Dynamic of cyclic automata over $\mathbb{Z}^2$

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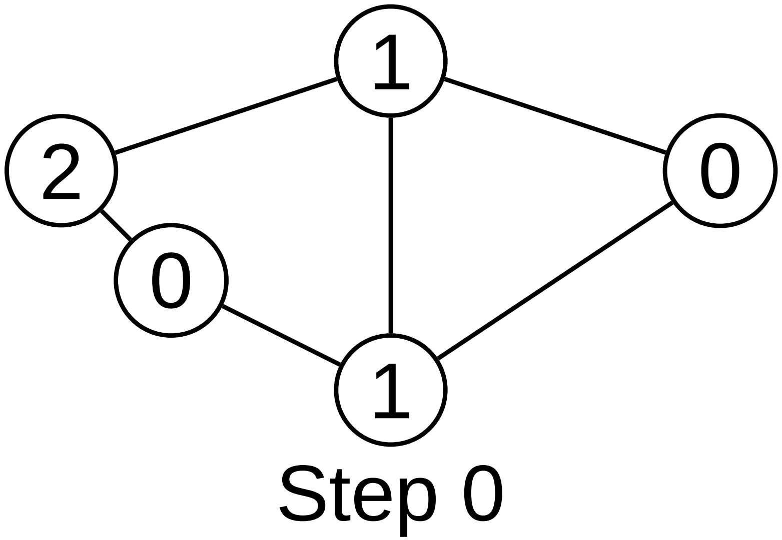
Cyclic Automata Network

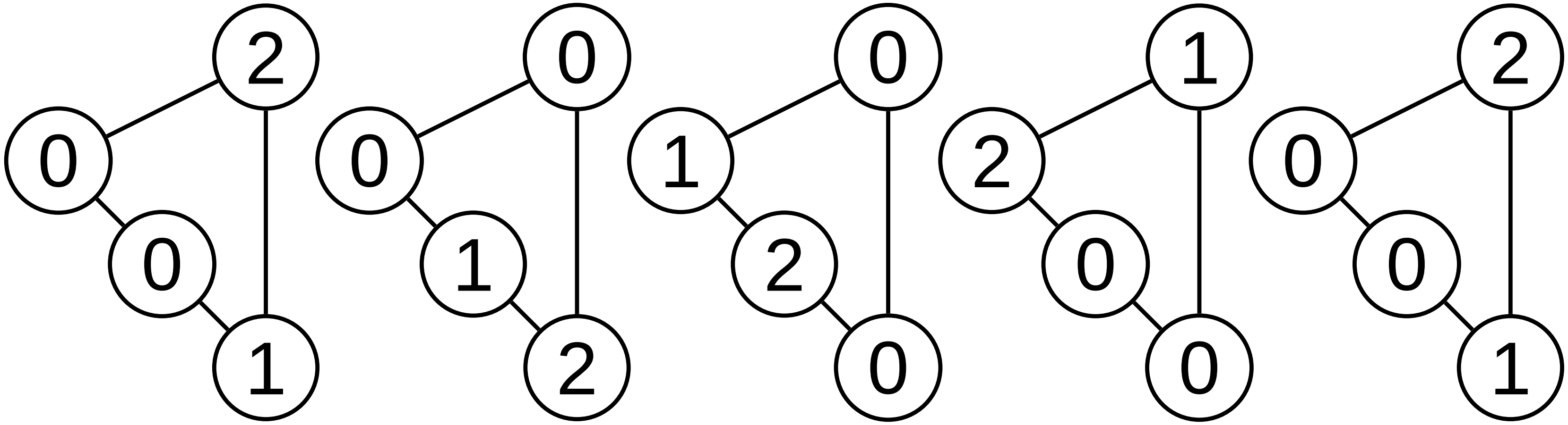
- Graph $G = (V, E)$
- Set of states: $Q = \{0, 1, \dots, q-1\}$
- Each vertex v has a state $x_v \in Q$
- Dynamic F: If a vertex v with state x_v has a neighbor with state $s(x_v) = x_v + 1 \mod q$ then the new state of x_v is $s(x_v)$, if not, the state does not change.

