Polynomial approximation and floating-point numbers

MC2 workgroup

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Arenaire team

January, 10. 2007

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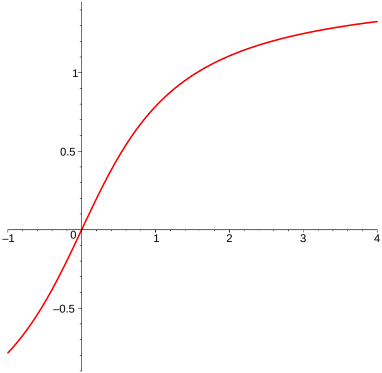
Why an approximation ?



Graph of $f : x \mapsto \arctan(x)$

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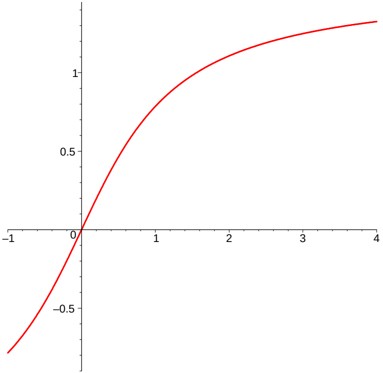

$\arctan(1) = \pi/4 = 0.78539\ldots$

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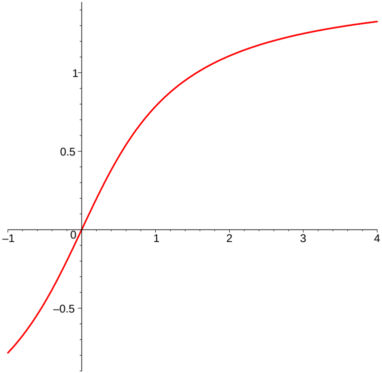

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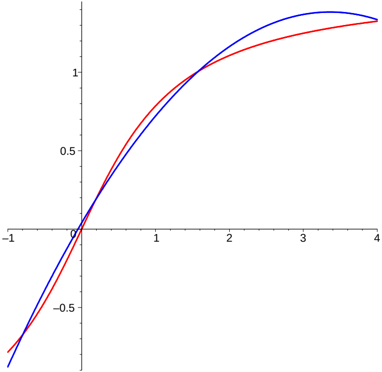
- We can only compute approximated values and hopefully bound the approximation error.

About the error of approximation



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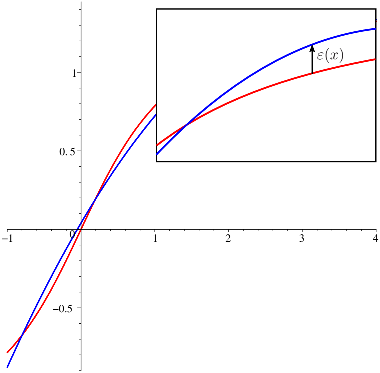

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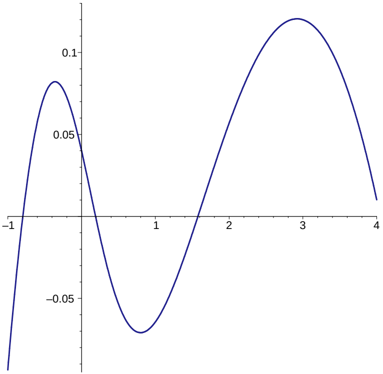

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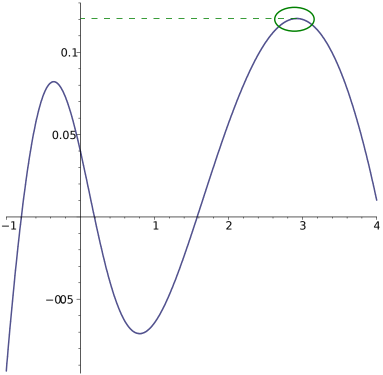

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$$\|\varepsilon(x)\|_\infty = \max_{x \in [a, b]} \{|\varepsilon(x)|\}$$

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- Natural question : what degree should have a polynomial to give a suitable approximation ?

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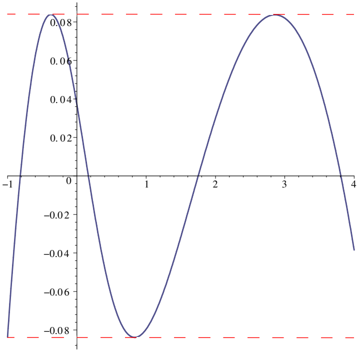
- Th. (Chebyshev) : given n and f there is a unique polynomial p of degree $\leq n$ minimizing $\|f - p\|_\infty$.

