Determinism in One-way Model: Finding Optimal Flows Efficiently

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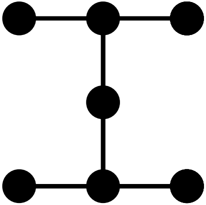
One-way model [Briegel - Raussendorf (00)]

One-qubit measurements over a large *entangled state* is universal for quantum computation.



Definition

For a given graph $G = (V, K)$,

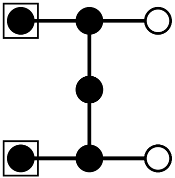
$$|G\rangle = \Pi_{(a,b)\in K} \Lambda Z_{a,b} |+\rangle_V$$

where $|+\rangle_V = \bigotimes_{u\in V} \frac{1}{\sqrt{2}}(|0\rangle_u + |1\rangle_u)$.

Definition (Open graph)

For any graph $G = (V, K)$, and for any $I, O \subseteq V$,

(G, I, O) is an open graph .



Definition (Initial state)

For any open graph (G, I, O) with $G = (V, K)$, for any $|I|$-qubit state $|\varphi\rangle$, the initial state is

$$\left|\Psi_{G,\varphi}\right\rangle = \left(\Pi_{(a,b)\in K}\Lambda Z_{a,b}\right)\left(|\varphi\rangle_I \otimes |+\rangle_{V\setminus I}\right)$$

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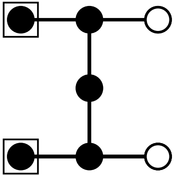

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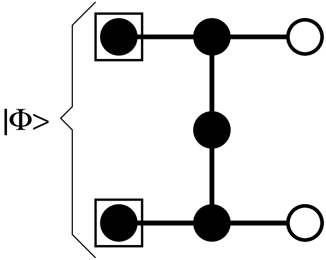
One-way quantum computation [Briegel, Raussendorf]



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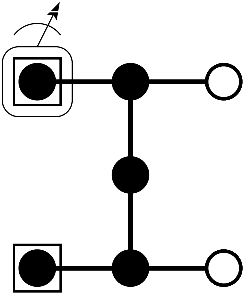

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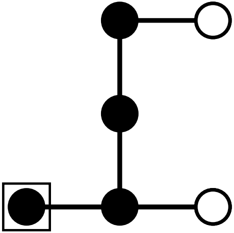

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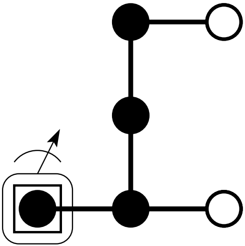

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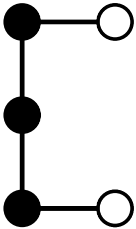

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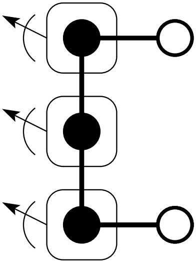

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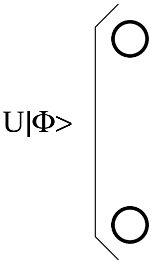
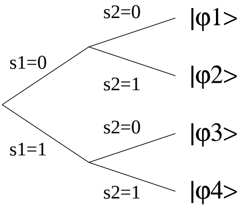




Definition (Determinism)

A One-way QC is **deterministic** iff all branches have the same output (up to a Pauli operator);

Theorem (Danos, Kashefi 05)

For a given (G,I,O), a deterministic One-way QC can be driven on the corresponding quantum state if (G,I,O) has a **causal flow**.

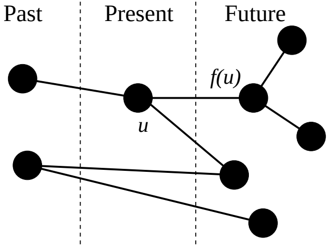
Definition (Causal Flow)

$(f,\prec)$ is a **causal flow** of (G,I,O), where $f : V\setminus O \to V\setminus I$, if for any u,

— $u \prec f(u)$

— $u \in N(f(u))$

— if $v \in N(f(u))\setminus\{u\}$ then $u \prec v$



Definition (Open Graph)

An **open graph** is a triplet (G,I,O), where $G=(V,E)$ is a undirected graph , and $I,O\subseteq V$, are respectively called input and output vertices.