Example



Some definitions

Cycles:



Step 0 Step 1 Step 2 Step 3 Step 4

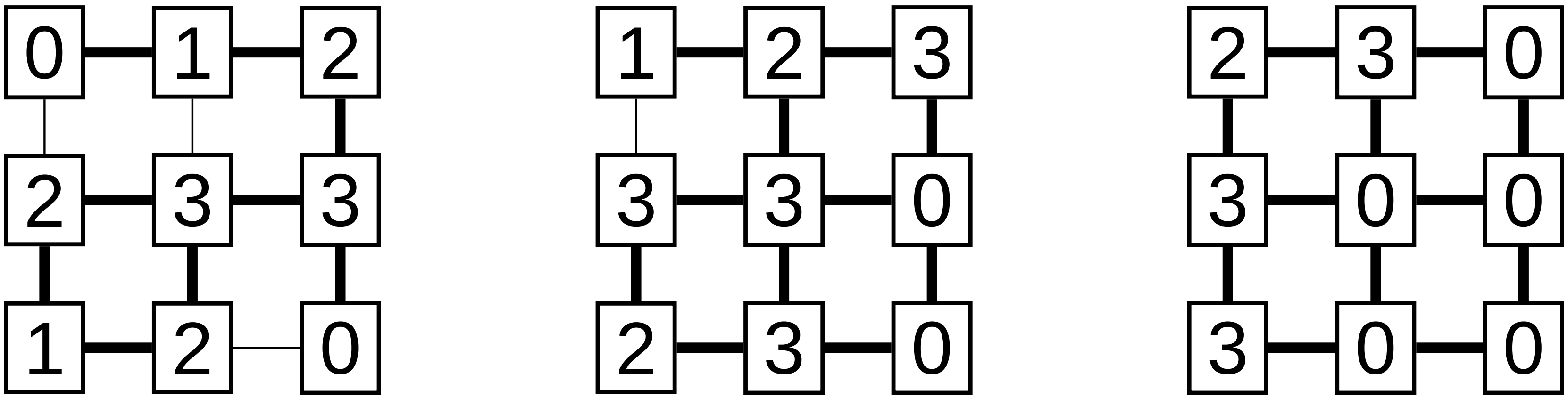
- Jump $J(C) = 3$ and Length $L(C) = 4$
- Period 4

Some definitions 2

Skeleton $Es(x)$: Subgraph of G with the same set of vertices. An edge connect two vertices with state p and p' if and only if $p = s(p')$, $p = p'$ or $s(p) = p'$.

- $Es(x) \subseteq Es\left(F\left(x\right)\right)$
- In a finite number of iteration, the skeleton becomes stable.

Example



Step 0 Step 1 Step 2

Predicting the future

Proposition 1 (Goles and Matamala). *Let x be an assignment and assume that $Es(x)$ is stable. Then for all $L \geq 1$ we have*

1. *If $a(v, L, x) \leq 0$ then $F_v^l(x) = x_v \; \forall l \in \{1, \ldots, L\}$.*
2. *If $a(v, L, x) > 0$ then $F_v^l(x) = s^{a(v,L,x)}(x_v) \; \forall l \in \{h(v, L, x), \ldots, L\}$.*

$a(v, L, x)$ the maximum of the jumps of all the walks of length less or equal than L starting in v on x.

$h(v, L, x)$ be the minimum of the lengths among all walks starting in x which reach the maximum jump $a(v, L, x)$.

Some definitions 3

- Efficiency of a path P: is the quotient between its jump and its length.

