Progresses in the analysis of Stochastic 2D cellular automata:

Asynchronous 2D Minority

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Joint work with

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Asynchronous Systems

Most of the real systems are asynchronous

- Networks, physical particles, biological cells...
- How does **randomness introduced by asynchonicity** affect the global behavior of these systems?

Example

A ring network

- where each node has **two states**:

  "has a token" or *"does not have a token"*

- running an algorithm that redistributes the tokens according to some **rules/constraints**:

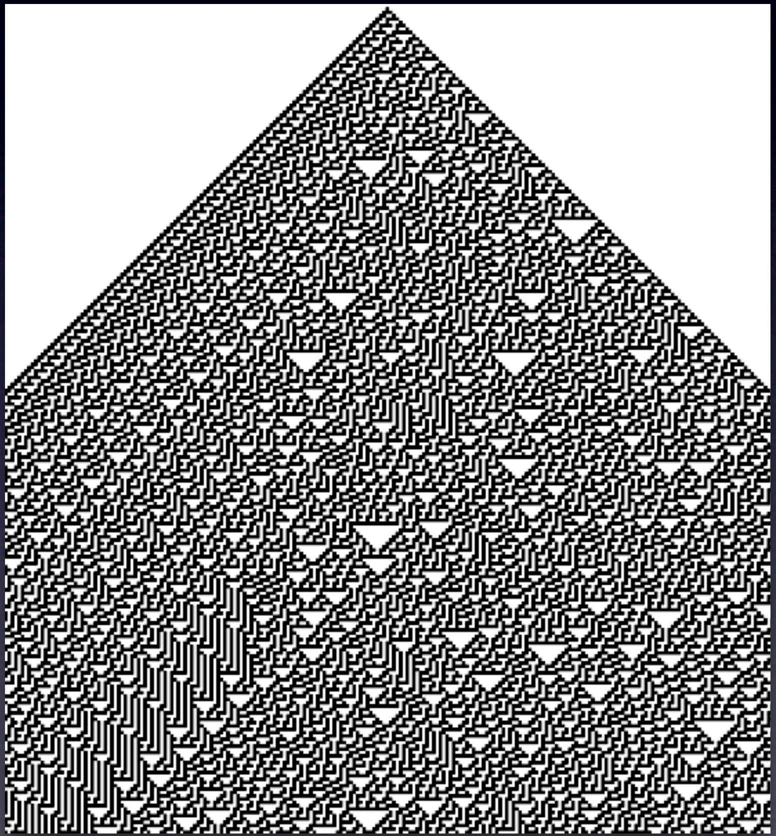
  "I get a token if none of my neighbors have one" or

  "I get a token if my right neighbor has one",...

- **Example of question**

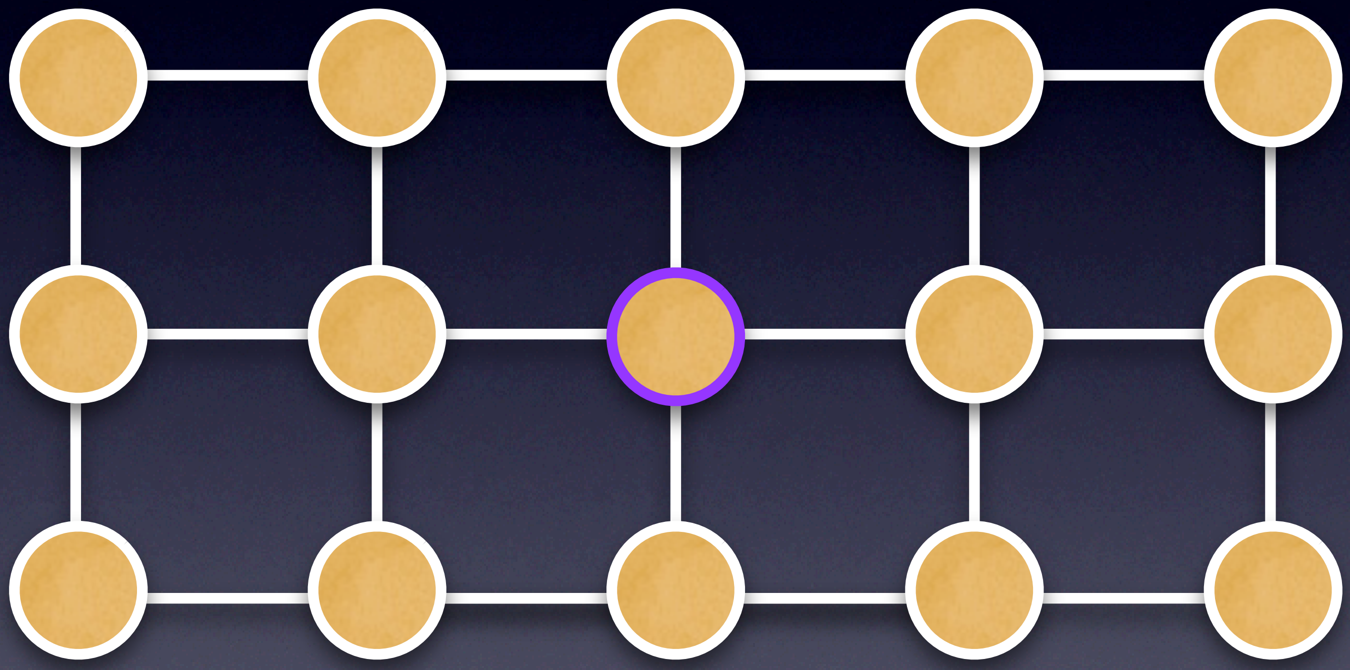
  How long does it take to reach a "stable configuration"?

Nature is an other example

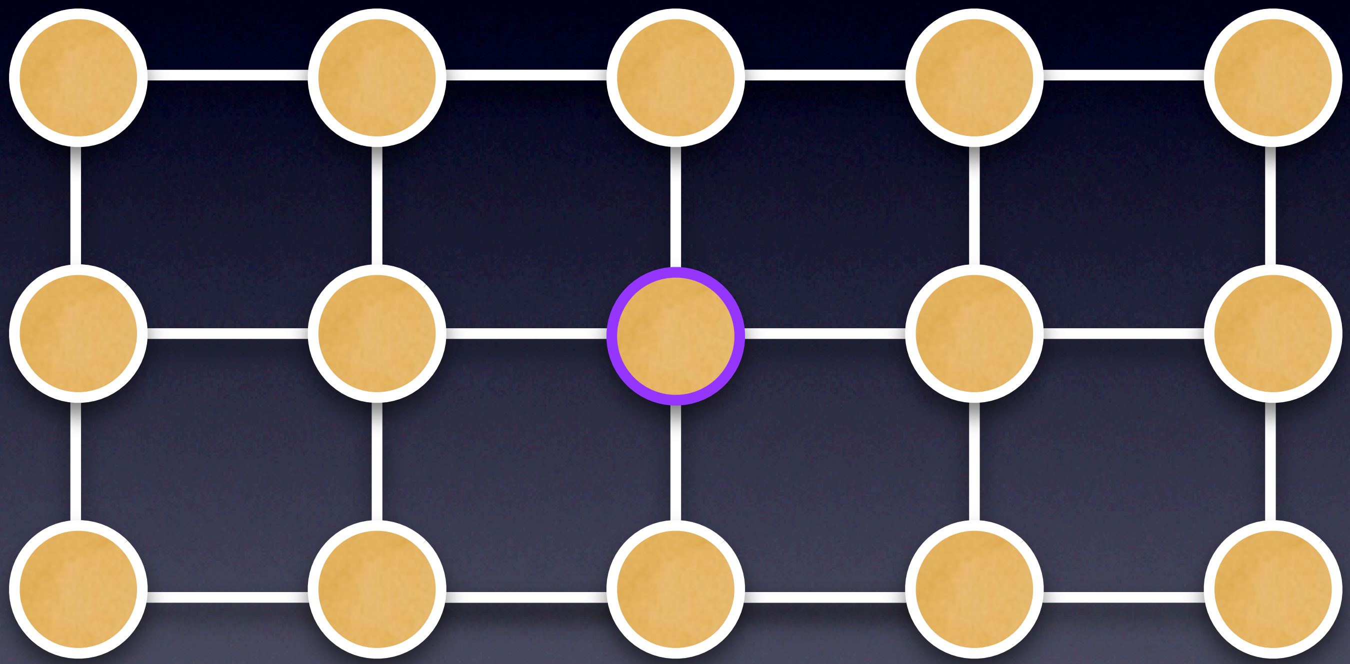


*Patterns **are** governed by rule 30 as the shell grows*

2D Cellular automata

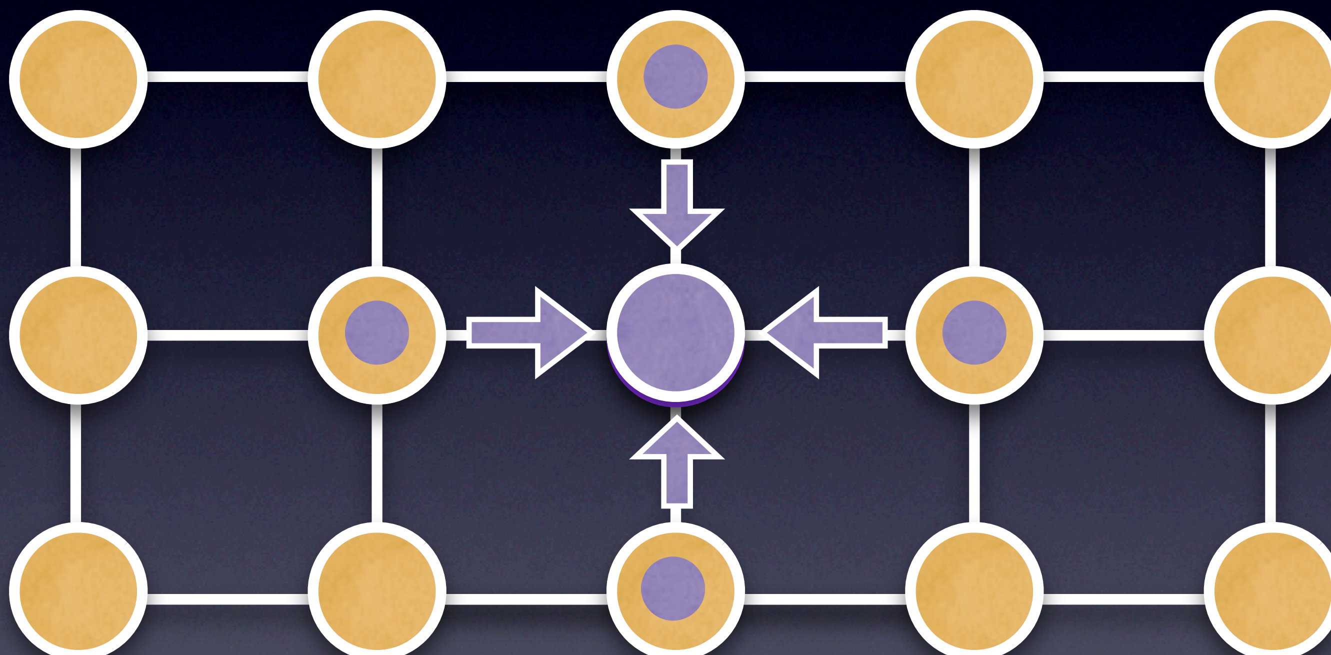


2D Cellular automata



At each time step, each cell updates its states according to the state of its neighbors

