Acyclicity of Preferences, Nash Equilibria, and Subgame Perfect Equilibria: a Formal and Constructive Equivalence

Stéphane Le Roux
stephane.le.roux@ens-lyon.fr
perso.ens-lyon.fr/stephane.le.roux/

LIP-École normale supérieure de Lyon, CNRS, INRIA, UCBL

27 June 2007, MC2 Seminar

Game Theory in Short

Developments I rely on

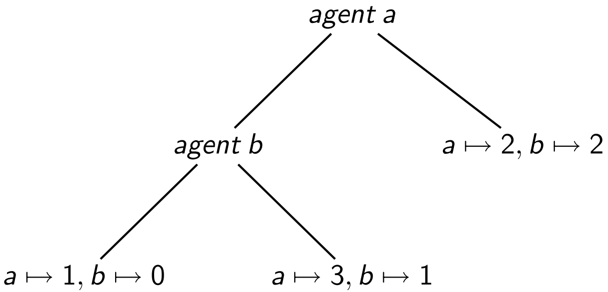
- 1950: Nash equilibrium for strategic games.
- 1953: Kuhn's Nash equilibrium existence for sequential games.
- 1965: Selten's subgame perfect equilibria.
- 2006: using Coq, Vestergaard proved part of Kuhn's result (binary trees instead of trees).

This Informal Presentation

1. Traditional sequential game theory.
2.
   - Abstraction of sequential games.
   - Generalisation of Kuhn's result.
   - Fully proved using Coq.
3. Applications.
4. Another proof.

Sequential Game

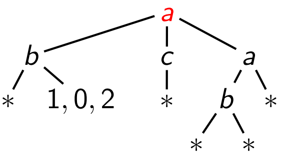
- Finite rooted tree.
- Internal nodes labelled with agents.
- Leaves labelled with payoff functions (from agents to real numbers).



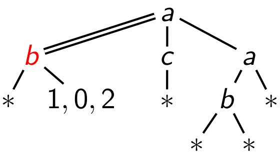
Play in a Sequential Game

*'s are arbitrary payoff functions.

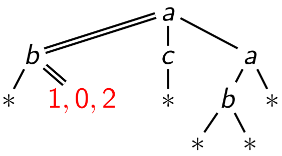
Start



First move

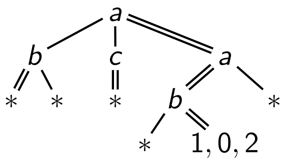


Last move

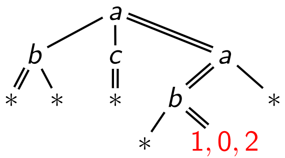


Strategy Profile and Induced Payoff Function

Strategy profile

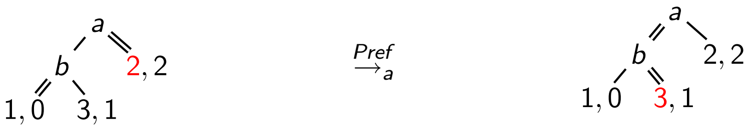


Induced payoff function

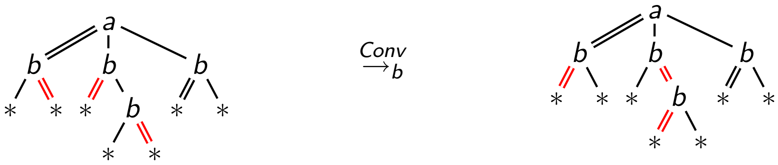


Preference and Convertibility

Preference of agent *a*



Conversion ability of agent *b*

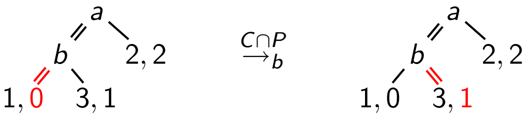


Nash Equilibrium

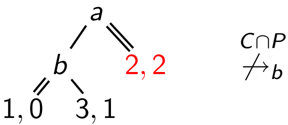
Given a strategy profile,

- Agent happiness: Conversion ability $\cap$ Preference $= \emptyset$.
- Nash equilibrium: Every agent is happy.

A strategy profile not Nash equilibrium



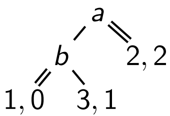
A Nash equilibrium

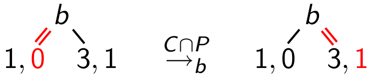


Subgame Perfect Equilibrium (S.P.E.)

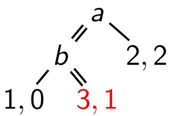
Nash equilibrium each of whose child is an S.P.E.

A Nash equilibrium
not S.P.E.



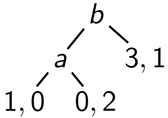


A Subgame perfect equilibrium

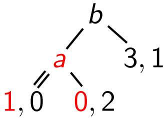


“Backward Induction”

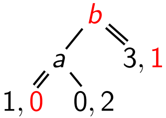
Input: game below



First step



Last step: ouput



Theorem (Kuhn)

Every sequential game has a Nash equilibrium.

Proof.

"Backward induction" yields S.P.E. □

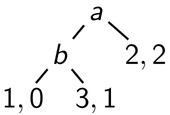
Towards Abstraction

- Why only real-valued payoff functions?
- Why payoff functions?