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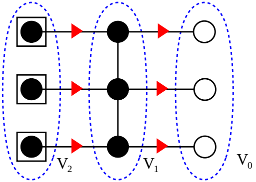
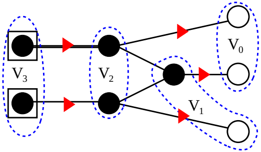
Definition (Layers and Depth)

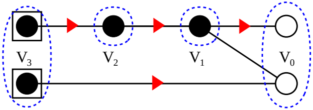
– $(f,\prec)$ induces a partition $(V_k^{\prec})_{k=0..d^{\prec}}$ of the vertices into $d^{\prec}+1$ **layers**, where $V_0^{\prec}=max(V)$ and $V_1^{\prec}=max(V\setminus V_0^{\prec})$, etc...
– $d^{\prec}$ is the **depth** of the flow.



depth 2



depth 3



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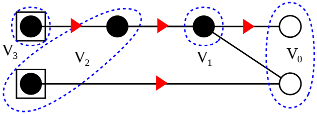

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Theorem (De Beaudrap 06)

There exists a $O(nm)$[a] algorithm for finding causal flow of a given open graph (G,I,O), when $|I| = |O|$.

[a] where n (resp. m) is the number of vertices (resp. edges) of G

Open Question (Danos, Kashefi 05 & De Beaudrap 06)

Is there an efficient (poly-time) algorithm for finding a causal flow if $|I| \neq |O|$?

Theorem (Mhalla, Perdrix 07)

$O(m)$-algorithm for finding causal flow of a given open graph (G,I,O), whatever the numbers of inputs and outputs are.

Theorem (De Beaudrap 06)

There exists a $O(nm)^a$ *algorithm for finding causal flow of a given open graph* (G,I,O)*, when* $|I| = |O|$.

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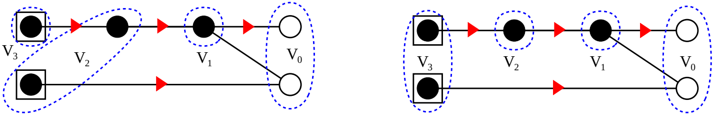
Definition (Maximally Delayed)

$(f, \prec)$ is **maximally delayed** if for any causal flow $(f', \prec')$ of the same open graph,

$$\forall k, |\cup_{i=0..k} V_i^{\prec}| \geq |\cup_{i=0..k} V_i^{\prec'}|$$



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If $(f, \prec)$ *is a maximally delayed causal flow of* (G, I, O) *then*
$$V_0^{\prec} = O$$

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$$V_1^{\prec} = \{u \in V \setminus V_0^{\prec}, \exists v \in V_0^{\prec}, N(v) \setminus V_0^{\prec} = \{u\}\}$$

Property (Inductive Structure)

If $(f, \prec)$ *is a maximally delayed causal flow of* (G, I, O) *then*
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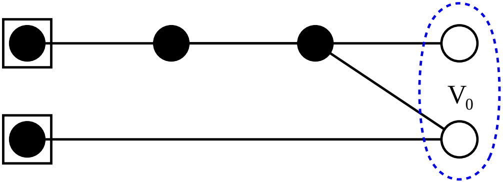
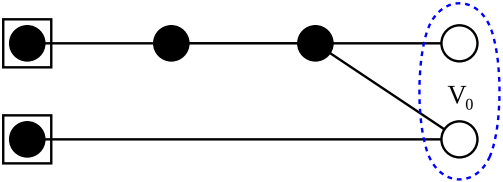
Greedy backward algorithm

$$V_0^{\prec} := O$$



$$V_1^{\prec} := \{u \in V \setminus V_0^{\prec} \text{ s.t. } \exists v \in V_0^{\prec}, N(v) \setminus V_0^{\prec} = \{u\}\}$$



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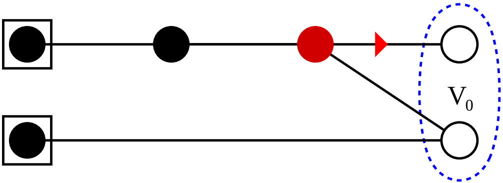

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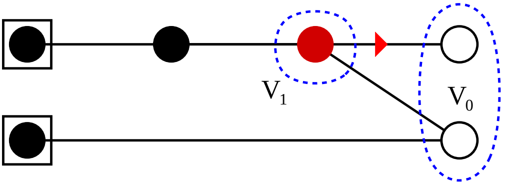
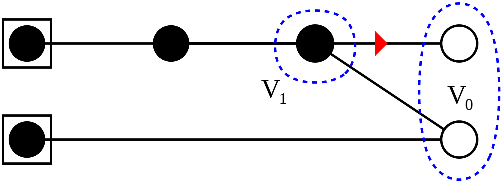


