Algorithmic problems in free groups

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The free group on A, $F(A)$

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- ... but a subgroup may have a larger rank than the group in which it sits!
- Every subset of $\{a^n b a^{-n} \mid n \geq 0\}$ is a basis of a subgroup of $F(a, b)$.

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- The general idea for subgroups: to represent the finitely generated subgroups of a free group by combinatorial objects (certain kinds of *automata*)
- results in effective (often efficient) computations on subgroups of free groups

Representation of subgroups

- $H = \langle abab^{-1}, aba^{-1}b^{-1}, aba^3b^{-1} \rangle$

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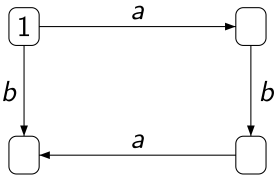

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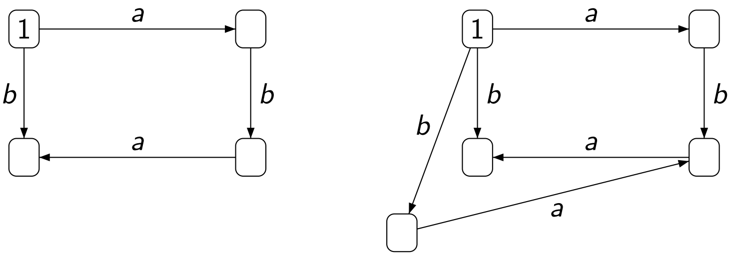

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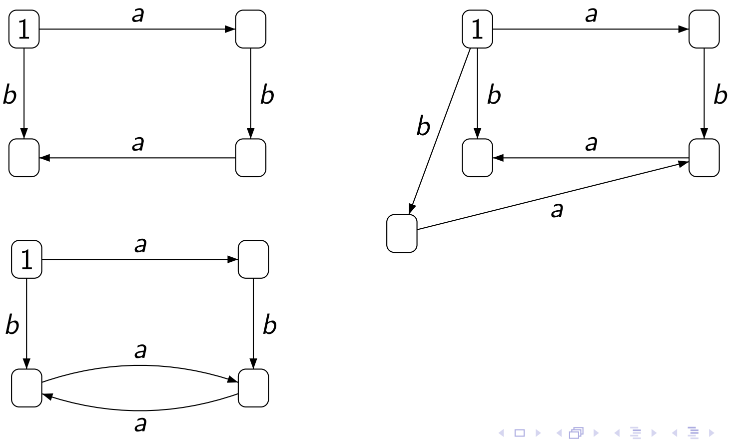

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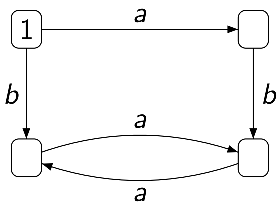

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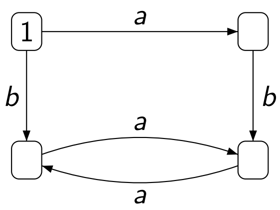
- H = all reduced words that read from 1 to 1



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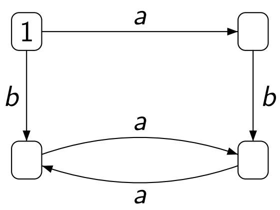

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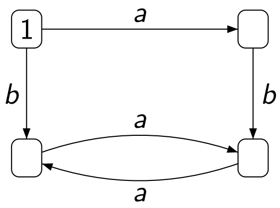

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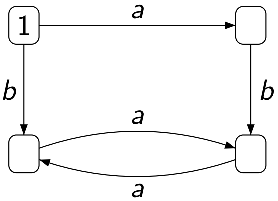

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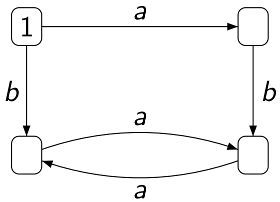

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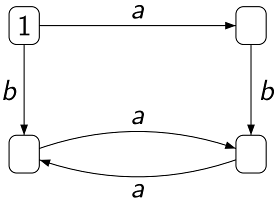

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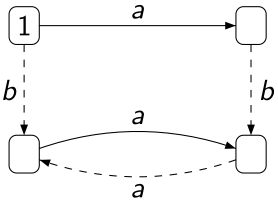

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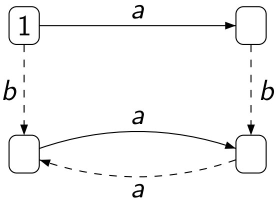

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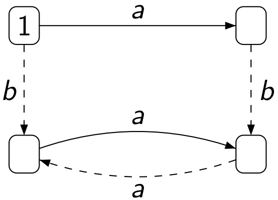

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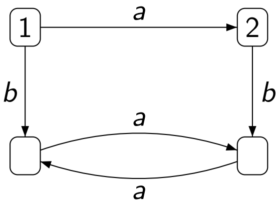
- $\text{rank}(H) = |E(H)| - |V(H)| + 1$.