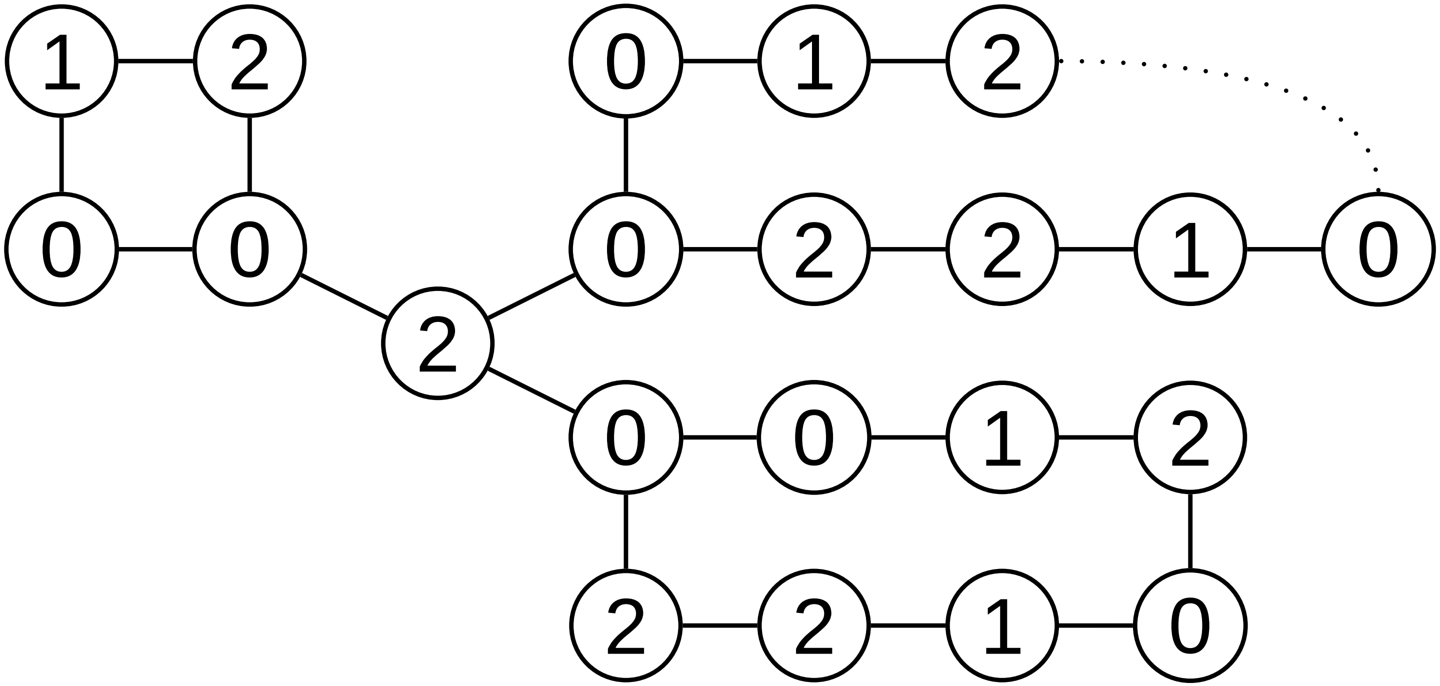
$$e(P, x) = \frac{J(P, x)}{L(P)}$$

- Efficiency of the system or *global efficiency*: is the maximum efficiency over all closed walks in the skeleton.

Previous results [G. & M.]

- Closed walks of global efficiency take control of the dynamic.
- The length of the period divides the least common multiple of the length of the closed walks with global efficiency. [non-polynomial upper bound]
- This upper bound can be reached.

Reaching the upper bound



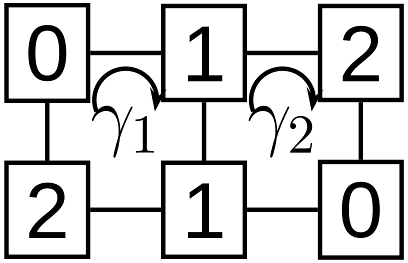
Our goal

To obtain non-polynomial periods in the 2-dimensional lattice $\mathbb{Z}^2$.

