E \to \mathbb{R}$. Then, $\exists x \in \prod_{e \in E} \{l_e, u_e\}$, $\forall v \in V$

$$|x(\delta_v)| \leq \max \{u_e - l_e : e \in \delta_v\}$$


Theorem D: Weighted $\lambda$–factors always exists.
A $w$–weighted $\lambda$–factor always exists. 

**Theorem C:** Let $G = (V, E)$ be a graph and $l \leq 0 \leq u : E \to \mathbb{R}$. Then, $\exists x \in \prod_{e \in E} \{l_e, u_e\}$, $\forall v \in V$

$$|x(\delta_v)| \leq \max \{u_e - l_e : e \in \delta_v\}$$


**Theorem D:** Weighted $\lambda$–factors always exists.
Decomposition in several $\lambda-$factors

Let $G = (V, E)$ be a graph and let $\lambda_1, \ldots, \lambda_k \in (0, 1)$ be such that $\sum_{i=1}^{k} \lambda_i = 1$.

Can $E$ be partitioned into $F_1, \ldots, F_k$ such that for all $v \in V$: $|\deg_{F_i}(v) - \lambda_i \deg_E(v)| \leq 1$?

Combining $\lambda-$factor Theorem with Correa and Goemans’ previous ideas, we show $|\deg_{F_i}(v) - \lambda_i \deg_E(v)| < 3$. 
Let $G = (V, E)$ be a graph and let $\lambda_1, \ldots, \lambda_k \in (0, 1)$ be such that $\sum_{i=1}^{k} \lambda_i = 1$.

Can $E$ be partitioned into $F_1, \ldots, F_k$ such that for all $v \in V$:

$$|\deg_{F_i}(v) - \lambda_i \deg_{E}(v)| \leq 1?$$

Combining $\lambda$–factor Theorem with Correa and Goemans’ previous ideas, we show $|\deg_{F_i}(v) - \lambda_i \deg_{E}(v)| < 3$. 

---

Decomposition in several $\lambda$–factors
Let $G = (V, E)$ be a graph and let $\lambda_1, \ldots, \lambda_k \in (0, 1)$ be such that $\sum_{i=1}^{k} \lambda_i = 1$.

Can $E$ be partitioned into $F_1, \ldots, F_k$ such that for all $v \in V$: $|\deg_{F_i}(v) - \lambda_i \deg_E(v)| \leq 1$?

Combining $\lambda$–factor Theorem with Correa and Goemans’ previous ideas, we show $|\deg_{F_i}(v) - \lambda_i \deg_E(v)| < 3$. 

Martín Matamala
Factors
Feasibility
A necessary condition

- $G$ is $A$–feasible if for each $S \subseteq A$, $G$ has a $a_S$–factor, where $a_s = \sum_{a \in S} a$.
- In particular, $a_A$ must be the degree function of $G$.
- Feasibility can be tested in polynomial time.
Feasibility
A necessary condition

- $G$ is $A$–feasible if for each $S \subseteq A$, $G$ has a $a_S$–factor, where $a_s = \sum_{a \in S} a$.
- In particular, $a_A$ must be the degree function of $G$.
- Feasibility can be test in polynomial time.
Feasibility in the complete graph

\[ K_n = (V, E) \]

- Each function \( a \in A \) must be a graphical function, i.e., there exists a graph \( G \) on \( V \) such that \( d_G = a \).
- Can be tested in linear time: Erdös, Gallai (1960).
Feasibility in the complete graph

Each function $a \in A$ must be a **graphical function**, i.e., there exists a graph $G$ on $V$ such that $d_G = a$.

Can be tested in linear time: Erdös, Gallai (1960).
Feasibility is not sufficient

$G = K_5$, $|A| = 3$ and $A$ feasible but $G$ has not $A$-decomposition.

$$A = \begin{bmatrix}
1 & 0 & 0 & 2 & 1 \\
3 & 2 & 2 & 1 & 0 \\
0 & 2 & 2 & 1 & 3
\end{bmatrix}$$
Feasibility and Roomy
Using feasibility

- $e$ is **forced** if there is $a \in A$ such that $G - e$ has no $a$–factor.
- $A$ is roomy for $G$ if no edge of $e$ is forced.
- $p$–decomposition problem $\equiv$ its restriction to roomy matrices, in any class $\mathcal{C}$ closed under subgraphs.
Feasibility and Roomy

Using feasibility

- $e$ is **forced** if there is $a \in A$ such that $G - e$ has no $a$–factor.
- $A$ is roomy for $G$ if no edge of $e$ is forced.
- $p$–decomposition problem $\equiv$ its restriction to roomy matrices, in any class $C$ closed under subgraphs.
Complete graphs are not closed under subgraphs.
Complete bipartite graphs are not closed under subgraphs.