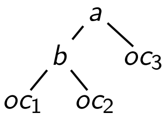
Abstract Sequential Game

- For leaves: abstract objects named outcomes instead of concrete payoff functions.
- For preferences: arbitrary binary relations instead of the usual total order over the reals.

Traditional



Abstract

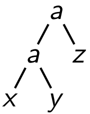


Similar notions of strategy profile, induced outcome, convertibility, Nash equilibrium with respect to given preferences.

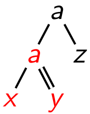
“Backward Induction” $\not\Rightarrow$ Nash equilibrium

The preference for agent a is defined by $z \overset{Pref}{\longrightarrow}_a x$ only.

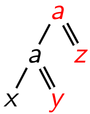
Game input



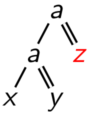
First step



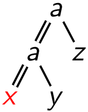
Last step



Not Nash equilibrium



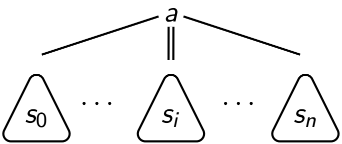
$\overset{C \cap P}{\longrightarrow}_a$



Key Lemma

Lemma

If s_i is a Nash equilibrium, and if agent a is happy with the following strategy profile, then it is a Nash equilibrium.



Total Ordering of Preferences Guarantees S.P.E.

Lemma

If preferences are total orders, then all games have S.P.E.

Proof.

By structural induction on strategy profiles and the key lemma (akin to Kuhn's proof). □

Acyclicity of Preferences Guarantees S.P.E.

Lemma

If preferences are acyclic, then all games have S.P.E.

Proof.

- acyclicity $\Rightarrow$ linear extensions (total order bigger preferences).
- There is S.P.E. with respect to the linear extensions (previous slide).
- It is also S.P.E. with respect to the original preferences, since smaller preferences imply more equilibria.

$\square$

Nash Equilibria Requires Acyclicity of Preferences

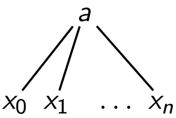
Lemma

If there exists a cycle in the preferences, then there exists a game without Nash equilibrium.

Proof.

Assume a cycle $x_0 \overset{Pref}{\longrightarrow}_a x_1 \ldots x_n \overset{Pref}{\longrightarrow}_a x_0$.

The following game has no Nash equilibrium.



□

Triple Equivalence

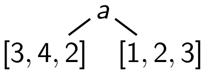
Theorem

The following three propositions are equivalent:

- *The preferences are acyclic.*
- *Every game has a Nash equilibrium.*
- *Every game has a subgame perfect equilibrium.*

Multi-criteria sequential games

- 1950's: Simon and Blackwell's vector payoff.
- For each agent, payoffs are vectors of fixed length.



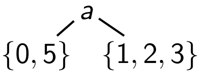
$$\frac{\forall i \leq n, x_i \leq y_i \qquad \exists k \leq n, x_k < y_k}{(x_0, \ldots, x_n) <_{vect} (y_0, \ldots, y_n)}$$

Corollary

Every multi-criteria sequential game has a Nash equilibrium/subgame perfect equilibrium.

Non-Deterministic Payoffs

Payoffs: real number $\rightsquigarrow$ set of real numbers.



$$\frac{\forall x \in X, y \in Y,\ x \leq y \qquad \exists x \in X, y \in Y,\ x < y}{X <_{set} Y}$$

Corollary

Every sequential game with non-deterministic payoffs has a Nash equilibrium/subgame perfect equilibrium.

Benevolent Selfishness

Traditional sequential games, but different preferences.
Let P_1 and P_2 be two payoff functions.

$$\frac{P_1(a) < P_2(a)}{P_1 <_a P_2} \qquad \frac{\forall b, P_1(b) \leq P_2(b) \qquad \exists b', P_1(b') < P_2(b')}{P_1 <_a P_2}$$

Corollary

Every sequential game with selfish-benevolent agents has a Nash equilibrium/subgame perfect equilibrium.

Second Proof

Lemma

If preferences are partial orders, then all games have Nash equilibria.

Proof.

- By induction on the number of nodes in the strategy profile.
- Apply induction hypothesis to the leftmost child.
- Case split on the root agent *a* being "satisfiable".
- Yes case, invoke the key lemma.
- No case, track agent *a* down the induced path.
- Cut-and-paste, apply induction hypothesis in between.
- Agents other than *a* are happy.
- Agent *a* is happy too, by transitivity.

□

Acyclicity of Preferences Guarantees Nash Equilibrium

Lemma

If preferences are acyclic, then all games have a Nash equilibrium.

Proof.

- acyclicity $\Rightarrow$ partial orders by transitive closure (bigger preferences).
- There is a Nash equilibrium with respect to the transitive closures (previous slide).
- It is also a Nash equilibriu with respect to the original preferences, since smaller preferences imply more equilibria.

□

Double Equivalence

Theorem

The following two propositions are equivalent:

- *The preferences are acyclic.*
- *Every game has a Nash equilibrium.*

Nothing about "every game has a subgame perfect equilibrium".

Summary

- Abstraction of sequential tree games.
- Generalisation of Kuhn's result, by structural induction.
- Converse property.
- Fully proved using Coq.
- Applications to various natural preferences.
- Second proof, by induction on the size.