Complete Skeleton

We can decompose the skeleton in “tiles” (cycles of length 4). Its jumps could be 0, 3 or 4.
Property: We can sum the jump of some cycles

$$J(\gamma, x) = J(\gamma_1, x) + J(\gamma_2, x)$$



Some results

If the skeleton is complete, the efficiency and periods possibles are:

	$q \geq 5$	$q = 4$	$q = 3$
$e(x)$	0	0 or 1	$\geq 3/4$
Periods	1	1 or 4	any even ≥ 18

also

For $q = 3$ *and for any* $r \in \mathbb{Q} \cup [\frac{3}{4}, 1]$ *exists a* Q*-assignment* x *such that* $e(x) = r$

Our main result

How to "repeat" the idea of previous construction?

Main problems:

- Construction of cycles with a desired length
- Independent evolution of cycles
- Assure the stability of the skeleton

Steps of the construction

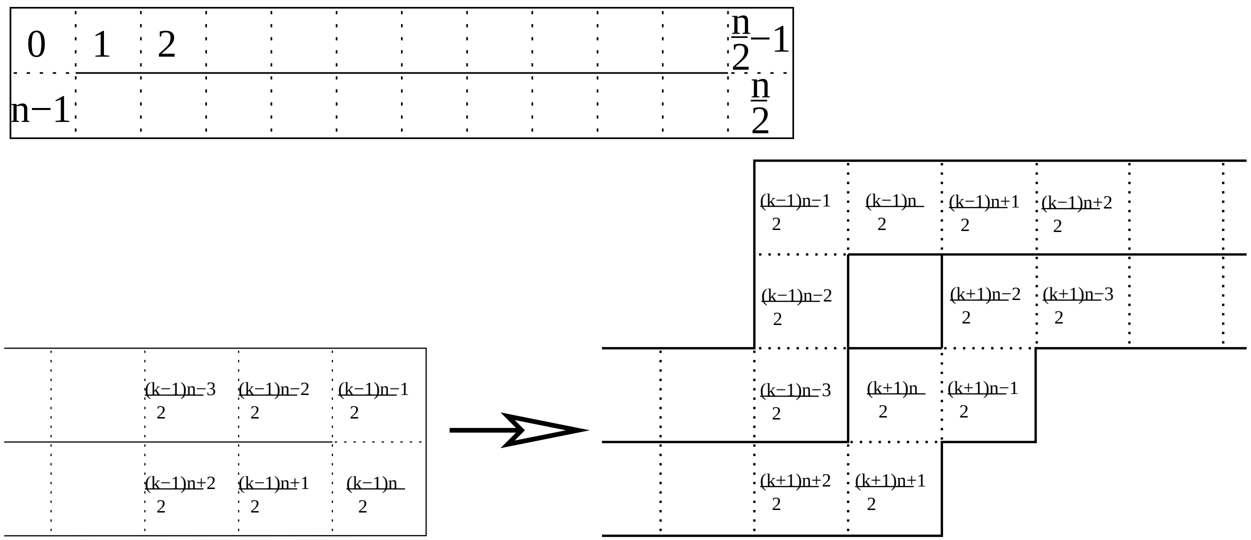
1. Good sequences of states
2. Construction (iteratively) of a cycle with a desired length
3. Completion to a rectangle
4. Extension for to embed the construction in the plane
5. Interaction between different cycles

Good sequences $\mathcal{A}$

1. For every $i = 0, \ldots, kp-1$, $a_{i+1 \mod kp} \in \{a_i, s(a_i)\}$ and $a_{i+3 \mod kp} \notin \{a_i, s(a_i)\}$.
2. The jump $J(a)$ of the sequence a is kq, where $J(a) := \sum_{i=0}^{kp-2} \mu(a_i, a_{i+1})$.
3. For every $j \in \{0, \ldots, k-1\}$ and for every $i = 0, \ldots, kp-1$ the jump of the subsequence $(x_i, \ldots, x_{i+jp \mod kp})$ belongs to $\{jq-1, jq, jq+1\}$.