If $V_1^{\prec} \cup V_0^{\prec} = V$ then *'YES'*, if $V_1^{\prec} = \emptyset$ then *'NO'*

$$V_2^{\prec} := \{u \in V \setminus L_2 \text{ s.t. } \exists v \in L_2, N(v) \setminus L_2 = \{u\}\}$$

where $L_2 = V_1^{\prec} \cup V_0^{\prec}$



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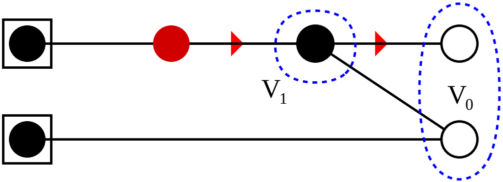

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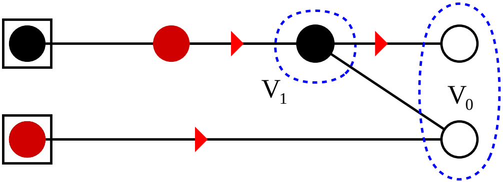

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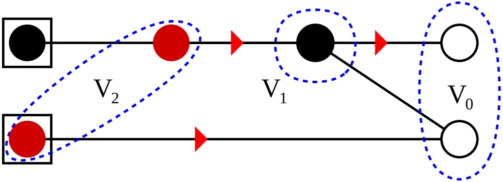

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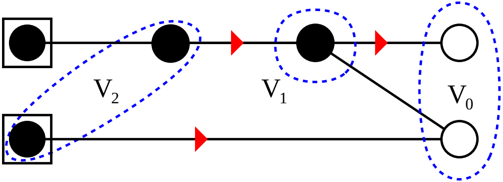

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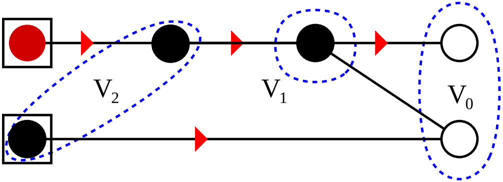

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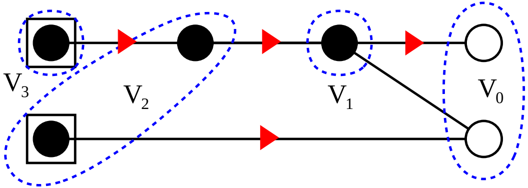

Greedy backward algorithm

For a given open graph (G,I,O)

$L := O$;
while $L \neq V$ **do**
  $C := \emptyset$
  for all $v \in L$ **do**
    if $|N(v) \setminus L| = 1$ **then** $C := C \cup (N(v) \setminus L)$;
  endfor
  if $C = \emptyset$ **then** 'no' ;
  $L := L \cup C$;
endwhile
'yes'

Correction

The produced flow $(f, \prec)$ is a causal flow of the input open graph (G, I, O).

Completeness

The algorithm produces a flow if the input open graph has a causal flow.

Complexity

$O(m)$ operations (model of adjacency list and adapted data structures)

Theorem (Optimal Flow)

The causal flow produced by the algorithm is **optimal***: there is no causal flow for the same open graph which has a smaller depth.*

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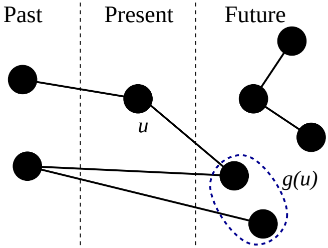
$(g, \prec)$ is a **generalized flow** of (G, I, O), where $g : V \setminus O \to \wp(V \setminus I) \setminus \{\emptyset\}$, if for any u,

— if $v \in g(u)$, then $u \prec v$

— $u \in Odd(g(u)) = \{v \in V, |N(v) \cap g(u)| = 1[2]\}$

— if $v \in Odd(g(u)) \setminus \{u\}$ then $u \prec v$



Theorem (Perdrix 06)

Generalized flow is sufficient for determinism.

Theorem (Browne, Kashefi, Mhalla, Perdrix 07)

Generalized flow is necessary for uniform, strong and stepwise determinism.