2D Cellular automata



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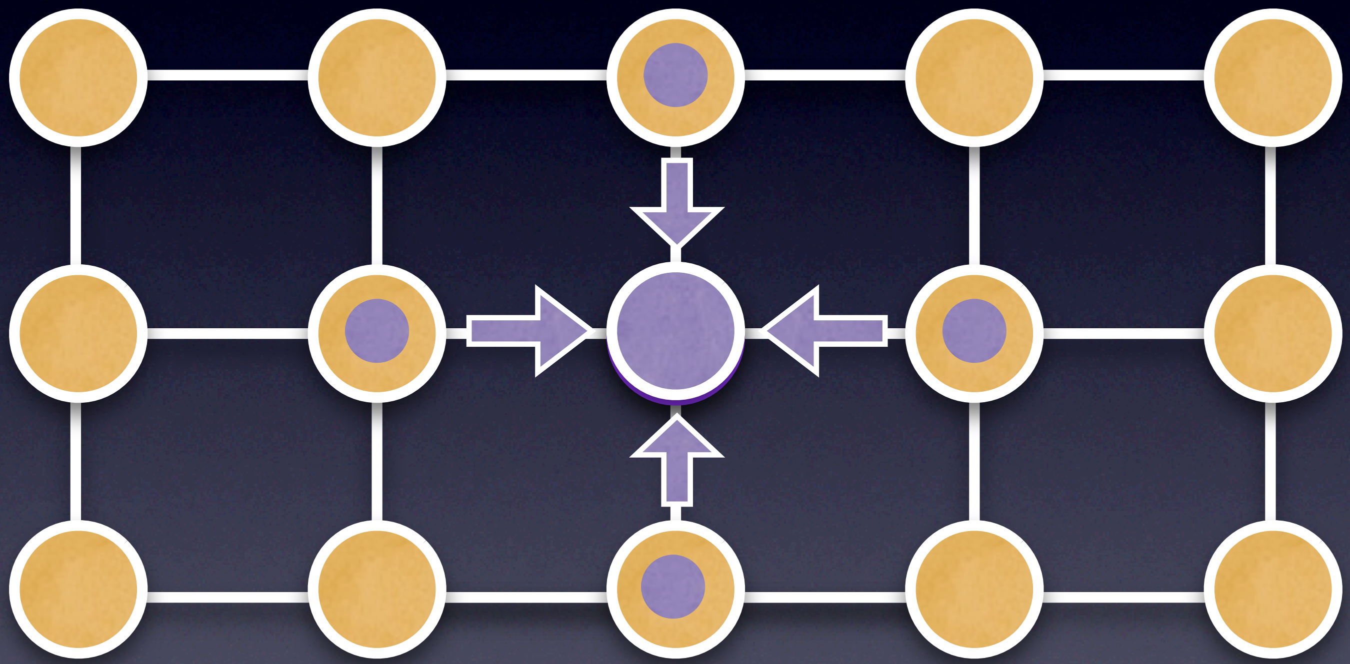
2D Cellular automata



Extensively used in physics, biology,...

What happens in asynchronism regims?

Cellular automata, here

- **0/1** state (**0** = white & **1** = black)
- **Full asynchronism**
  A deamon chooses a random cell uniformly at random and updates it
- **α-asynchronism**
  Each cell is independently updated with probability $\mathbf{0 < \alpha < 1}$

Full synchronism: $\boldsymbol{\alpha = 1}$

Full asynchronism: "limit" for $\boldsymbol{\alpha \to 0}$

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Full synchronism: ***$\alpha = 1$***

Full asynchronism: "limit" for ***$\alpha \to 0$***

Demo

Historic

Ergodicity of deterministic CA with random noise

- Toom - Gasc - Gray - Park - Louis (1974-)

Indecidability of independance to update history

- Gacs (2002)

Empiral studies of asynchronism

- Buvel, Ingerson - Bersini, Detour - Schönfish, de Roos (1995-)

Study of particular automata or classes of automata

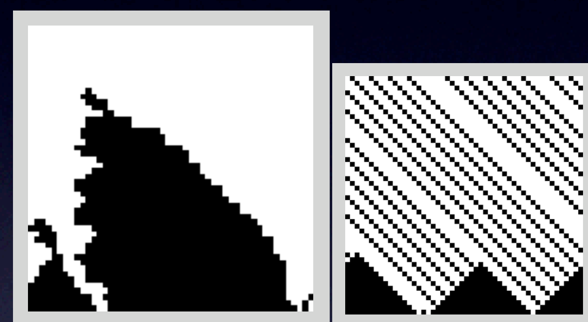
- Fuks (2004 -)
- Fatès, Morvan, Regnault, S., Thierry (2005-)
- Chassaing, Gerin (2007)

Historic: 1D automata

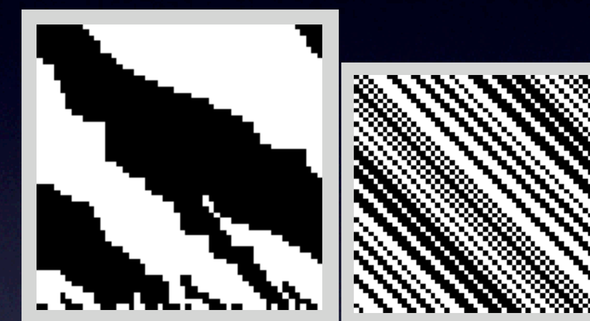
6 types of relaxation times

[FMST 2005, FRST 2006, CG 2007]

(a) LOGARITHMIC (232)



(b) LINEAR (130)



(c) QUADRATIC (170)

(c') QUADRATIC (146)

(d) EXPONENTIAL (210)

(e) DIVERGING (150)

232

130

170

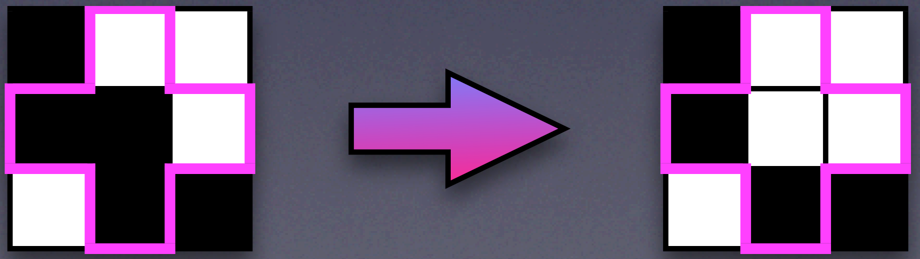
210

150

Behavior	ACE (#)	Rule	01	10	010	101	Convergence
Identity	204 (1)	∅	•	•	•	•	0
Coupon collector	200 (2)	E	•	•	✔	•	$\Theta(n \ln n)$
	232 (1)	DE	•	•	✔	✔	
Monotone	206 (4)	B	←	•	•	•	$\Theta(n^2)$
	222 (2)	BC	←	→	•	•	
	234 (4)	BDE	←	•	✔	✔	
	250 (2)	BCDE	←	→	✔	✔	
	202 (4)	BE	←	•	✔	•	
	192 (4)	EF	→	•	✔	•	
	218 (2)	BCE	←	→	✔	•	
	128 (2)	EFG	→	←	✔	•	
Biased random walks	242 (4)	BCDEF	⟷	→	✔	✔	
	130 (4)	BEFG	⟷	←	✔	•	
Random walks	226 (2)	BDEF	⟷	•	✔	✔	$\Theta(n^3)$
	170 (2)	BDEG	←	←	✔	✔	
	178 (1)	BCDEFG	⟷	⟷	✔	✔	
	194 (4)	BEF	⟷	•	✔	•	
	138 (4)	BEG	←	←	✔	•	
	146 (2)	BCEFG	⟷	⟷	✔	•	
Biased random walk	210 (4)	BCEF	⟷	→	✔	•	$\Theta(n2^n)$
Diverging	198 (2)	BF	⟷	•	•	•	Diverging
	142 (2)	BG	←	←	•	•	
	214 (4)	BCF	⟷	→	•	•	
	150 (1)	BCFG	⟷	⟷	•	•	

Fully asynchronous 2D Minority

- **0/1** states
- **n x m toric** configurations
- A daemon selects **uniformly at random** a cell and the cell updates to the **minority state among its 4 neighbors and itself.**



Energy function

Potential. $\mathbf{v_{ij}}$ = #{neighbors in the same state as (i,j)}

Cell (i,j) is **active** if $\mathbf{v_{ij}} \geq 2$.

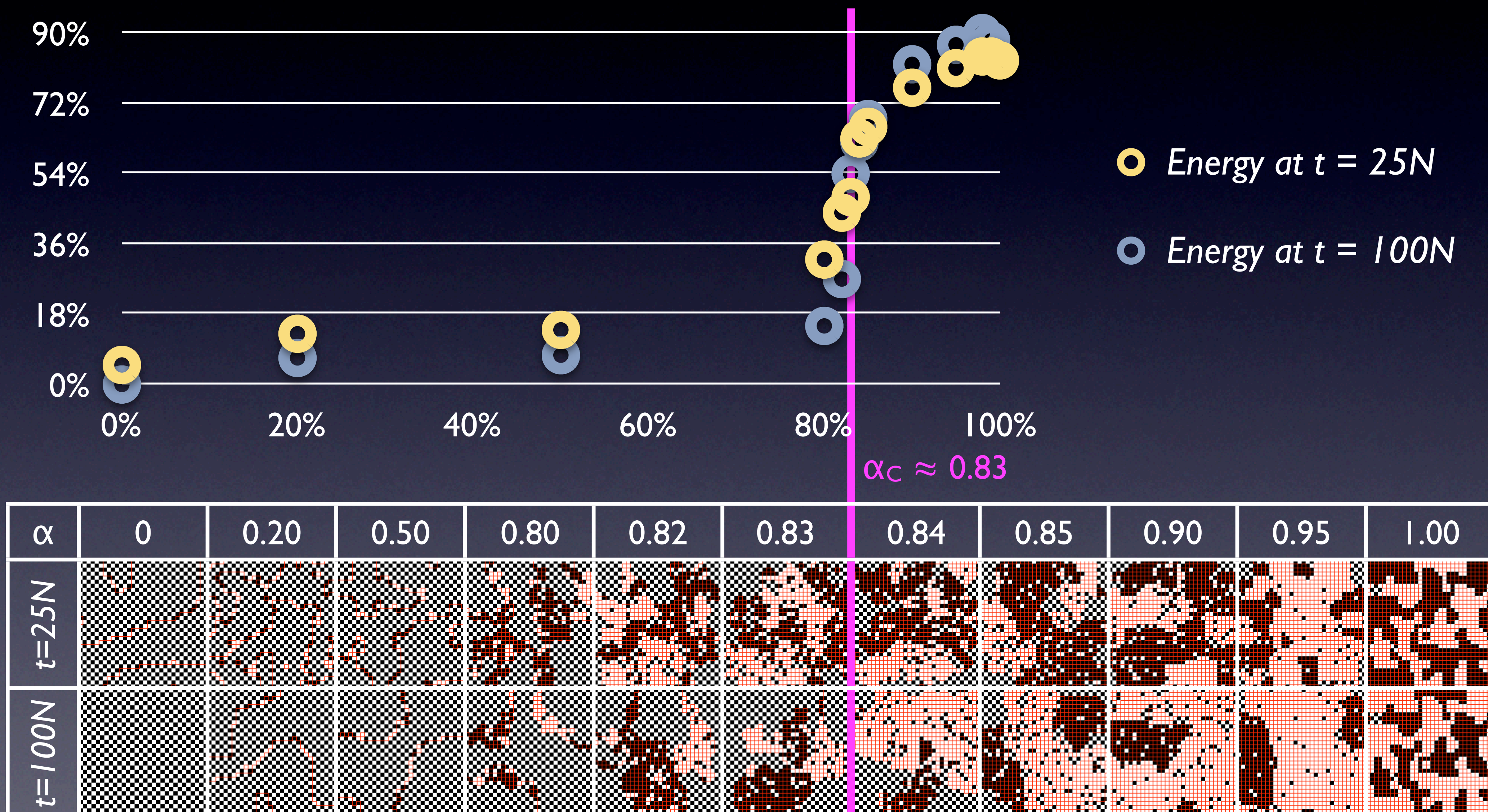
Energy of configuration *c*:

$$\mathbf{Energy}(\boldsymbol{c}) = \sum_{(i,j)} \mathbf{v_{ij}}$$

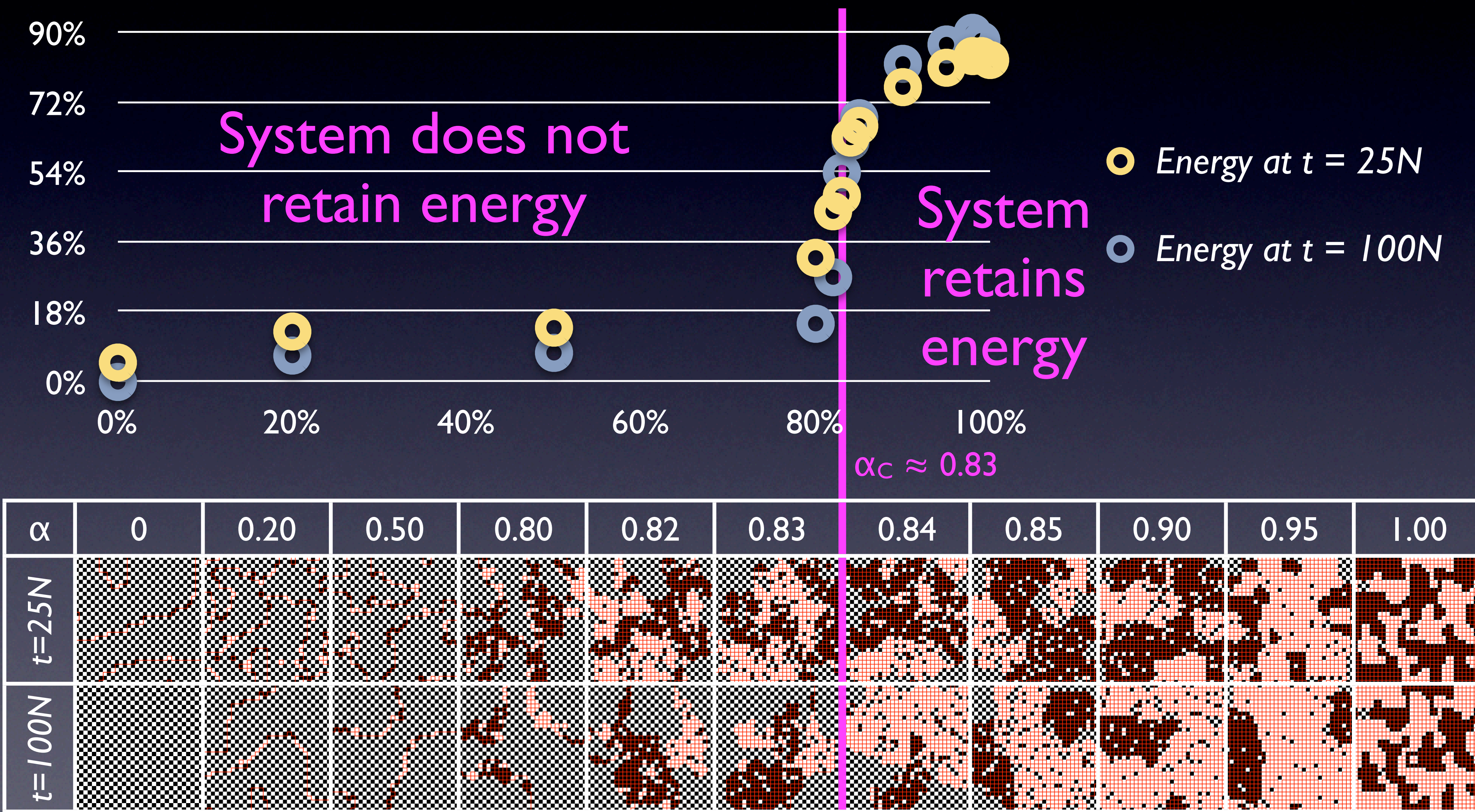
Fact. The expected energy of a random configuration is

2 N,   where N = n m

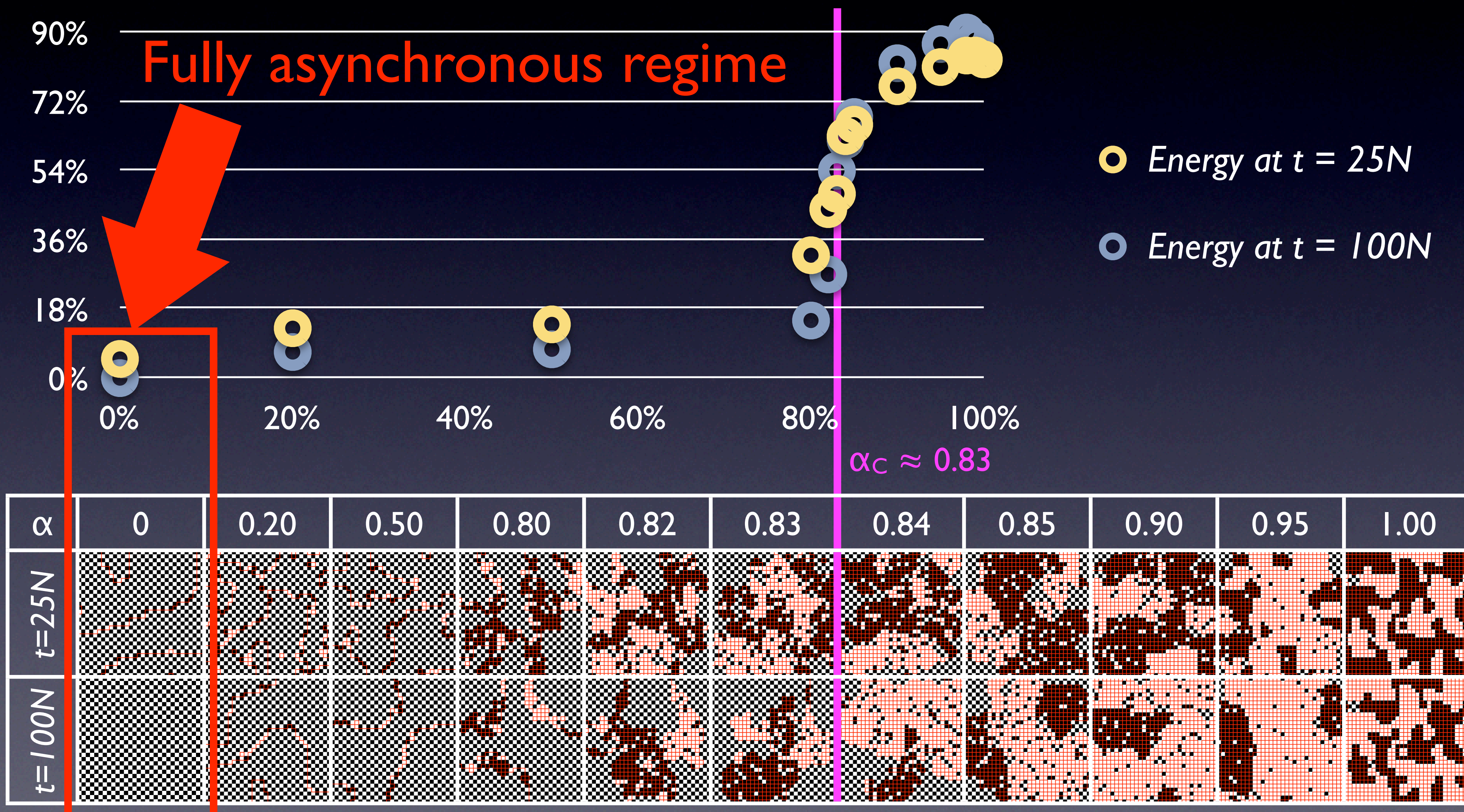
Observed phase transition



Observed phase transition



Observed phase transition



Energy is non-increasing when fully asynchronous

Theorem. The energy of a configuration is a **non-increasing** function of time in fully asynchronous dynamic.

Energy decreases by **at least 4** each time a cell with **potential at least 3** is updated.

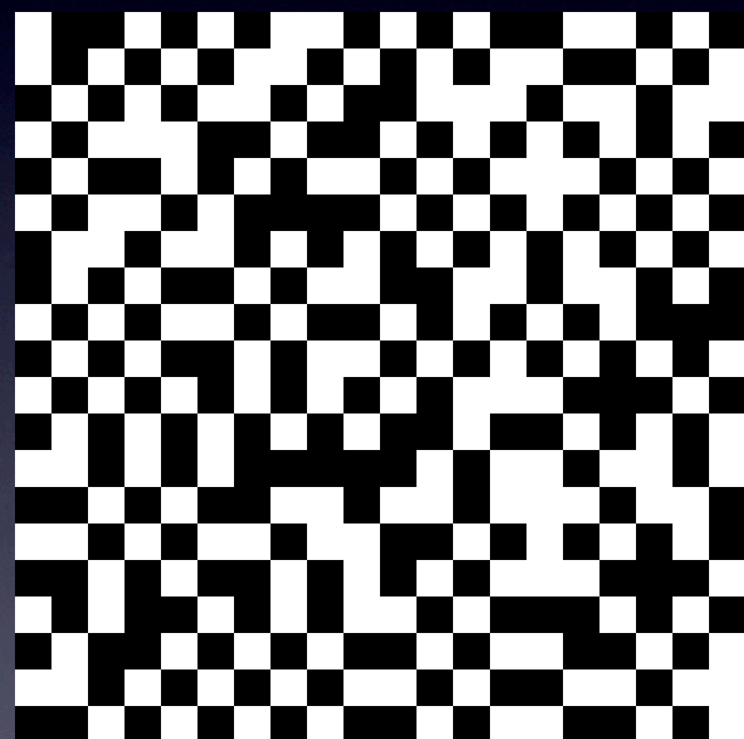
proof. Δ**Energy** $= 8 - 4\,\mathbf{v_{ij}}$, when (i,j) is updated.

Initial energy drop

Theorem. The energy of any configuration of size N is at most N+2N/3 after $O(N^2)$ updates on expectation.

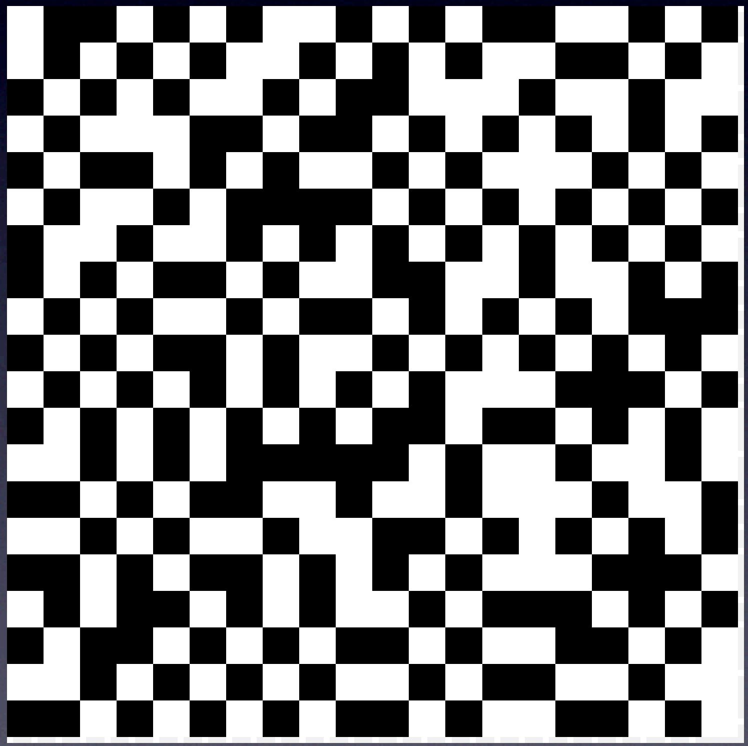
proof. Any such configuration contains a neighborhood in which a finite sequence of updates decreases the energy by at least 1.

Borders



Borders

There is a **red border** between two **neighboring** cells in the **same** state.