Reminder of approximation theory (2)



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$n + 2$ oscillations

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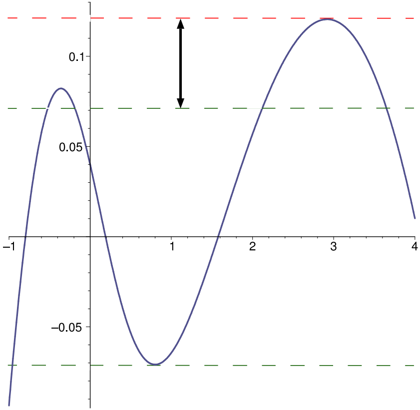

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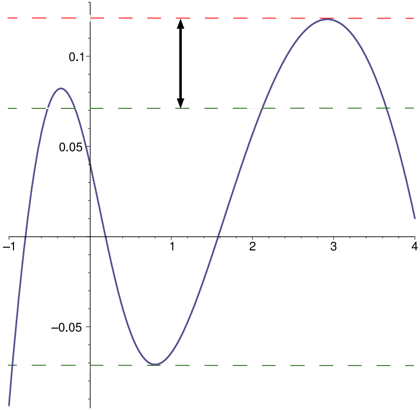

- Th. (Chebyshev) : characterization of the optimal error.
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- Remez' algorithm : given n, computes the optimal polynomial of degree $\leq n$ (called minimax).

Representing real numbers in computers

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- IEEE `double` format : $\beta = 2$, $t = 53$, and $e \in [\![-1074, 971]\!]$.
- From now, we will assume that $[e_{\min}, e_{\max}] = [-\infty, +\infty]$.

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- $\widehat{p}$ may be far from being optimal.
- Example with $f(x) = \log_2(1 + 2^{-x})$, $n = 6$, on $[0; 1]$ with single precision coefficients (24 bits).

Minimax	Naive method	Optimal
$8.3 \cdot 10^{-10}$	$119 \cdot 10^{-10}$	$10.06 \cdot 10^{-10}$

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- D. Kodek has studied a similar problem in signal processing. Limited to small precision and degree (typically $t < 10, n < 20$).
- N. Brisebarre, J.-M. Muller and A. Tisserand have proposed an approach by linear programming (the implementation relies on P. Feautrier's tool PIP).

Polytope approach (Brisebarre, Muller, Tisserand)

- Idea of the method :
  - let f be a continuous function on $[a, b]$;
  - we try to minimize $\|f - p\|_\infty$, $p \in \mathcal{P}$.
    ($\mathcal{P}$ may be $\mathbb{R}_n[X]$ or the subset of polynomials with floating-point coefficients, for instance)
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- Let $K \in \mathbb{R}_+$.
  The set $\mathcal{C}_K$ of every $(a_0, \cdots, a_n) \in \mathbb{R}^{n+1}$ such that

$$\forall x \in [a, b], \quad f(x) - K \leq \sum_{i=0}^{n} a_i\, x^i \leq f(x) + K$$

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  - if $K = \overline{\varepsilon}$, it corresponds to the solution set.

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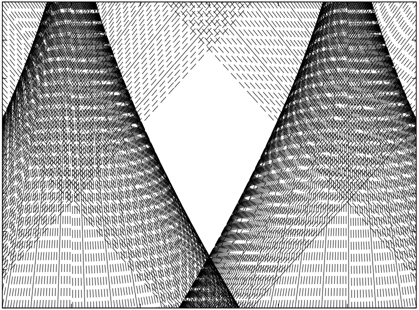

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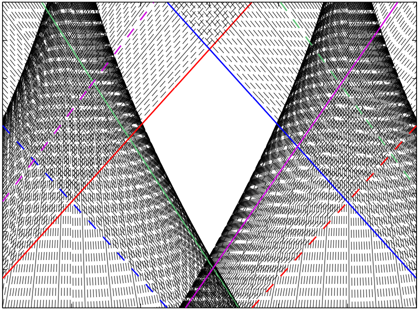

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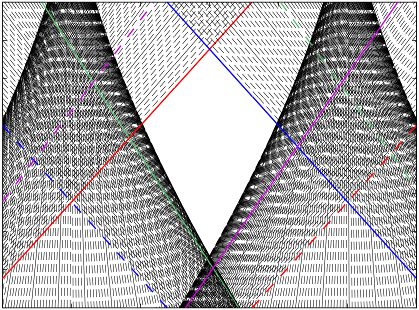

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  - requires to know a pretty tight over-estimation of $\overline{\varepsilon}$
- To find such an estimation, we developed a new method :
  - fast (it is proven to run in polynomial time) ;
  - heuristic (there is no proof that the result is always tight) ;
  - with good practical results.

