Let $\mathfrak{A}$ and $\mathfrak{B}$ be categories. A functor $F: \mathfrak{A} \to \mathfrak{B}$ consists of

(2.6) a function which to each object A of $\mathfrak{A}$ assigns an object FA of $\mathfrak{B}$;

(2.7) a function which to each morphism $f: A_1 \to A_2$ in $\mathfrak{A}$ assigns a morphism $Ff: FA_1 \to FA_2$ in $\mathfrak{B}$.

The following axioms are postulated.

(2.8) $F(gf) = (Fg)(Ff)$.

(2.9) $F1_A = 1_{FA}$.

Given functors $F: \mathfrak{A} \to \mathfrak{B}$ and $G: \mathfrak{B} \to \mathfrak{C}$, the composite functor $GF: \mathfrak{A} \to \mathfrak{C}$ is defined in the obvious way.

3. EXAMPLES OF CATEGORIES

The category S of sets has sets as objects and functions as morphisms with composition defined as composition of functions. Two objects in S will play special roles: the empty set $\emptyset$ and the set I consisting of the number 1 alone. For any set A there are unique morphisms

$$\emptyset \to A, \qquad A \to I.$$

We say that $\emptyset$ is an *initial* object and I is a *terminal* object for S.

A set A and an element $a \in A$ determine a unique morphism $I \to A$ with a as value. We shall denote this morphism by the same letter a. Thus morphisms $I \to A$ and elements of A will be identified.

For each integer $n = 0, 1, \cdots$ we denote by $[n]$ the set $\{1, \cdots, n\}$. Thus $[0] = \emptyset$ and $[1] = I$. The sets $[n]$, $n = 0, 1, \cdots$ together with all morphisms between them form a subcategory S_0 of S.

4. THEORIES

A *theory* T is a category such that

(4.1) the objects of T are $[n]$ for $n = 0, 1, \cdots$;

(4.2) S_0 is a subcategory of T; i.e., every morphism in S_0 is also a morphism in T, composition of morphisms in S_0 agrees with that in T, and the identity morphisms $1_{[n]}$ in S_0 are also identity morphisms in T;

(4.3) given morphisms

$$\phi_i: I \to [p] \quad \text{in} \quad T, \quad i = 1, \cdots, n,$$

there exists a unique morphism

$$\phi: [n] \to [p]$$

such that ϕ_i is the composition

$$I \xrightarrow{i} [n] \xrightarrow{\phi} [p]$$

for every $i \in [n]$.

We shall write $\langle \phi_1, \cdots, \phi_n \rangle$ for the morphism ϕ given in (4.3). Thus for any morphism $\phi: [n] \to [p]$ in T, we have $\phi = \langle \phi 1, \cdots, \phi n \rangle$.

It should be noted that it follows from the above axioms that in T there is only one morphism $O_n: \emptyset \to [n]$ for every n, just as in the case of S_0. However, (contrary to what takes place in S_0) there may be in T morphisms $\phi: I \to \emptyset$. In fact, these "0-ary operations" play a fundamental role in the sequel.

5. ALGEBRAS

Let T be a theory. A T-algebra A consists of a set A and a rule which to each $\phi: [n] \to [p]$ in T and each p-tuple $(x_1, \cdots, x_p)$ of elements of A assigns an n-tuple

$$(x_1', \cdots, x_n') = (x_1, \cdots, x_p)\phi$$

of elements of A, subject to the following two axioms:

(5.1) if ϕ is in S_0, then $x_i' = x_{\phi i}$;

(5.2) if $\psi: [k] \to [n]$ in T, then

$$(x_1', \cdots, x_n')\psi = (x_1, \cdots, x_p)(\phi\psi).$$

If we write $x = (x_1, \cdots, x_p)$, then (5.2) may be rewritten as

$$(x\phi)\psi = x(\phi\psi). \tag{5.2'}$$

A morphism $f: A \to B$ of T-algebras is a mapping from A to B satisfying

$$f[(x_1, \cdots, x_p)\phi] = (fx_1, \cdots fx_p)\phi, \tag{5.3}$$

or in abbreviated form

$$f(x\phi) = (fx)\phi, \tag{5.3'}$$

where $fx = (fx_1, \cdots, fx_p)$.

With composition of morphisms of algebras defined in the ordinary fashion, there results the category T^b of T-algebras.

We note that if $\phi: I \to [p]$ in T, then

$$(x_1, \cdots, x_p)\phi \in A$$

so that ϕ yields a mapping $A^p \to A$ where A^p is the p-fold Cartesian product $A \times \cdots \times A$.

If in the above $p = 0$, i.e., $\phi: I \to \emptyset$, then $(\ )\phi \in A$ is an element of A determined by ϕ, independent of any "inputs" $a_1, \cdots, a_p$. We denote this element by ϕ_A.