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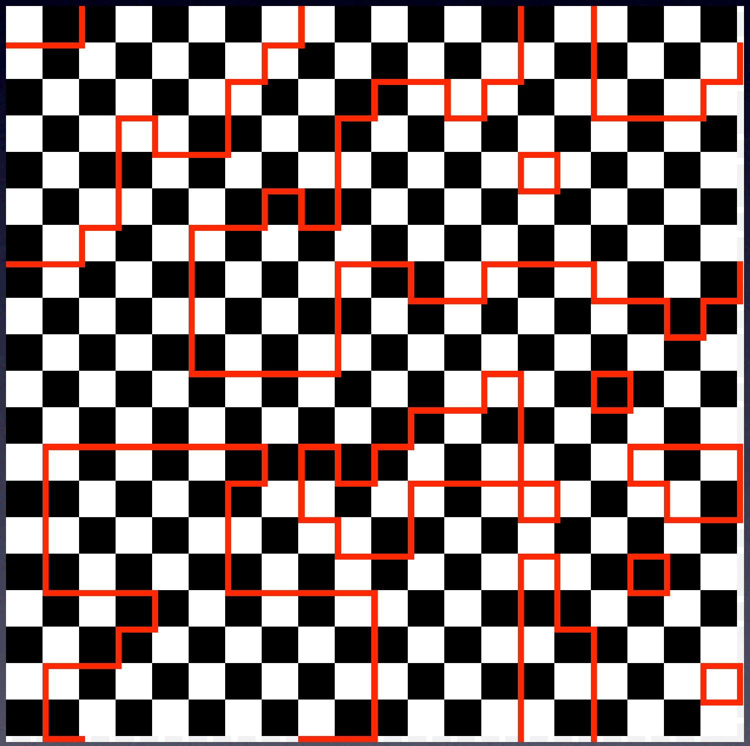

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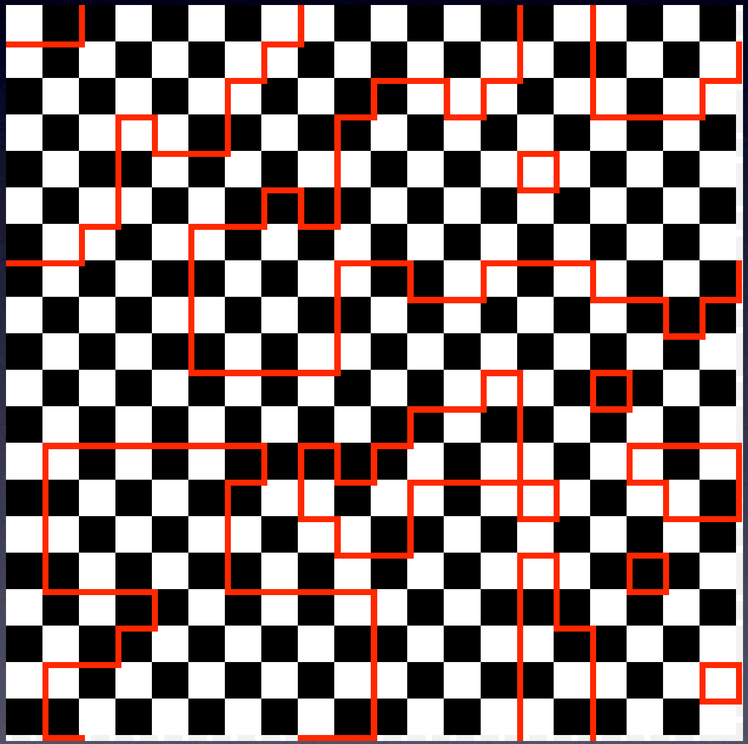
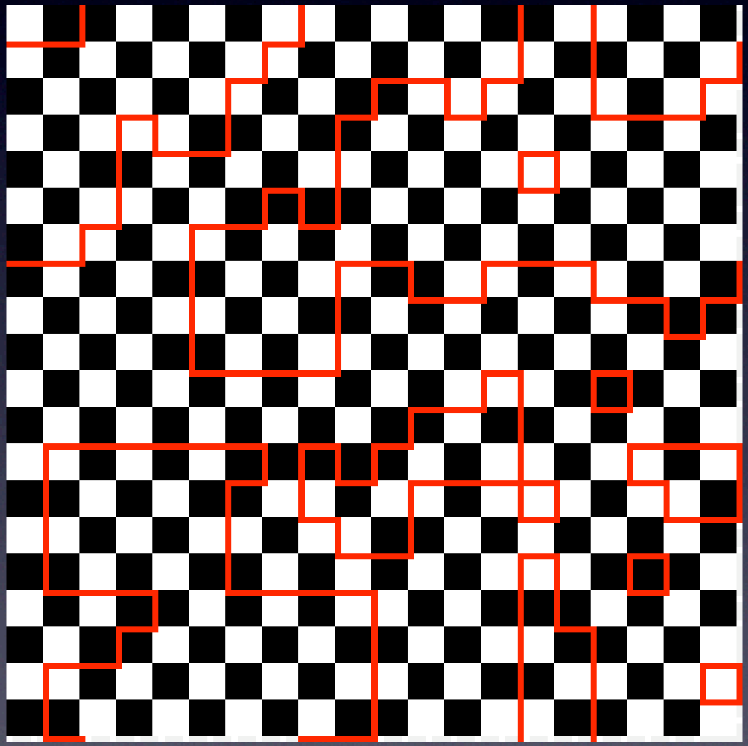


Fact. The borders are the **boundaries** of the covering **homogeneous checkerboards regions**

Dual configurations

(from now on, *n* and *m* are even)

dual configuration = configuration **XOR checkerboard**



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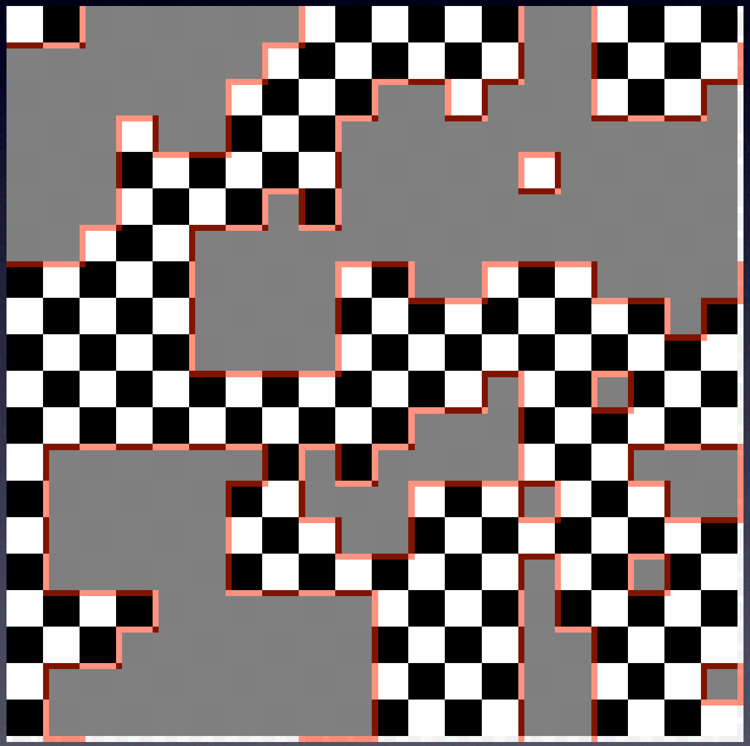

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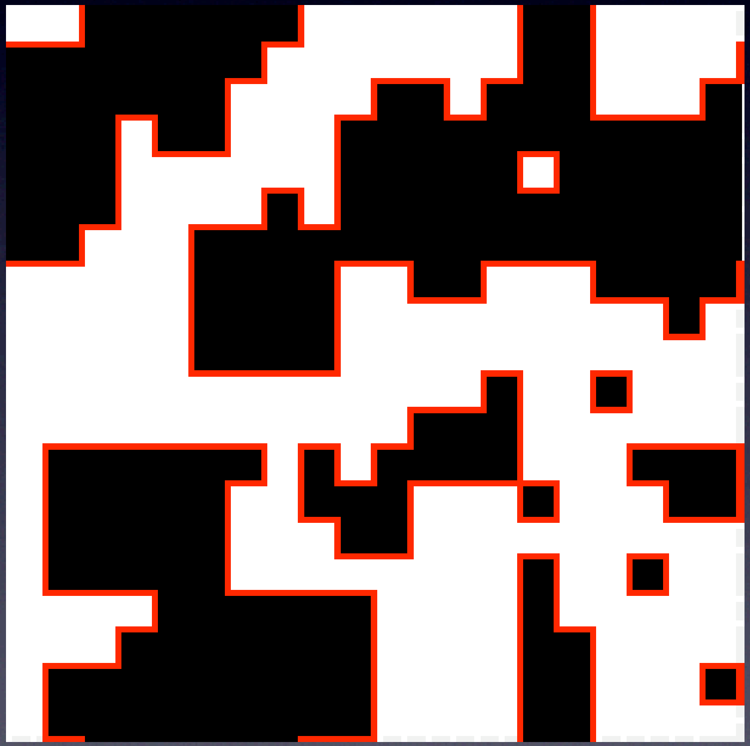
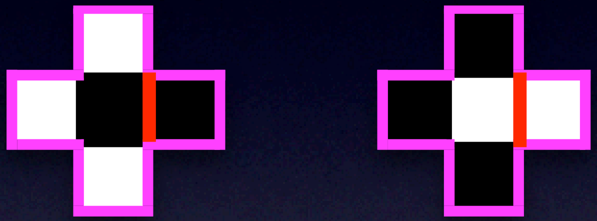
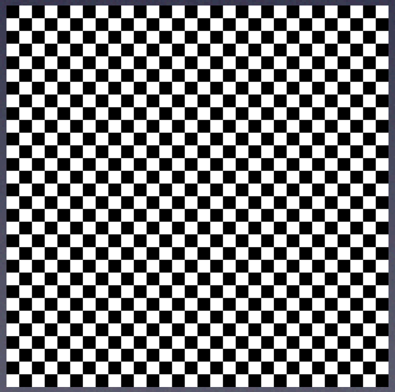


Stable configurations

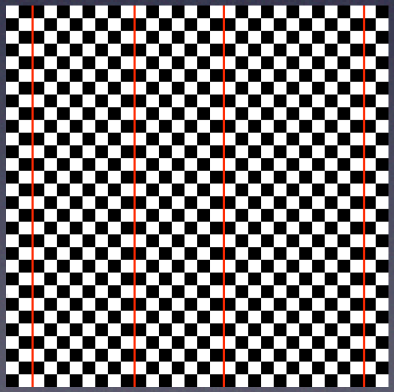
Stable configurations are made of cells with **potential ≤ 1**, i.e. in contact with at most one border:



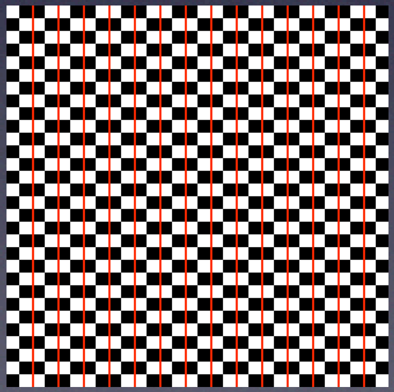
Fact. Stable configurations are made of an **even** number of **bands of width ≥ 2** tiled by alternating **checkerboards**.



of lowest energy



of higher energy



of highest energy

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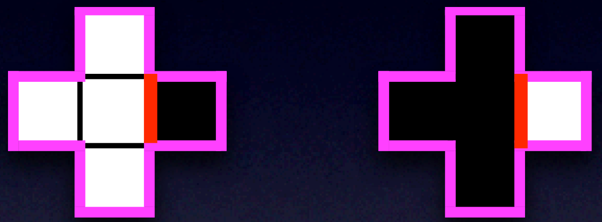

