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Automata in General Algebras*

SAMUEL EILENBERG

Columbia University, New York I.B.M. Watson Research Center, Yorktown Heights, New York

JESSE B. WRIGHT

I.B.M. Watson Research Center, Yorktown Heights, New York

1. INTRODUCTION

The study of automata and of context-free languages usually deals with monoids or semigroups. The purpose of this paper is to propose an extension of the domain of study to other algebraic systems. In the process, a better understanding of what takes place in monoids can be achieved.

The basic language is that of the theory of categories. The basic facts are reviewed in sections 2 and 3. The basic ideas of universal algebra are then introduced. The key notions are that of a "theory" and of algebras belonging to a theory. These ideas were laid down by Lawvere. A streamlined version of Lawvere's theory is given in sections 4–10. This part of the paper is regarded as expository and no proofs are given.

Recognizable sets and (deterministic) automata are discussed briefly in sections 11 and 12. In order to consider the analogs of nondeterministic automata, a restriction must be imposed upon the "theories" considered. The normal habitat for this notion is the so-called "linear theories". However, since the main result (Theorem III) is valid only for "free theories" (which are linear), we accept this restriction starting with section 13 and do not introduce linear theories at all. A full treatment of the subject is scheduled to appear in a book by the first of the authors.

Relational algebras (section 13) and relational automata (section 14) supply then the analog of nondeterministic automata, while the notion

of a polynomial (section 15) and of an algebraic set (section 16) are the generalizations of that of a grammar and of a context-free language. The main result asserts that, for free theories, recognizable sets and algebraic ones coincide. This result is due to Mezei and Wright. The proofs are given in sections 17 and 18.

It is clear from this introduction that this paper contains nothing that is essentially new, except perhaps for a point of view.

2. CATEGORIES

A category $\mathfrak{a}$ consists of

(2.1) a class of elements called *objects* of $\mathfrak{a}$ and denoted by A, A_1, A_2, A', etc.;

(2.2) a set $\mathfrak{a}(A_1, A_2)$ defined for any pair A_1, A_2 of objects of $\mathfrak{a}$. The elements $f \in \mathfrak{a}(A_1, A_2)$ are called *morphisms* and are written as $f: A_1 \to A_2$ or

$$A_1 \xrightarrow{f} A_2;$$

(2.3) a composition law which to morphisms

$$A_1 \xrightarrow{f} A_2 \xrightarrow{g} A_3$$

assigns a morphism

$$A_1 \xrightarrow{gf} A_3.$$

The following axioms are postulated.

(2.4) Associativity: Given morphisms

$$A_1 \xrightarrow{f} A_2 \xrightarrow{g} A_3 \xrightarrow{h} A_4,$$

we have $h(gf) = (hg)f$.

(2.5) Identity: For every object A there exists a morphism $1_A : A \to A$ such that in

$$A_1 \xrightarrow{f} A \xrightarrow{1_A} A \xrightarrow{g} A_2$$

we have

$$1_A f = f \quad \text{and} \quad g 1_A = g.$$

The uniqueness of 1_A follows from (2.4) and (2.5).

A morphism $f: A_1 \to A_2$ is called an isomorphism if there exists a morphism $g: A_2 \to A_1$ such that $gf = 1_{A_1}$, $fg = 1_{A_2}$. The uniqueness of g follows from (2.4) and (2.5) and we write $g = f^{-1}$.