Formalization of the problem

- Problem : given n and a floating-point format, find (one of) the polynomial(s) of degree $\leq n$ with floating-point coefficients minimizing $\|p - f\|_\infty$.

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- Remark : the existence is still ensured. The unicity may be lost.
- A simplification : we may try to guess the value of each e_i (assuming that $\widehat{a_i}$ and a_i have the same order of magnitude) $\hookrightarrow$ if e_i is correctly guessed, we are reduced to find $m_i \in \mathbb{Z}$ such that
$$\left\| f(x) - \sum_{i=0}^{n} \frac{m_i}{\beta^{e_i}} x^i \right\|_\infty$$
is minimal.

Description of our method

Our goal : find p of the form $\dfrac{m_0}{\beta^{e_0}} + \dfrac{m_1}{\beta^{e_1}} X + \cdots + \dfrac{m_n}{\beta^{e_n}} X^n$ with $m_i \in \mathbb{Z}$ which well approximates f.

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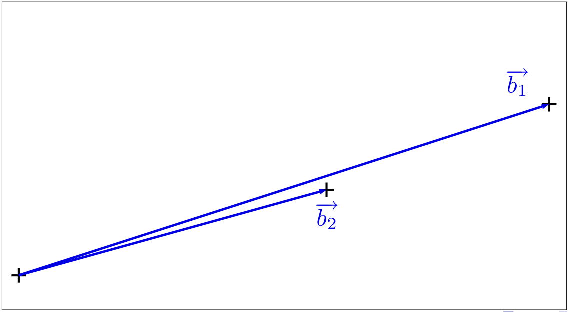
  - Rewritten with vectors :

$$\underbrace{m_0 \begin{pmatrix} 1/\beta^{e_0} \\ 1/\beta^{e_0} \\ \vdots \\ 1/\beta^{e_0} \end{pmatrix} + \cdots + m_n \begin{pmatrix} x_0^n/\beta^{e_n} \\ x_1^n/\beta^{e_n} \\ \vdots \\ x_n^n/\beta^{e_n} \end{pmatrix}}_{\Gamma \text{ of the form } \mathbb{Z}\overrightarrow{b_0}+\mathbb{Z}\overrightarrow{b_1}+\cdots+\mathbb{Z}\overrightarrow{b_n}} \simeq \underbrace{\begin{pmatrix} f(x_0) \\ f(x_1) \\ \vdots \\ f(x_n) \end{pmatrix}}_{\overrightarrow{t} \in \mathbb{R}^{n+1}} \quad .$$

Notions about lattices

Let $(\overrightarrow{b_1}, \cdots, \overrightarrow{b_n})$ be a basis of a real vector space.



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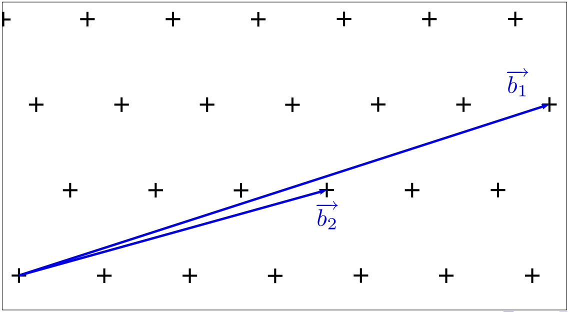
$$\Gamma = \mathbb{Z}\overrightarrow{b_1} + \mathbb{Z}\overrightarrow{b_2} + \cdots + \mathbb{Z}\overrightarrow{b_n} \quad .$$



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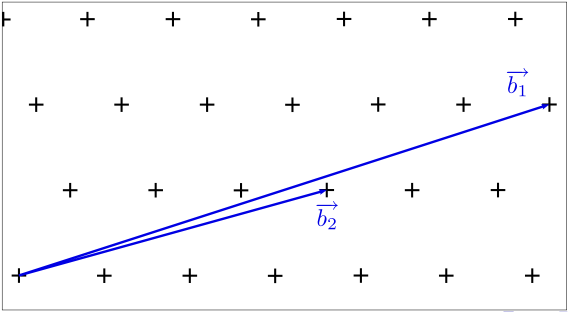