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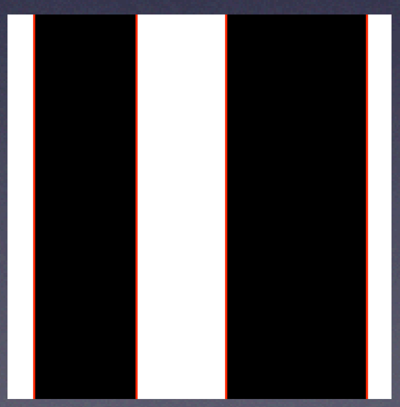

of higher energy

of highest energy

Energy of configurations

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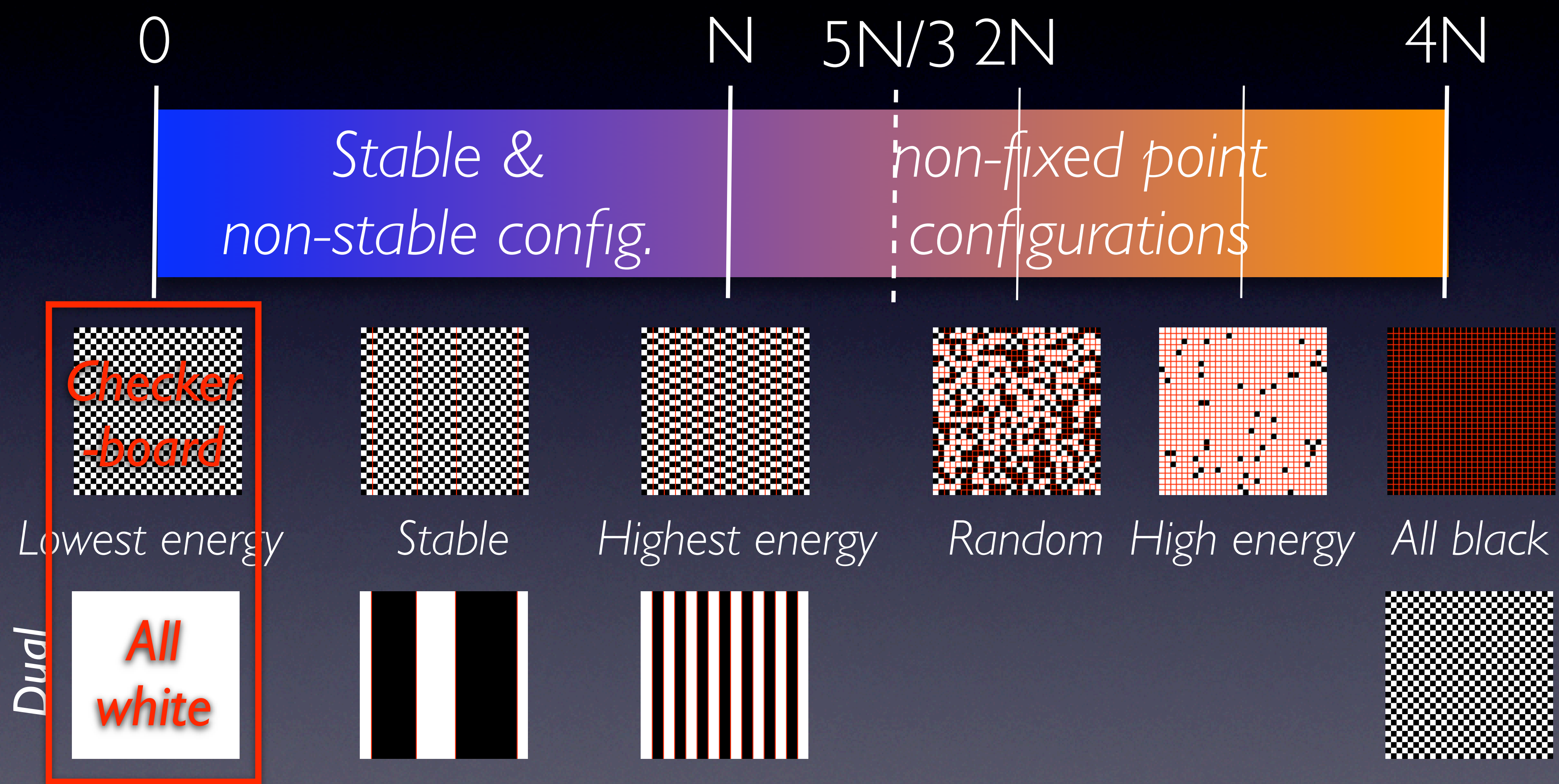


Typical asynchronous 2D minority

t = 0

t = *N*

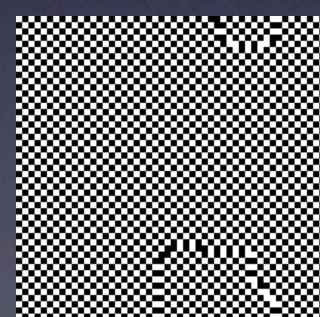
t = 5*N*

t = 20*N*

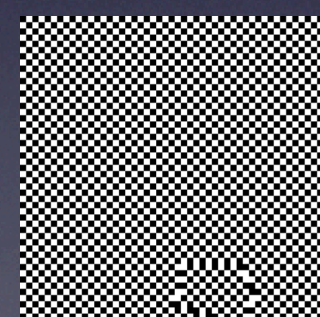
t = 50*N*

t = 75*N*

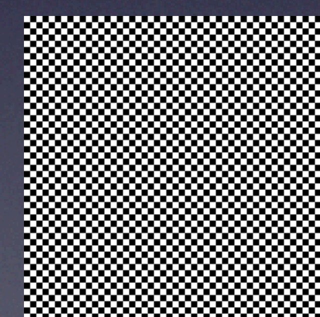
t = 150*N*



t = 300*N*



t = 350*N*



t = 381*N*

Convergence is almost sure

Definition. The dynamics **converges** from an **initial configuration** c^0 if $\mathbf{T} = \min\{t : c^t \text{ is stable}\}$ is almost surely **finite**.

Theorem. For all c^0, $\mathbf{E}(\mathbf{T}) \leq 2N \bullet N^{2N}$.

Proof. The following is a sequence of at most **2N** updates that stabilizes any configuration:

1. **As long as** there is an active black cell, **flip** it;
2. **As long as** there is an active white cell, **flip** it.

This sequence is followed with probability $\mathbf{1/N^{2N}}$.

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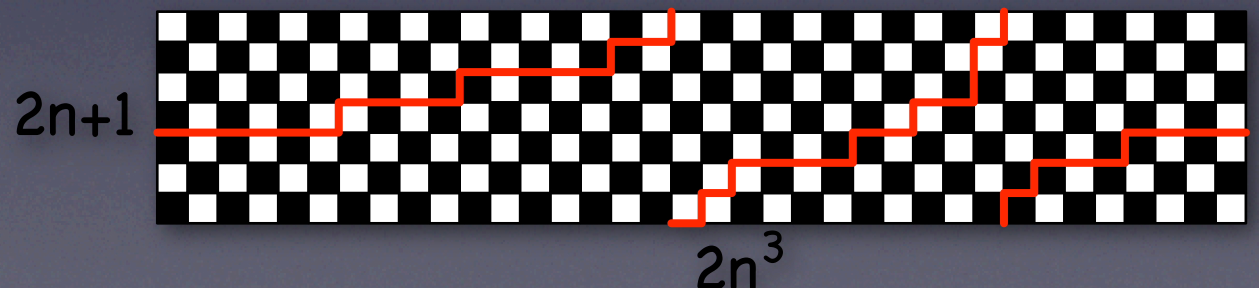
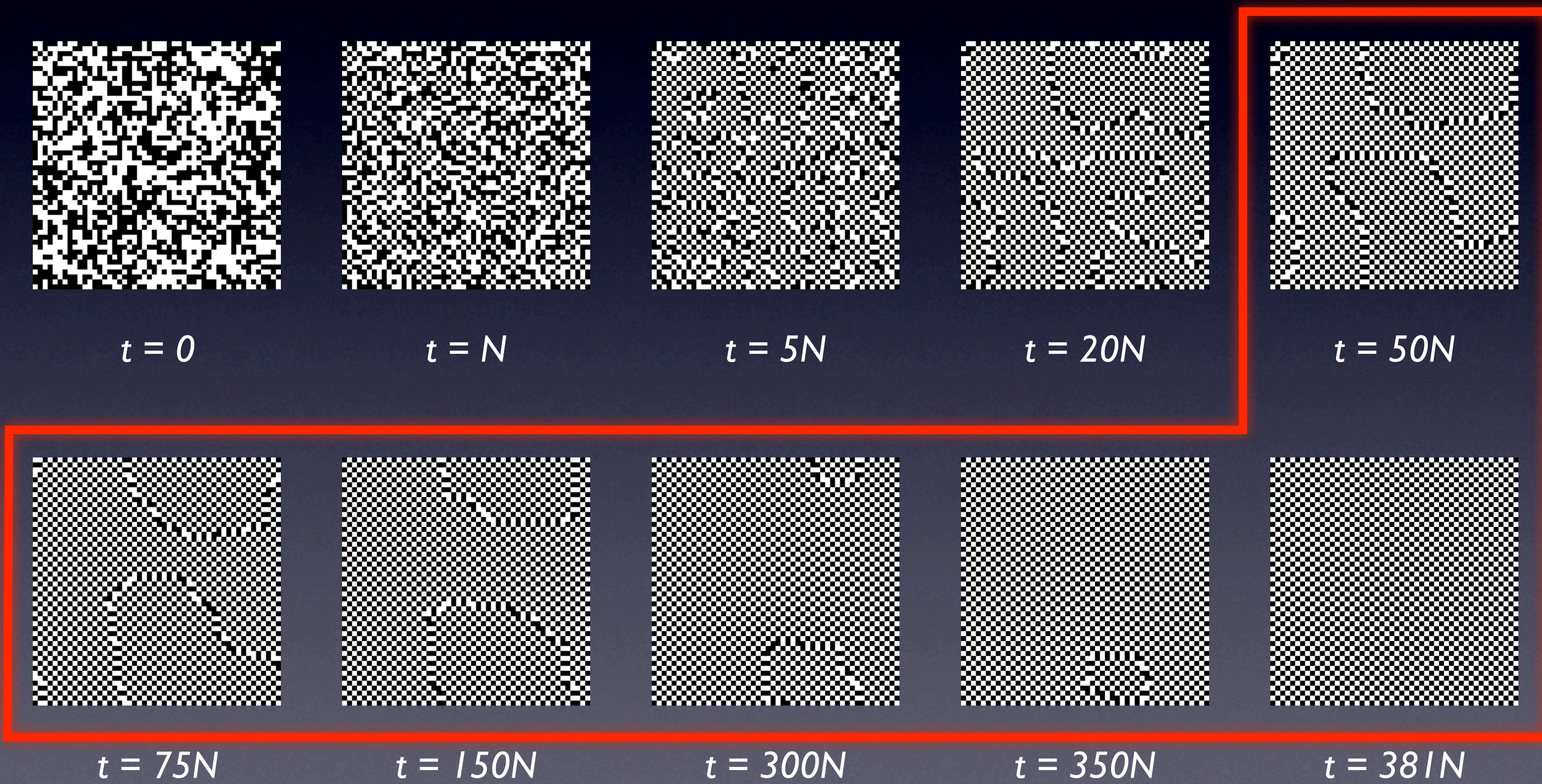
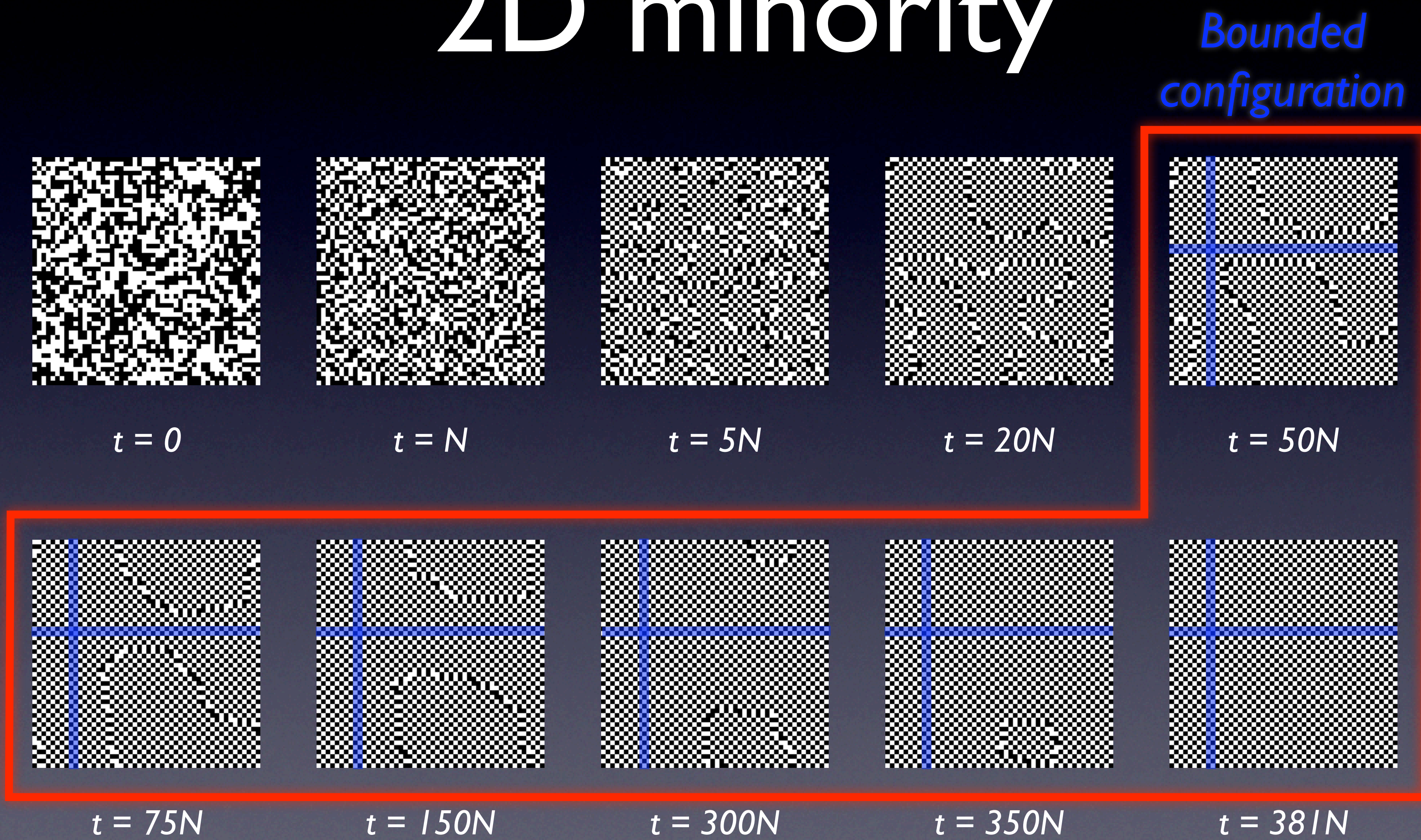
Conjecture. If one of *n* or *m* is **odd**, the convergence time can indeed be **exponential**.