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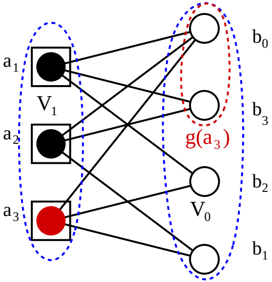
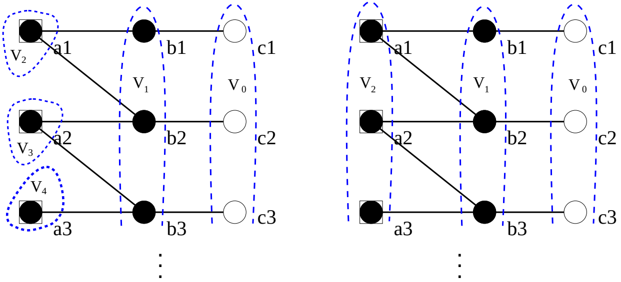


$$\forall i = 1 \ldots 3, g(a_i) = \{b_0, b_i\}$$

Speed up

There exists an open graph of size $3n$ having an optimal causal flow of depth $n+1$ but a generalized flow of depth 2.



$f(a_i) = b_i$ and $f(b_i) = c_i$ $\qquad$ $g(a_i) = \{b_k\}_{k \le i}$ and $g(b_i) = \{c_i\}$

Open Question [Browne, Kashefi, Mhalla, Perdrix 06]

Is there an efficient (poly-time) algorithm for finding a generalized flow ?

Generalized Flow Algorithm [Mhalla, Perdrix 07]

$O(n^4)$ algorithm for finding generalized flow of a given open graph (G, I, O).

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      $C := C \cup (Odd(X) \setminus L)$;
  endfor
  if $C = \emptyset$ **then** 'no' ;
  $L := L \cup C$;
endwhile
'yes'

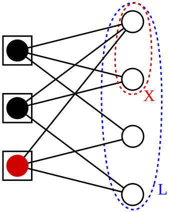
Greedy backward algorithm

For a given open graph (G, I, O)

$L := O$;
while $L \neq V$ **do**
  $C := \emptyset$
  for all $X \subseteq L$ **do**
    if $|Odd(X) \setminus L| = 1$ **then**
      $C := C \cup (Odd(X) \setminus L)$;
  endfor
  if $C = \emptyset$ **then** 'no' ;
  $L := L \cup C$;
endwhile
'yes'

$L := O$;
while $L \neq V$ **do**
  $C := \emptyset$
  for all $u \in V \setminus L$ **do**
    Solve $Odd(X) \setminus L = \{u\}$
    if there is a solution then
      $C := C \cup \{u\}$;
  endfor
  if $C = \emptyset$ **then** 'no' ;
  $L := L \cup C$;
endwhile
'yes'

Solve $Odd(X) \setminus L = \{u\}$



Definition (Cut-matrix)

$\Gamma_{(V\setminus L)\times L}$ is the $|V\setminus L| \times |L|$ sub adjacency matrix:

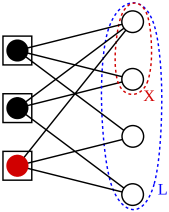
$$\Gamma = \left(\begin{array}{c|c} \cdot & \cdot \\ \hline \Gamma_{(V\setminus L)\times L} & \cdot \end{array}\right)$$

Linear System

$$Odd(X)\setminus L = \{u\} \iff \Gamma_{(V\setminus L)\times L}.\mathbb{I}_X^L = \mathbb{I}_{\{u\}}^{V\setminus L} \text{ in } \mathbb{F}_2$$

where $\mathbb{I}_X^A$ is $|A|$-vector such that $\forall u \in A, \mathbb{I}_X^A(u) = \begin{cases} 1 & u \in X \\ 0 & u \notin X \end{cases}$.

Solve $Odd(X) \setminus L = \{u\}$



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Theorem (Optimal Flow)

The generalized flow produced by the algorithm is **optimal***: there is no generalized flow for the same open graph which has a smaller depth.*

1 Context and Definitions

2 Causal Flow Algorithm

3 Generalized Flow

4 Conclusion

- Deterministic One-way QC as the existence of a *causal flow* or a *generalized flow* on a graph
- Polytime algorithm [De Beaudrap 07] for finding causal flow if $|I| = |O|$ (I: input vertices, O: output vertices)
- Faster polytime algorithm for finding causal flow (even if $|I| \neq |O|$)
- Polytime algorithm for finding generalized flow.
- These last two algorithms produce *optimal flows*, minimizing the depth of the One-way QC.
- Automatic parallelisation of one-way QCs (or quantum circuits).