→ Known complexity results do not given information when $A$ is roomy.

An open problem.

When $G$ is complete or complete bipartite and $A$ is roomy, is $p$–decomposition problem polynomially solvable?
Complete graphs are not closed under subgraphs.

Complete bipartite graphs are not closed under subgraphs.

→ Known complexity results do not given information when $A$ is roomy.

An open problem.

When $G$ is complete or complete bipartite and $A$ is roomy, is $p$–decomposition problem polynomially solvable?
Feasibility and Roomy
An open problem

- Complete graphs are not closed under subgraphs.
- Complete bipartite graphs are not closed under subgraphs.
  → Known complexity results do not give information when $A$ is roomy.
- An open problem.

When $G$ is complete or complete bipartite and $A$ is roomy, is $p$-decomposition problem polynomially solvable?
Related open problems
Kundu’s conjecture, (1973)

Definition (tree-functions)

A graphical function $a : V \rightarrow \mathbb{N}$ is a tree-function if there is a tree $T$ on $V$ such that $a = d_T$.

- Kundu’s conjecture:
  When $|A| = n$, each $a \in A$ is a tree-function and $a_A$ is $2n - 1$,
  then $K_{2n}$ have a $A$–decomposition?.
- In this case $A$ is roomy!
Related open problems

Kundu’s conjecture, (1973)

Definition (tree-functions)

A graphical function $a : V \rightarrow \mathbb{N}$ is a tree-function if there is a tree $T$ on $V$ such that $a = d_T$.

- Kundu’s conjecture:

  When $|A| = n$, each $a \in A$ is a tree-function and $a_A$ is $2n - 1$,

  then $K_{2n}$ have a $A$–decomposition?.

- In this case $A$ is roomy!
Related open problems

Kundu’s conjecture, (1973)

Definition (tree-functions)

A graphical function \(a : V \rightarrow \mathbb{N}\) is a \textit{tree-function} if there is a tree \(T\) on \(V\) such that \(a = d_T\).

- Kundu’s conjecture:
  
  When \(|A| = n\), each \(a \in A\) is a tree-function and \(a_A\) is \(2n - 1\),
  
  then \(K_{2n}\) have a \(A\)–decomposition?.

- In this case \(A\) is roomy!
**Related open problems**

*Kundu’s conjecture, (1973)*

---

**Definition (tree-functions)**

A graphical function $a : V \rightarrow \mathbb{N}$ is a **tree-function** if there is a tree $T$ on $V$ such that $a = d_T$.

- **Kundu’s conjecture:**
  
  When $|A| = n$, each $a \in A$ is a tree-function and $a_A$ is $2n - 1$,
  
  then $K_{2n}$ have a $A$–decomposition?.

- In this case $A$ is roomy!
Kundu’s conjecture
Partial results

- **Kundu** (1974), $|A| = 3$ and two functions are tree-functions.
- **Kundu** (1975), $|A| = 4$, three functions are tree-functions + fourth is upper bounded by $2n - 5$.
- **Kleitman, Koren, Li** (1977), $|A| = 3$ and two of them are forest-functions.
- **M., Zamora** (2006), $A$ is obtained by a cyclic rotation of a function $a$.

Uses graceful labeling of some trees (caterpillar).

Related to another conjecture of Rosa 1960...
Kundu’s conjecture
Partial results

- **Kundu** (1974), $|A| = 3$ and two functions are tree-functions.
- **Kundu** (1975), $|A| = 4$, three functions are tree-functions + fourth is upper bounded by $2n - 5$.
- **Kleitman, Koren, Li** (1977), $|A| = 3$ and two of them are forest-functions.
- **M., Zamora** (2006), $A$ is obtained by a cyclic rotation of a function $a$.
  
  Uses graceful labeling of some trees (caterpillar).
  
  Related to another conjecture of **Rosa** 1960...
Conjectures

Too much for today!

Merci Beaucoup!
Conjectures

Too much for today!

Merci Beaucoup!
Decomposition in factors
Computational Complexity
Scaled factors always exists
Open Problems

$\lambda -$ factors
Complete graphs and special $A$

??????