Typical asynchronous 2D minority



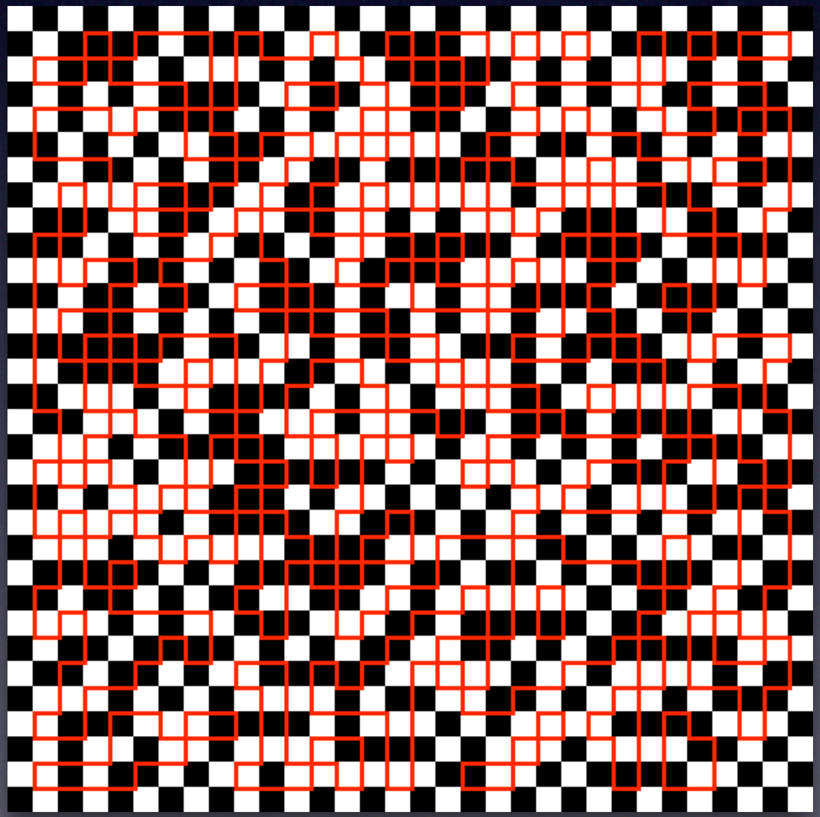
Typical asynchronous 2D minority



Polynomial convergence

(**conjecture.** It is true as soon as n and m are even)

We study ***bounded configuration***, which are surrounded by a checkerboard of width ≥ 2:



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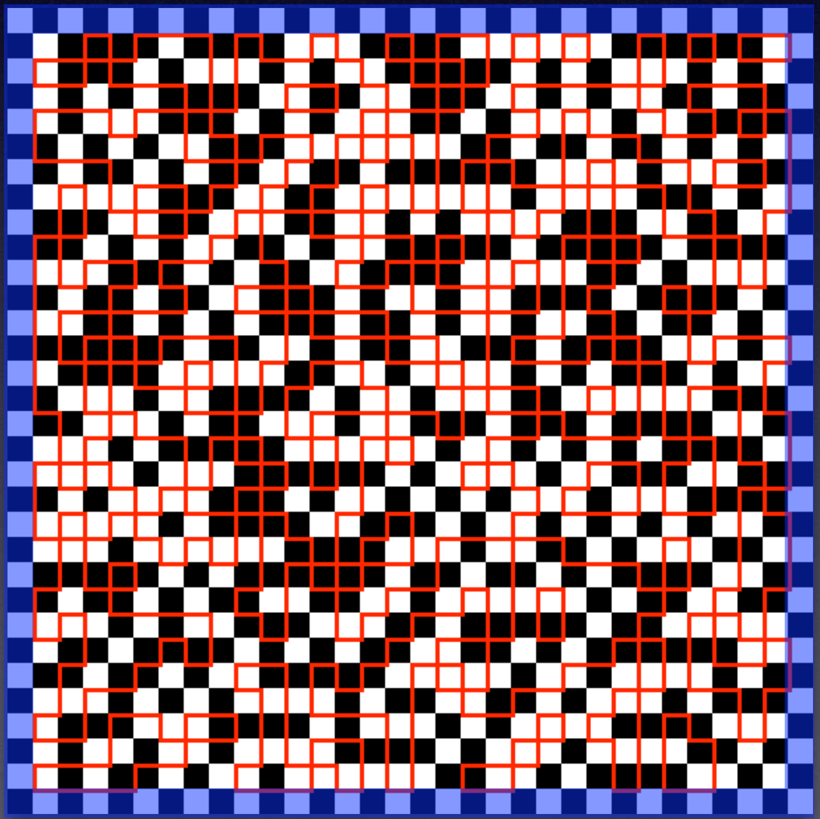
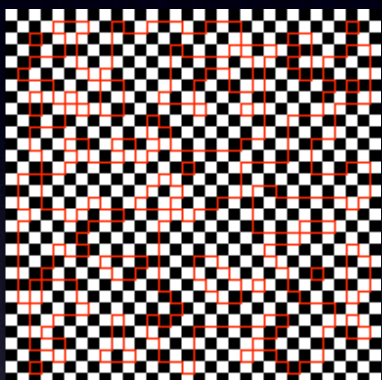
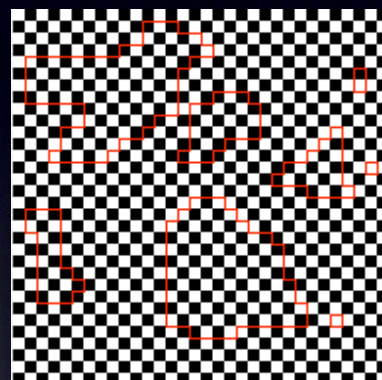


The surrounding checkerboard is **stable**

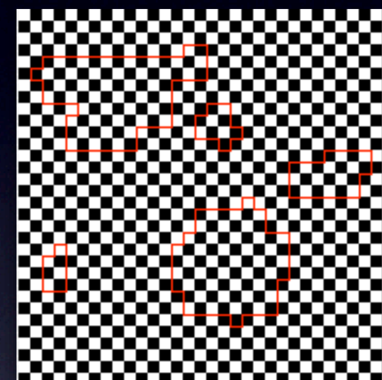
Typical evolution of a bounded configuration