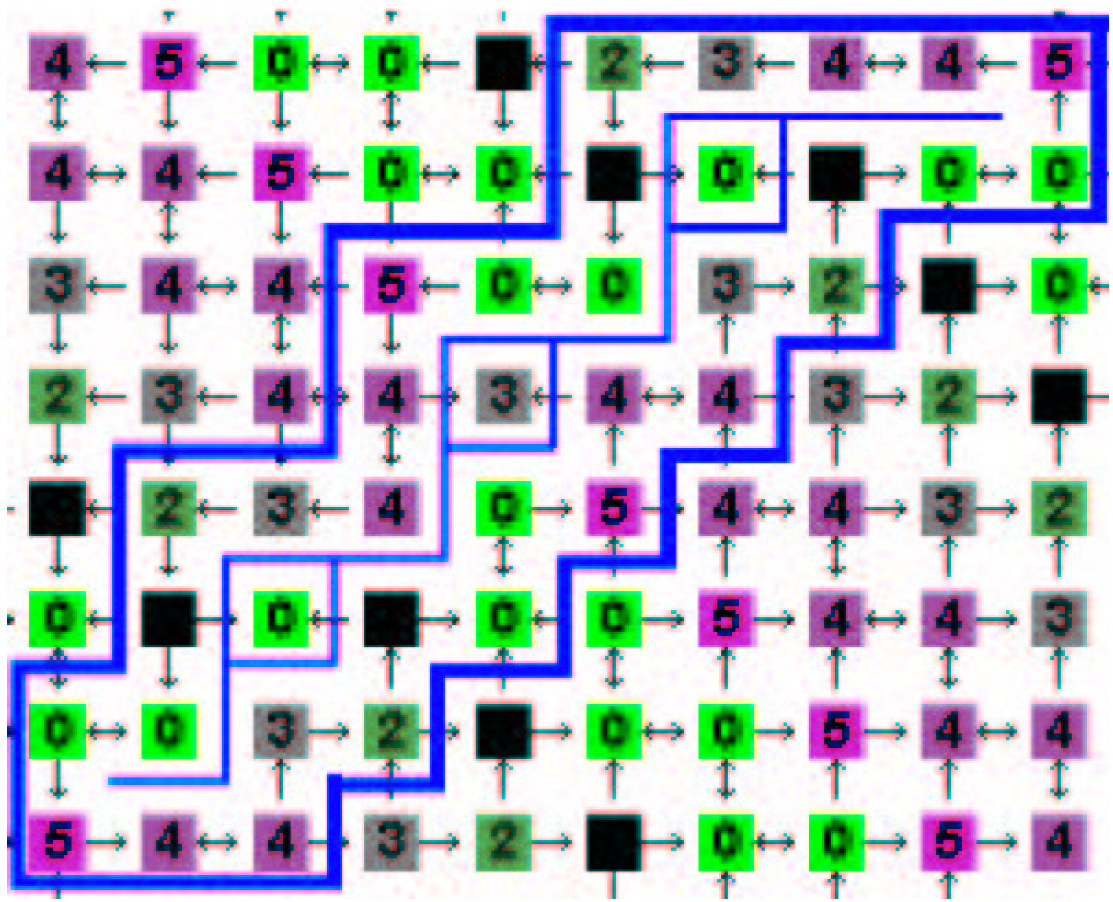
Example: $(q = 6)$ 001234450012344500123345

