6. FREE ALGEBRAS

Let $A_k = T(I, [k])$ be the set of all morphisms $I \to [k]$. We convert A_k into a T-algebra as follows: Given $\phi: [n] \to [p]$ in T and given $x_1, \cdots, x_p \in A_k$, we have $x_i : I \to [k]$ and therefore $\langle x_1, \cdots, x_p\rangle: [p] \to [k]$ in T. Thus, the composition $\gamma = \langle x_1, \cdots, x_p\rangle\phi$ is defined and is a morphism $\gamma: [n] \to [k]$ in T. We define

$$(x_1, \cdots, x_p)\phi = (\gamma 1, \cdots, \gamma n).$$

The verification that A_k is a T-algebra is immediate.

We note that each mapping $i: I \to [k]$ $i = 1, \cdots, n$ in S_0 is an element of A_k and thus in a natural fashion, $[k]$ becomes a subset of A_k. The following fact is fundamental:

(6.1) If A is any T-algebra, then every mapping $f: [k] \to A$ admits a unique extension $\bar{f}: A_k \to A$ to a morphism of T-algebras.

Indeed, we must have for $\phi \in A_k$

$$\bar{f}\phi = (f1, \cdots, fk)\phi \in A.$$

The above shows that the algebras A_k are "free" with $[k]$ as base. In particular, A_0 is the free algebra with an empty base and (6.1) asserts that for any T-algebra A there is a unique morphism

$$\zeta_A : A_0 \to A.$$

In fact, $\zeta_A\phi = \phi_A$ for $\phi \in A_0$; i.e., for $\phi: I \to \emptyset$. Thus, A_0 is an "initial object" in the category T^b.

7. FREE THEORIES

As is usual in algebra, theories will be defined by "generators and relations" or as "quotient" theories of "free" theories. We start out with this second notion.

Let $\Omega = \{\Omega_n\}$ $n = 0, 1, \cdots$ be a sequence of sets. Consider a theory T such that

$$\Omega_n \subset T(I, [n]).$$

Assume further that with each morphism $\phi: [n] \to [p]$ in T there is associated an integer $d\phi \geqq 0$ (called the *degree* of ϕ) satisfying the following conditions:

(7.1) $d\phi = 0$ if ϕ is in S_0.

(7.2) $d\phi = d(\phi 1) + \cdots + d(\phi n)$.

(7.3) If $\omega \in \Omega_n$, then $d(\phi\omega) = 1 + d\phi$.

(7.4) If $\phi: I \to [p]$ and $d\phi > 0$, then there exists a unique $k \geqq 0$ and a unique factorization

$$I \xrightarrow{\omega} [k] \xrightarrow{\psi} [p]$$

of ϕ with $\omega \in \Omega_k$ and ψ in T.

It is not too difficult to see that the above conditions virtually amount to the construction of a theory T which is unique. We call it the *free theory* with base Ω and denote it by $S_0[\Omega]$.

The theory $S_0[\Omega]$ has the following two important properties, both of which are easily provable by induction on the degree:

(7.6) Given any theory T', any family of functions

$$\Omega_n \to T'(I, [n]), \qquad n = 0, 1, \cdots$$

admits a unique extension to a morphism

$$S_0[\Omega] \to T'$$

of theories.

(7.7) Given a set A and functions

$$\bar{\omega}: A^n \to A \quad \text{for all} \quad \omega \in \Omega_n, \qquad n = 0, 1, \cdots,$$

there exists a unique $S_0[\Omega]$-algebra structure on A such that

$$(x_1, \cdots, x_n)\omega = \bar{\omega}(x_1, \cdots, x_n).$$

8. CONGRUENCES

Let A be a T-algebra. A congruence Q in A consists of an equivalence relation $\sim$ in A satisfying

$$(a_1, \cdots, a_p)\phi \sim (a_1', \cdots, a_p')\phi$$

for any $\phi: I \to [p]$ in T, provided $a_i \sim a_i'$ for $i = 1, \cdots, p$. It is then clear that the quotient set A/Q (i.e., the set of equivalence classes of A under the equivalence relation) acquires a structure of a T-algebra, uniquely determined by the condition that the natural factorization mapping $A \to A/Q$ be a morphism of T-algebras.

A congruence Q in a theory T is a family of equivalence relations, one in each set $T([n], [p])$ satisfying the following conditions:

(8.1) If $\phi_1, \phi_2 : [n] \to [p]$ and $\phi_1 \sim \phi_2$, then $\phi_1\psi \sim \phi_2\psi$ for every $\psi: [q] \to [n]$ and $\gamma\phi_1 \sim \gamma\phi_2$ for every $\gamma: [p] \to [q]$.

(8.2) If $\phi_1, \phi_2 : [n] \to [p]$ and $\phi_1 i \sim \phi_2 i$ for every $i = 1, \cdots, n$, then $\phi_1 \sim \phi_2$.