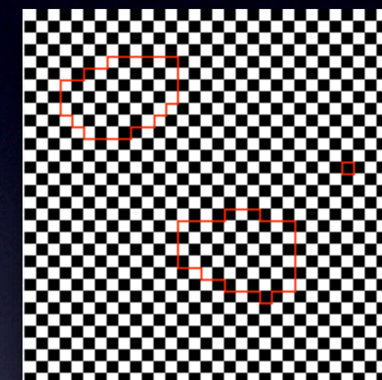
$t = N$



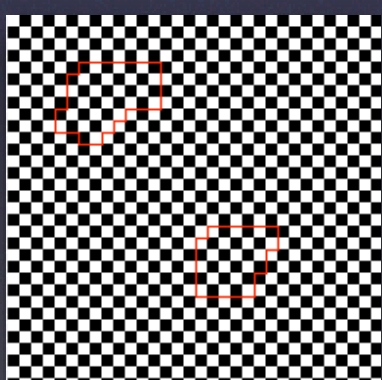
$t = 5N$



$t = 10N$



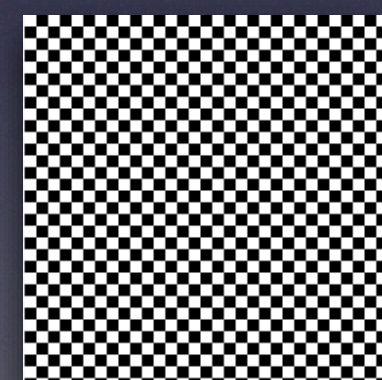
$t = 15N$



$t = 20N$

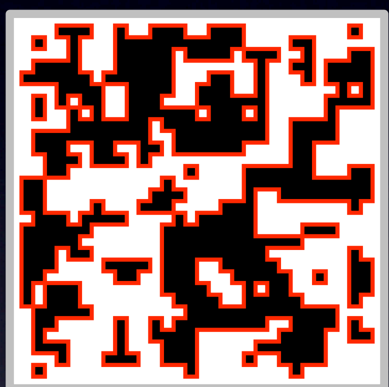
$t = 25N$

$t = 30N$

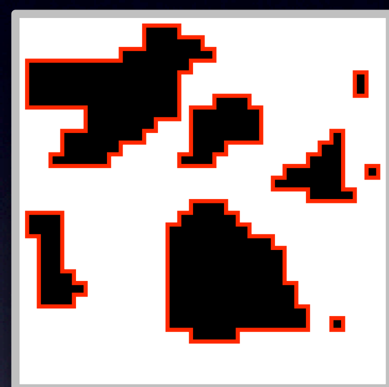


$t = 35N$

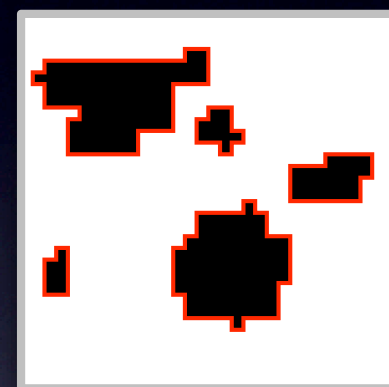
Dual evolution: Automaton **OT976**



$t = N$



$t = 5N$



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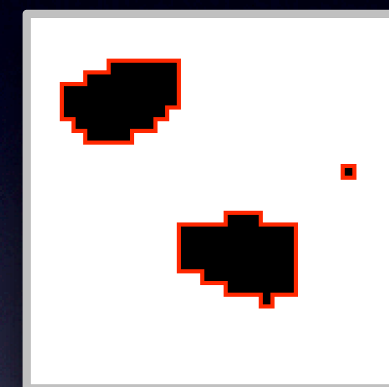
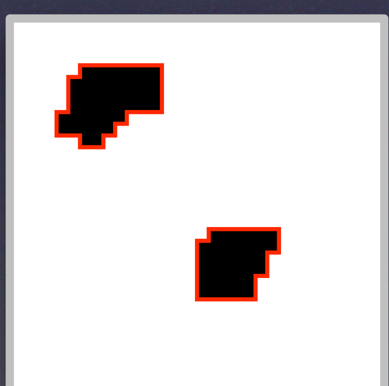
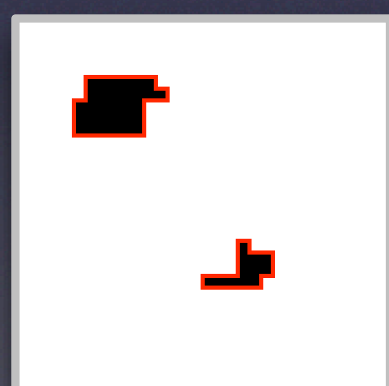


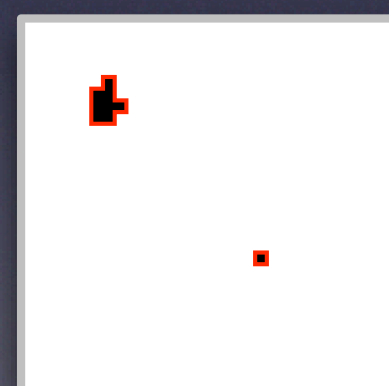
$t = 15N$



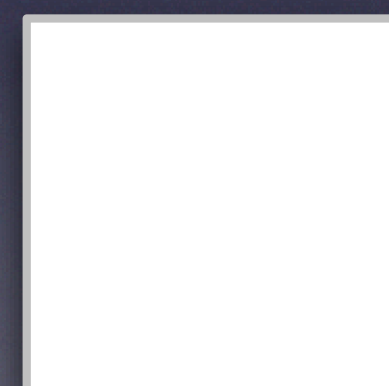
$t = 20N$



$t = 25N$

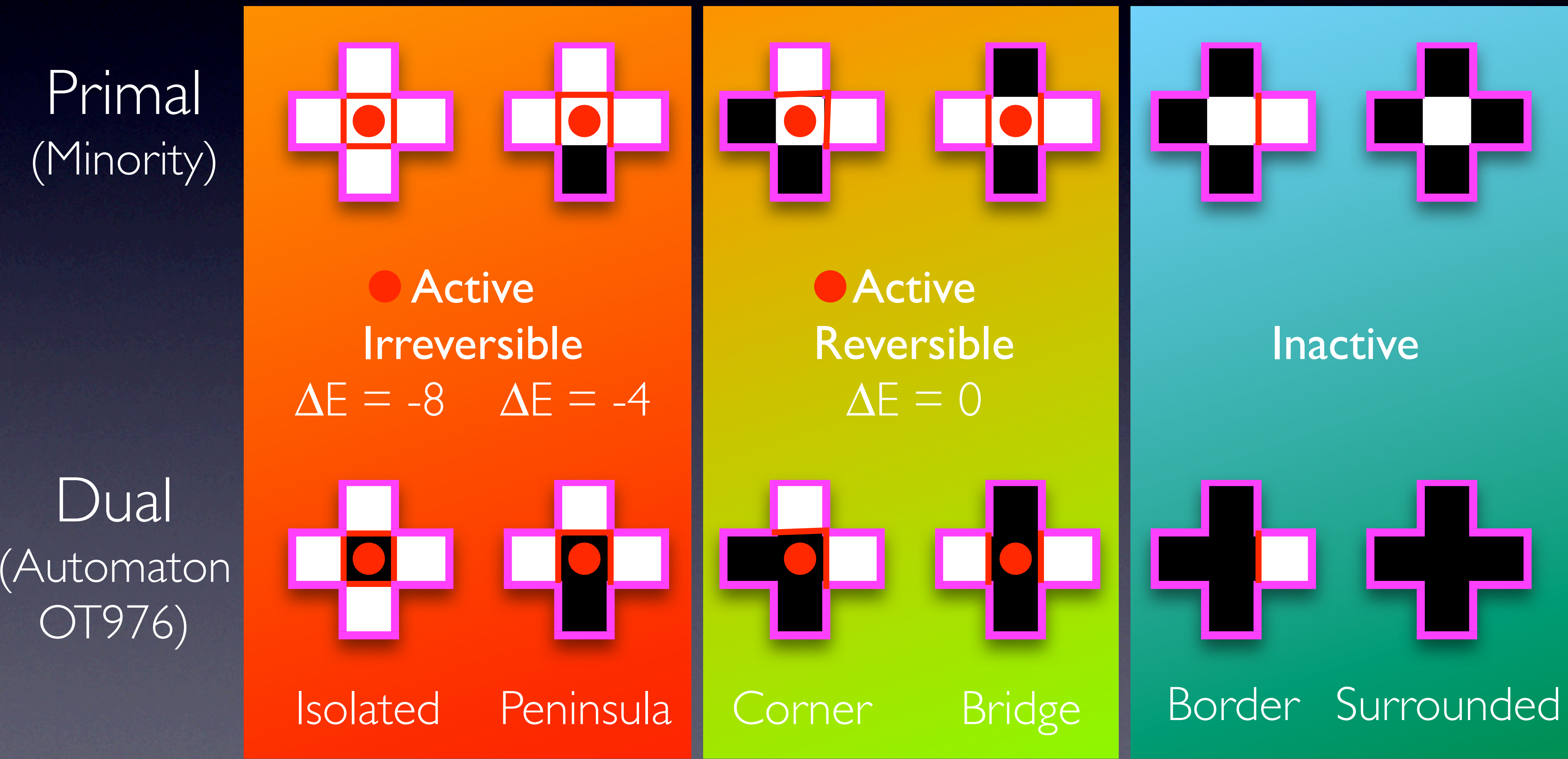


$t = 30N$



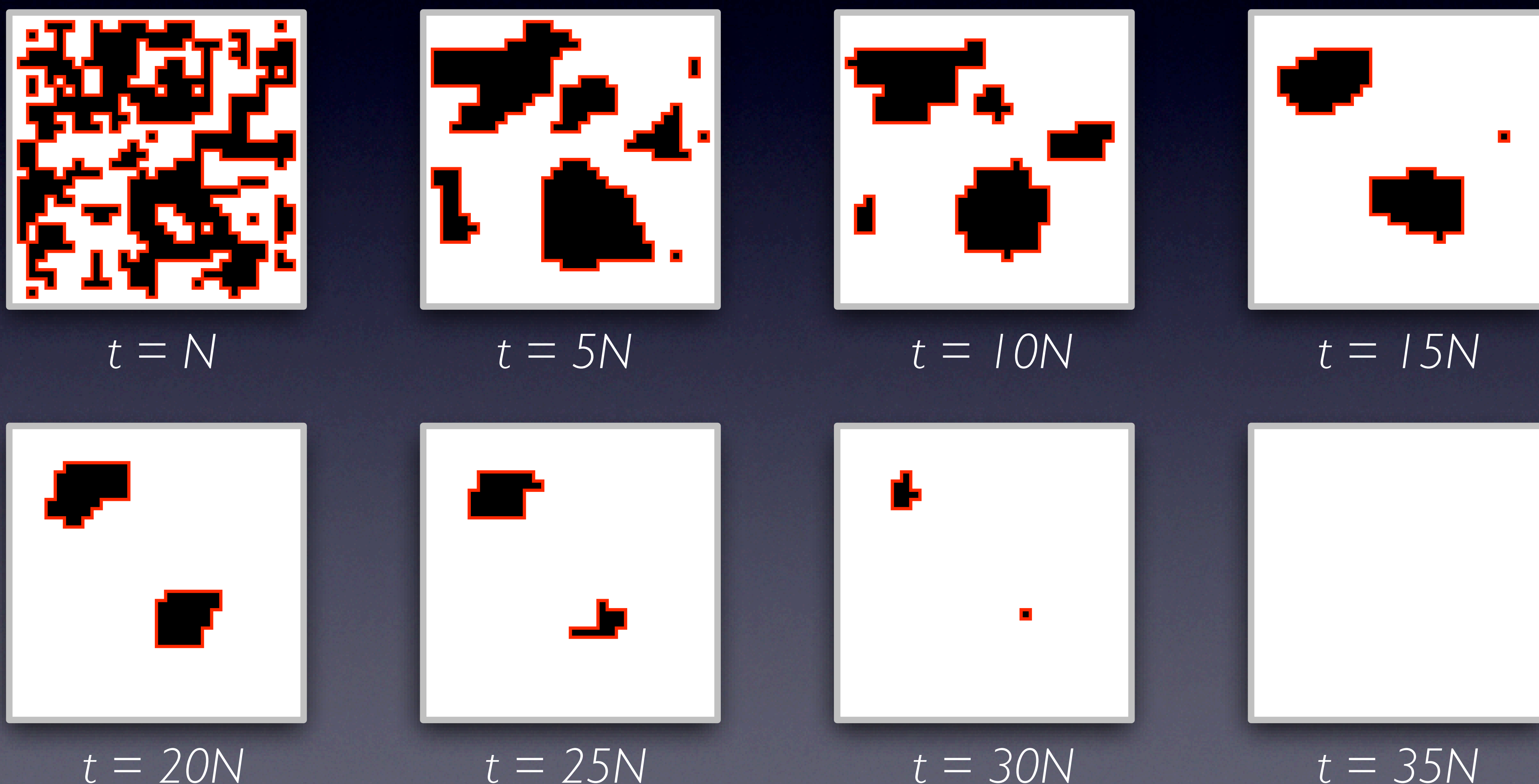
$t = 35N$

Rules of the primal and dual dynamics



neighborhoods are considered with white/black symetries & rotations

Coupling with the HV-convex hull



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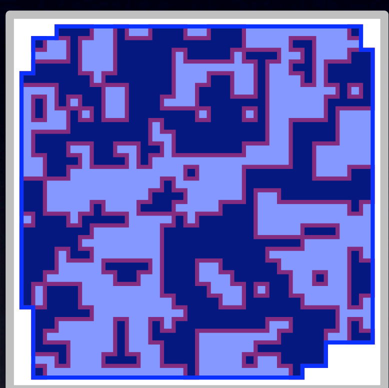
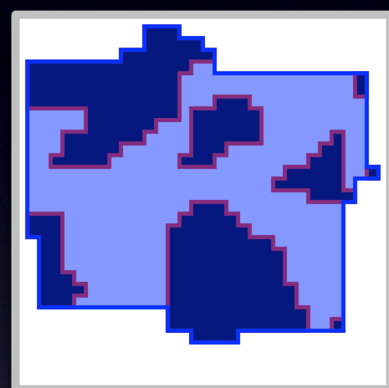
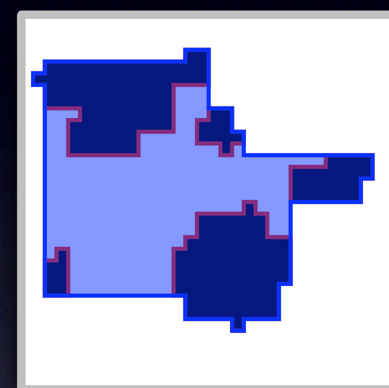