Finite-index subgroups

Each vertex is an H-coset:



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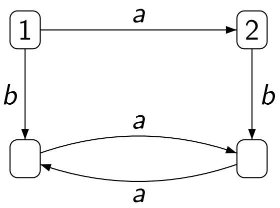

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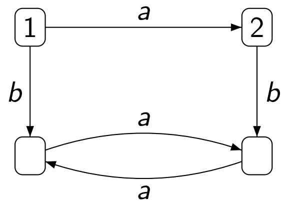

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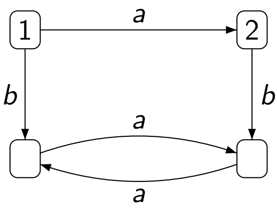

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- this yields a proof of the Nielsen-Schreier formula for finite index subgroups: if H has finite index $\imath(H)$, then $\mathrm{rank}(H) - 1 = \imath(H)(\mathrm{rank}(F(A)) - 1)$

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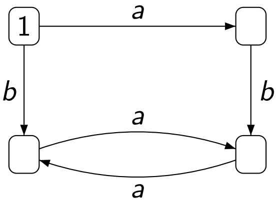

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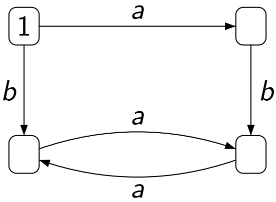

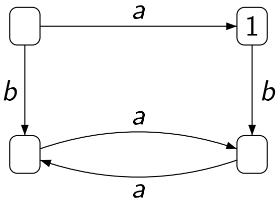

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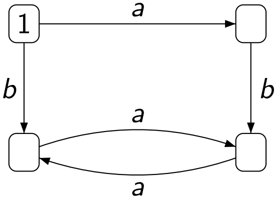

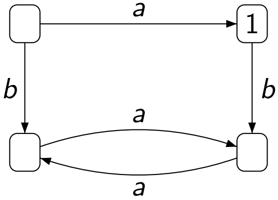

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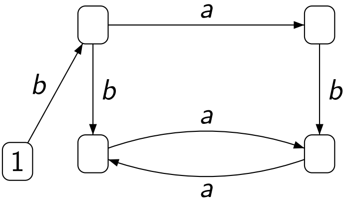

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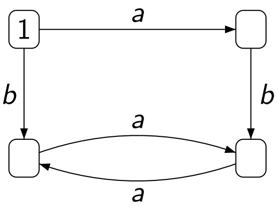

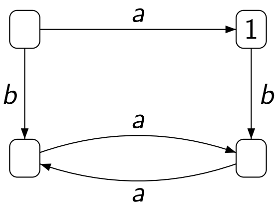

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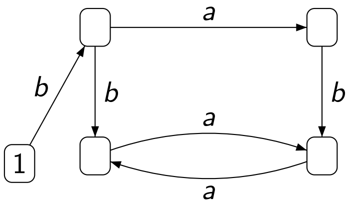

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- an algorithm to solve the conjugacy problem

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- Let $\widetilde{\mathrm{rk}}(H) = \max(\mathrm{rank}(H) - 1, 0)$, the *reduced rank* of H,
  Hanna Neumann Conjecture (**HNC**)

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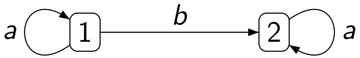
- How is this approached by means of automata? compute $\Gamma(H \cap K)$, knowing $\Gamma(H)$ and $\Gamma(K)$