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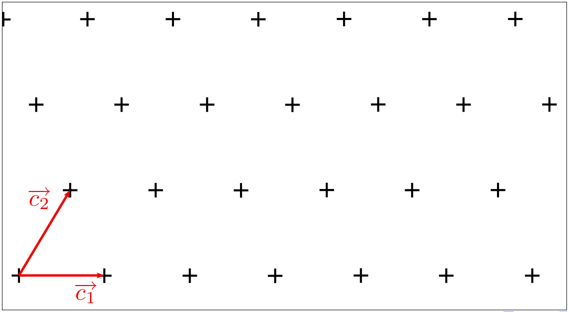

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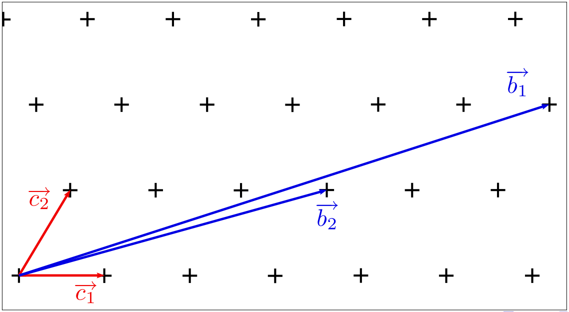

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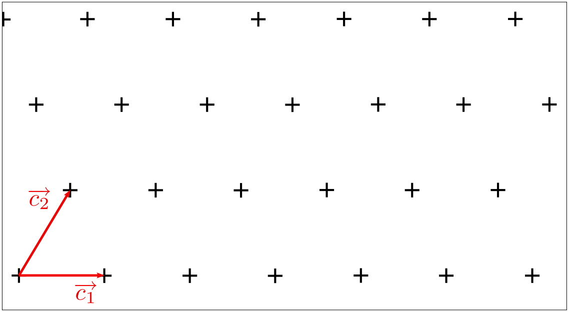

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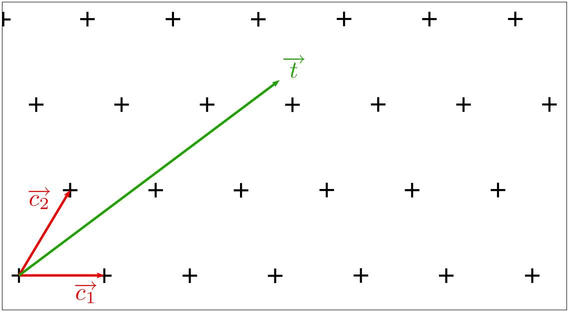

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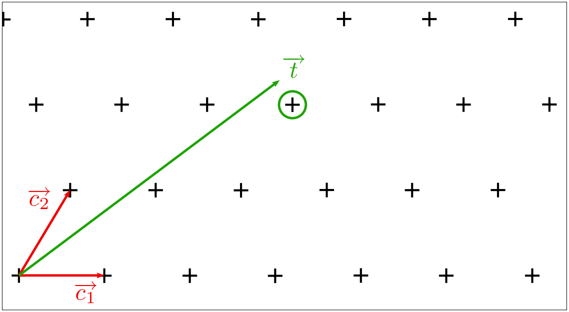

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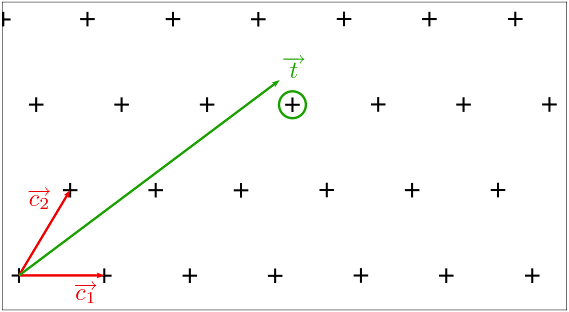

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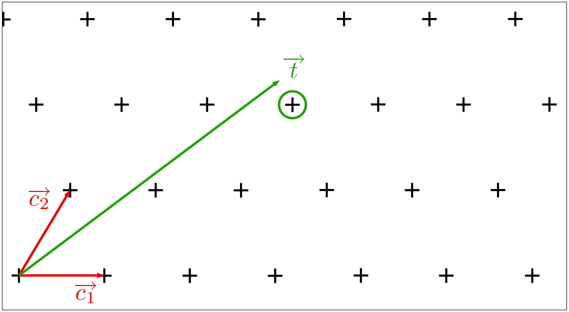