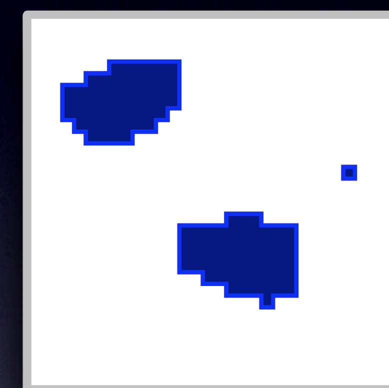
$t = N$



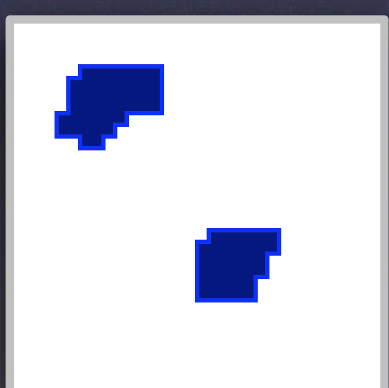
$t = 5N$



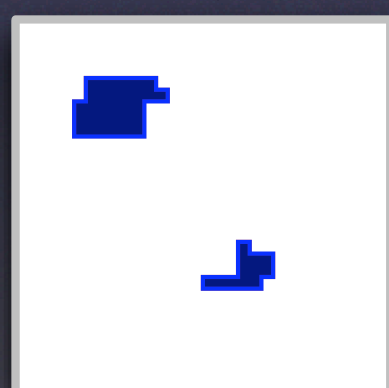
$t = 10N$



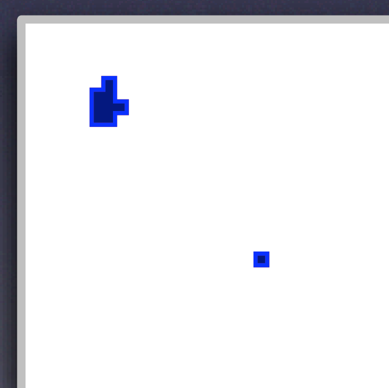
$t = 15N$



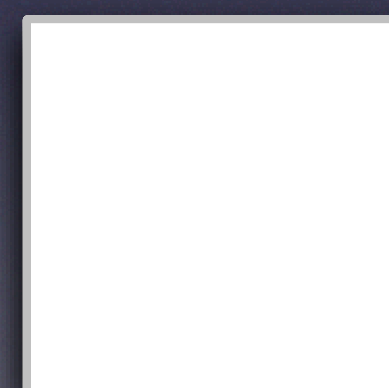
$t = 20N$



$t = 25N$

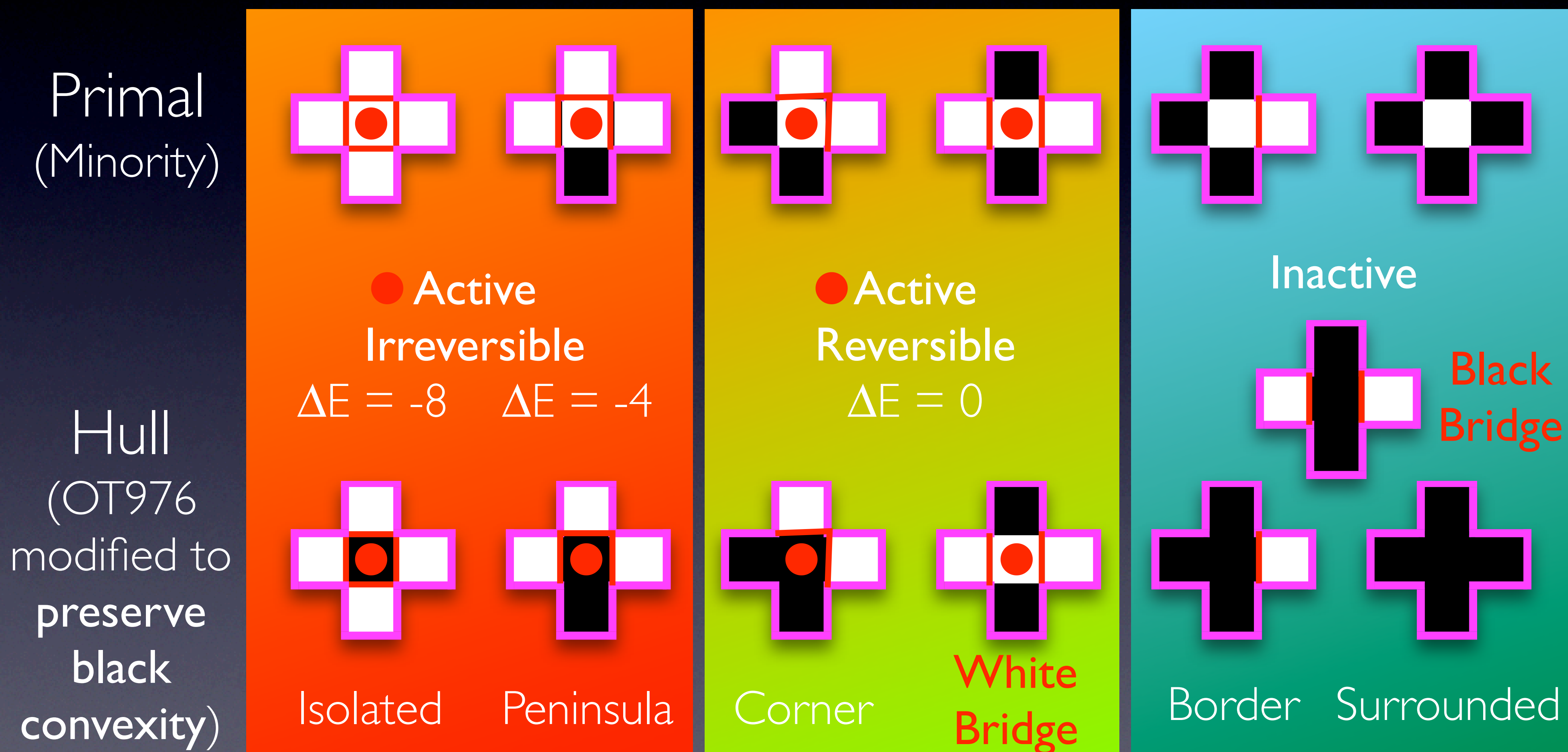


$t = 30N$



$t = 35N$

Rules of the hull dynamic

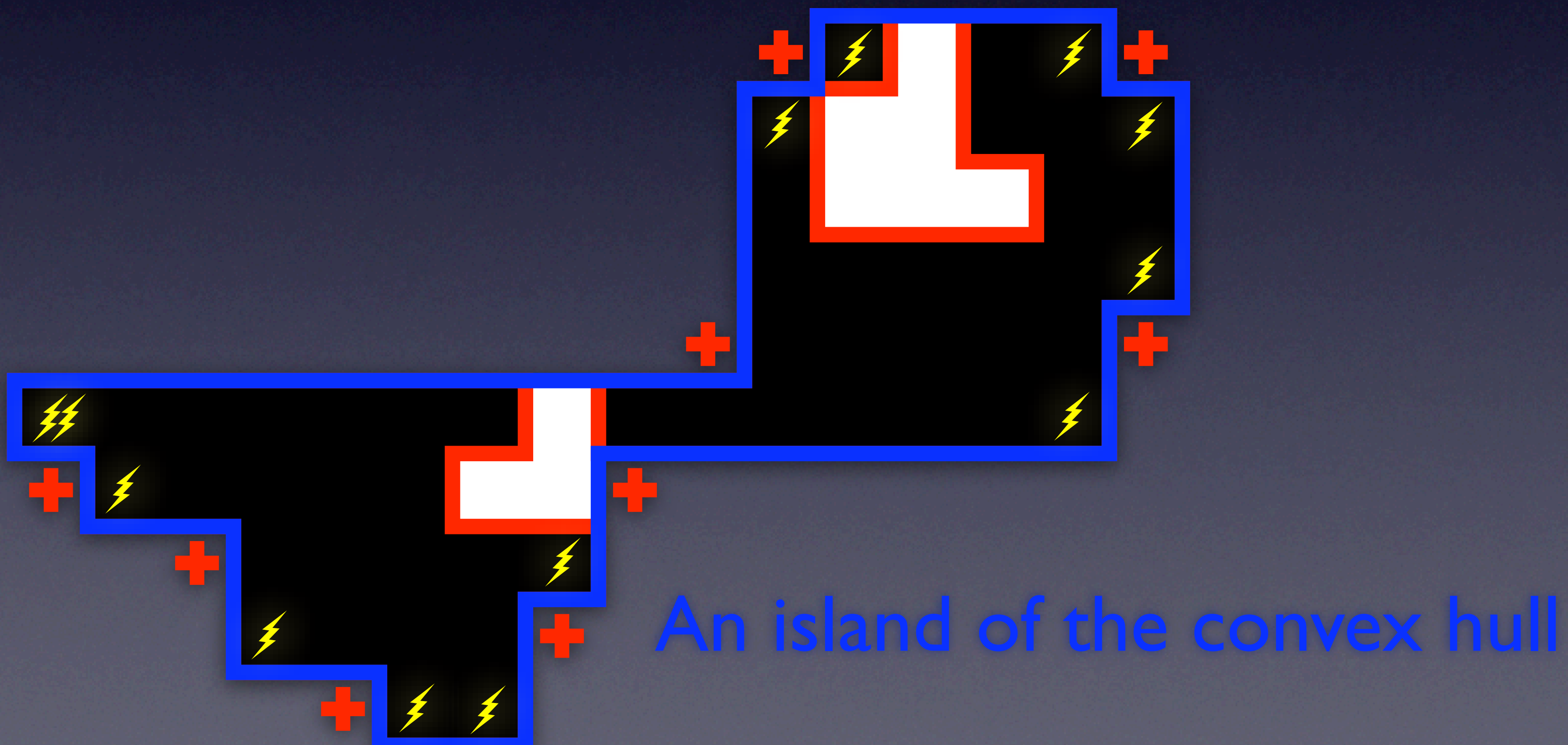


neighborhoods are considered with white/black symetries & rotations
(with exception of the bridge for the white/black symetry)

Convergence

Let $\boldsymbol{f}$ = #black dual hull cells + Energy/4.

Lemma. For any island of the dual hull, $E[\Delta f] \leq -3/N$.



Convergence

Let $\boldsymbol{f}$ = #black dual cells + Energy/4.

Lemma. For any island, $E[\Delta f] \leq -3/N$.

Proposition. For all bounded configuration,

$$E[\Delta f] \leq (2 \text{ \#island contacts} - 3 \text{ \#islands})/N$$
$$\leq -\text{\#islands}/N.$$

Theorem. Every bounded configuration **converges** to the checkerboard configuration in finite time a.s.. The expected convergence time is:

$O(N \bullet \textbf{Initial area})$.

Conclusion

- An **exponential upper bound** on the convergence time.
- **Polynomial convergence time for bounded configurations**
  (can be extended to configuration with a checkerboard band of width ≥ 2)
- An useful **energy** function that defines proper statistical physics
- Similar results for the **Moore-Neighborhood**

Conjectures

- Phase transition at $\alpha_c \approx 0.83$.
- $\alpha_c = \frac{\sqrt[3]{46+\sqrt{6969}}}{6} - \frac{16}{3 \cdot \sqrt[3]{46+\sqrt{6969}}} + \frac{2}{3}$
- Some ideas to prove polynomial convergence time to checkerboard for $\alpha < 1/3$ but...
- Strongly depends on the underlying network (a lot of differences on trees...)
- Generalization to the class of threshold automata

Thank you