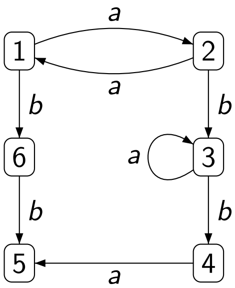
Example

$\Gamma(K)$



$$K = \langle a, bab^{-1} \rangle$$

$\Gamma(H)$



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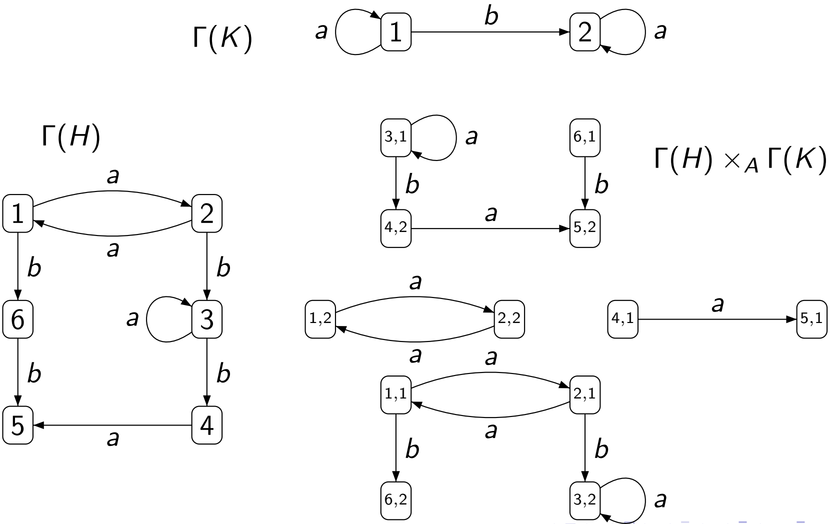
$\Gamma(K)$
a
1
b
2
a
$\Gamma(H)$
1
2
3
4
5
6
a
b
$\Gamma(H) \times_A \Gamma(K)$
3,1
6,1
4,2
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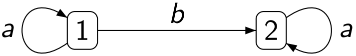

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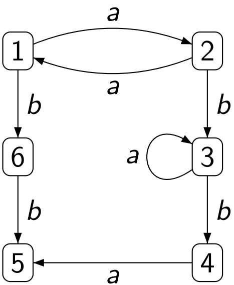

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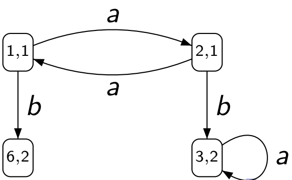
$\Gamma(H \cap K)$



Translation of HNC in graph-theoretic terms

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- a very simple, graph-theoretic problem. Very simple to state, yet very elusive...

The Whitehead minimization problem

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- there is an analogous problem for a finitely generated subgroup H instead of a word u

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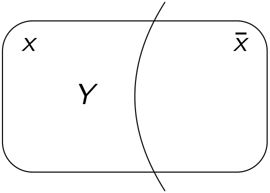

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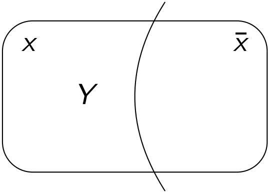

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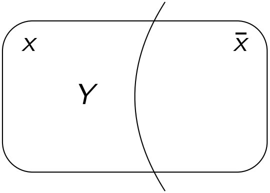

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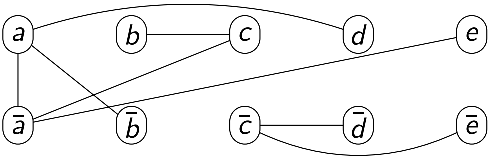

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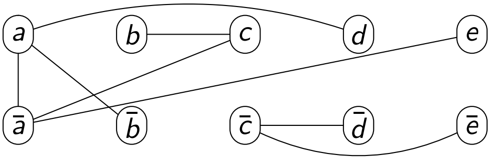

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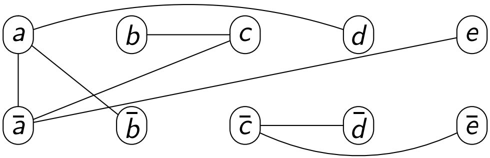

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- Evaluate $|\varphi([u])| - |[u]|$ in graph-theoretic terms, depending on x, Y and the Whitehead graph of $[u]$

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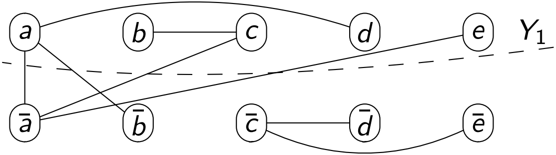

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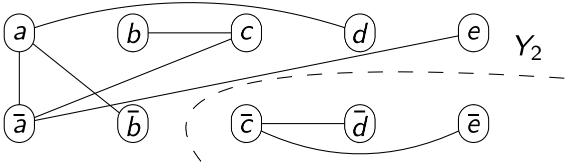