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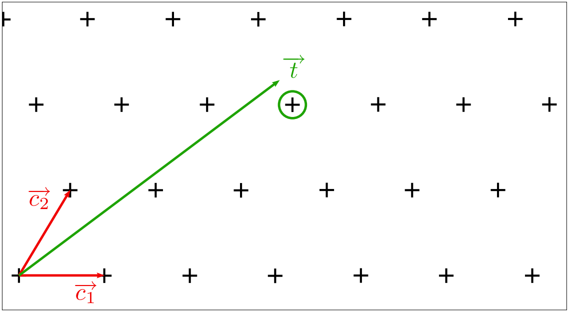

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- Idea of LLL algorithm : control the Gram-Schmidt basis to make $b_1^* = b_1$ minimal among the vectors of the orthogonal basis.

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  - other coefficients are `double extended` numbers.
- A `double extended` number has 64 bits of mantissa.
- He actually wants to have approximately 74 correct bits. (i.e. $\varepsilon \simeq 5.30\mathrm{e}{-23}$)

First try

Target	Degree 8 minimax	Degree 9 minimax
$5.30\mathrm{e}{-23}$	$40.1\mathrm{e}{-23}$	$0.07897\mathrm{e}{-23}$

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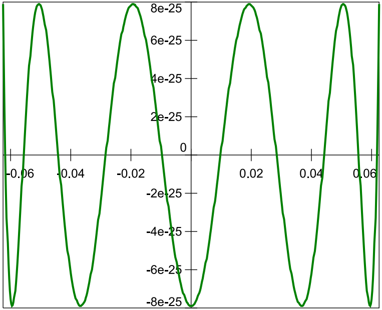

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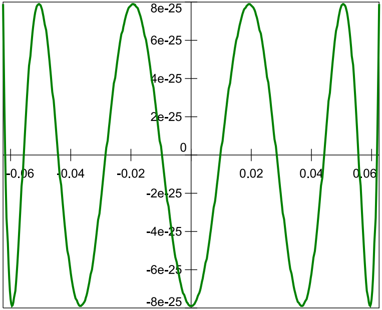

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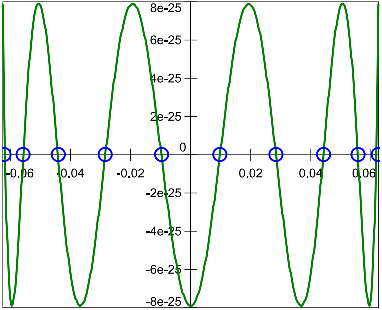

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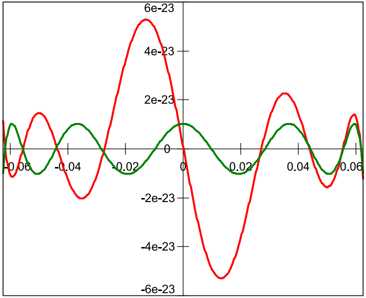

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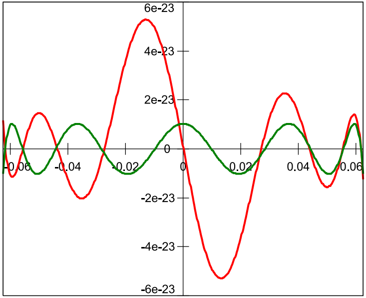

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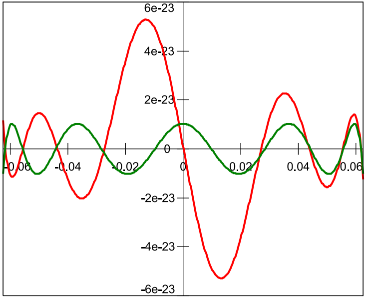

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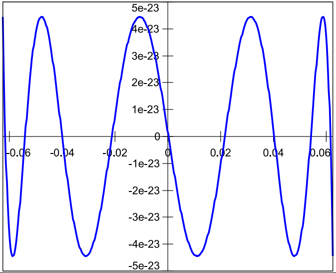

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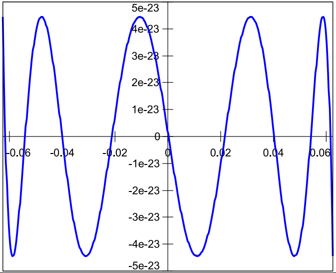

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