Main cycles

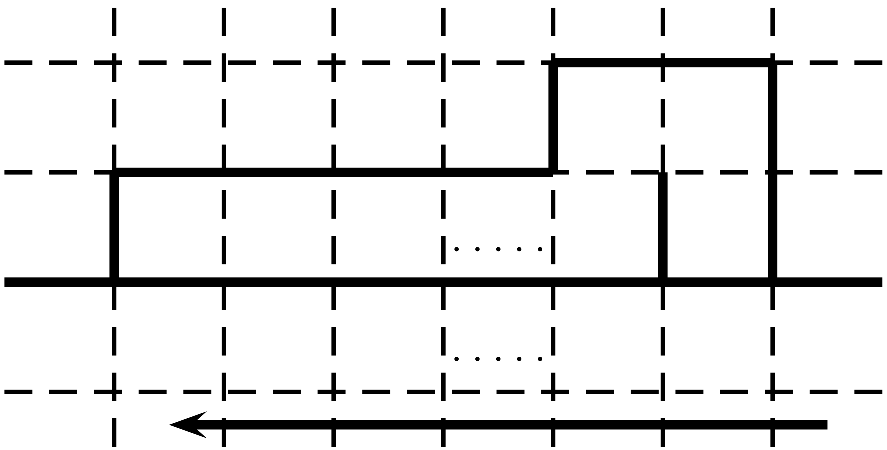


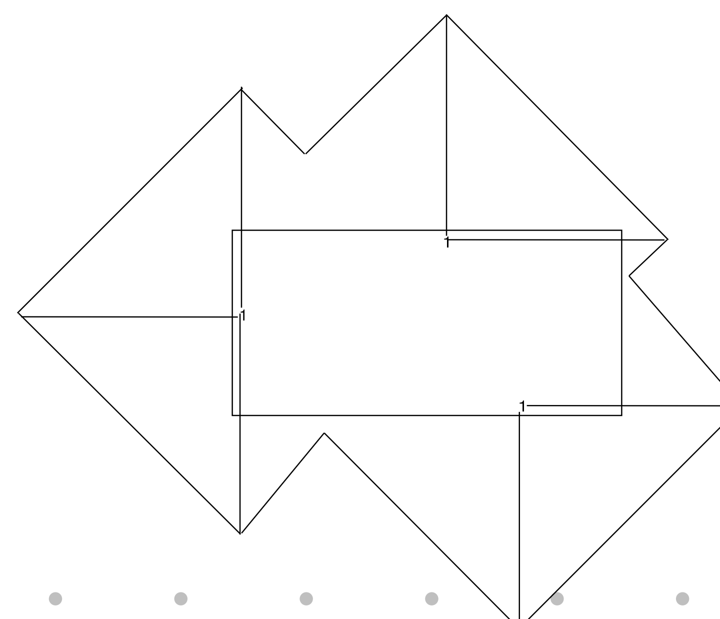
- $\forall a \in \mathcal{A}$ the skeleton is the same.
- $F(x(a)) = x(\sigma(a))$.
- $Es(G_k, F(x(a))) = Es(G_k, x(a))$.

Completion to a rectangle



Embeding the construction in the plane





Interaction between cycles

Theorem 1. *For every $q \geq 5$ and every even integer p such that $q < p \leq \lfloor \frac{3}{2} q \rfloor$ and, for every integer $m \geq 1$ and integers $k_1, k_2, \ldots, k_m$ there exists a Q-assignment y of K and a vertex v in K such that $T_v(y) = p \cdot \mathrm{lcm}_{i=1,\ldots,m}\{k_i\}$.*

Idea: Cycles with sequence a^i embedded equidistant to a particular vertex, where

$$a^i = (1, 2, 2, 3, \ldots, 0)^{(k_i - 1)}(1, 1, 2, 3, \ldots, 0)$$

Final Example

- $Q = \{0, 1, 2, 3, 4, 5\}$
- $e(x) = \frac{3}{4}$
- Sequences $(00123445)^{k_i - 1}(00123345)$
- $k_1 = 2$, $k_2 = 3$, $k_3 = 5$
- Period $= 8 \cdot \mathrm{lcm}(2, 3, 5) = 240$