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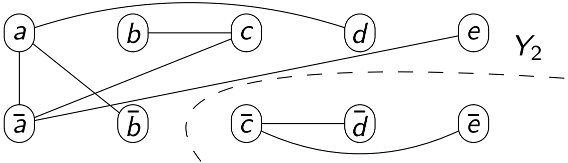

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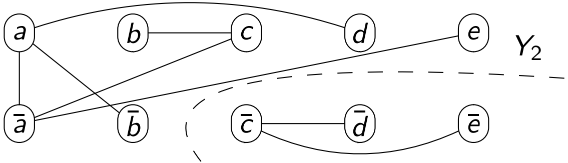

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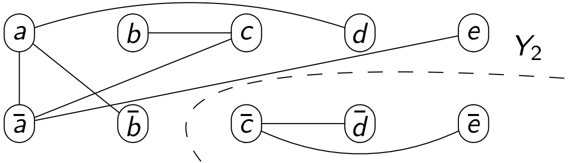

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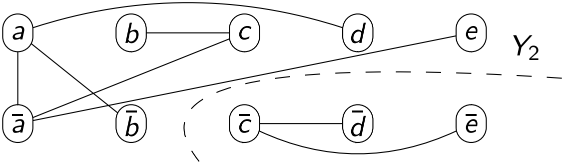

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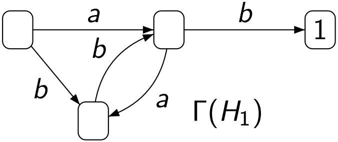

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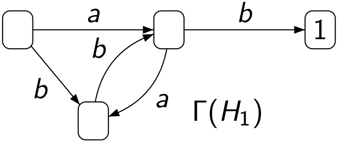

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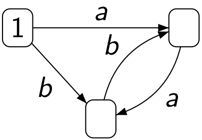

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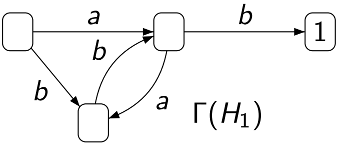

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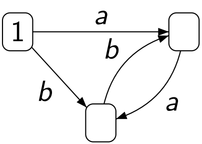

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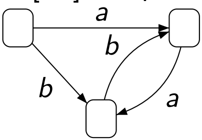

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- The cyclic word problem is a special case: if $H = \langle u \rangle$, then H (or $\Gamma([H])$) can be identified with the cyclic word $[u]$.

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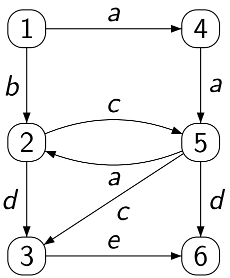
- 

$$d_1 = \{\bar{a}, \bar{b}\}$$
$$d_2 = \{a, b, \bar{c}, \bar{d}\}$$
$$d_3 = \{c, d, \bar{e}\}$$
$$d_4 = \{a, \bar{a}\}$$
$$d_5 = \{a, \bar{a}, c, \bar{c}, d\}$$
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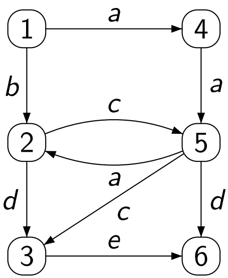
The cut formula still holds



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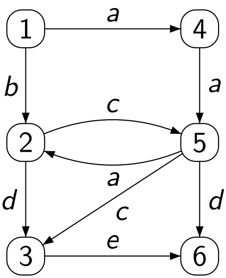
- 

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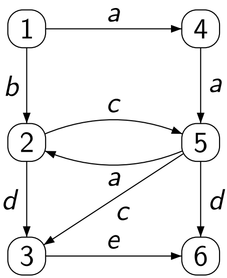
- 

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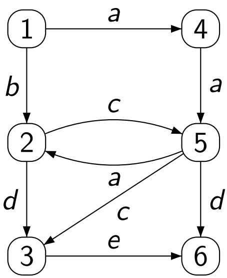
- 

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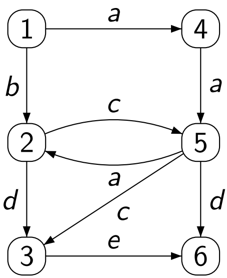
- 

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Thank you for your attention!