(8.3) If $\phi_1, \phi_2 : I \to [p]$ are in S_0 and if $\phi_1 \sim \phi_2$, then $\phi_1 = \phi_2$.

Condition (8.1) permits us to define a category T/Q in which morphisms are equivalence classes of morphisms in T. Condition (8.3) insures that S_0 is embedded in T/Q and condition (8.2) shows that T/Q is a theory. This is the *quotient theory* of T by the congruence Q. A T/Q-algebra A is simply a T-algebra satisfying

$$(a_1, \cdots, a_p)\phi_1 = (a_1, \cdots, a_p)\phi_2,$$

wherever $\phi_1 \sim \phi_2$. In this way, the category $(T/Q)^b$ becomes the subcategory of T^b determined by the T-algebras A that are "compatible" with Q.

Conditions (8.1) and (8.2) imply that Q is completely determined by knowing the equivalence relation in $T(I, [p])$ for every p. There result congruences Q_p in the free algebras A_p, $p = 0, 1, \cdots$. One can regard Q as the sequence $\{Q_p\}$ of these congruences and reformulate conditions (8.1)–(8.3) accordingly. The free T/Q-algebras are then simply the quotient algebras A_p/Q_p.

In practice, a congruence Q in T will seldom be given "in toto". It will usually be "generated" by designating for each p certain pairs $\phi_1, \phi_2 : I \to [p]$ and taking the least Q for which these pairs become congruent. It is easy to construct Q so that (8.1) and (8.2) are satisfied. Whether (8.3) is satisfied is impossible to predict. In this connection the following procedure is useful: Use Q (satisfying only (8.1) and (8.2)) to define congruences Q_p in A_p. Then Q satisfies (8.3) if and only if the algebra A_2/Q_2 has at least two points.

9. PRESENTATION OF THEORIES

The two operations described above, namely the formation of a free theory and the passage to the quotient theory by a congruence, form the basic two steps in the formation of theories. The procedure follows closely the method of presenting groups by generators and relations. Instead of discussing it in general, we shall illustrate by examples. The examples chosen are those particularly relevant to theory of automata.

The theory Sg whose algebras will be semigroups may be described as follows: We begin with a free theory T generated by a single morphism $\pi: I \to [2]$. More explicitly, $T = S_0[\Omega]$ where $\Omega_2 = \{\pi\}$ while $\Omega_n = \emptyset$ for $i \neq 2$. The T-algebras are then sets A with a binary multiplication $(a_1, a_2)\pi \in A$ subject to no conditions whatsoever. To introduce the

associative law, we must "equate" certain two morphisms $I \to [3]$ in T or equivalently certain two elements of the free T-algebra A_3. These two elements are

$$((1, 2)\pi, 3)\pi, (1, (2, 3)\pi)\pi.$$

This generates a congruence Q in T (*a priori* satisfying only (8.1) and (8.2)). A T-algebra A is compatible with Q if and only if

$$((a_1, a_2)\pi, a_3)\pi = (a_1, (a_2, a_3)\pi)\pi$$

holds for any $a_1, a_2, a_3 \in A$; i.e., if and only if A is a semigroup. The fact that there exist semigroups with more than one element implies that Q satisfies also (8.3). Then Sg is defined as T/Q.

To obtain the "monoid theory" M whose algebras will be monoids, one proceeds as above, but in addition to $\pi: I \to [2]$, one has an additional generator $\epsilon: I \to \emptyset$. Then in the free algebra A_1, one must "equate" the following three elements

$$(1, \epsilon_1)\pi, 1, (\epsilon_1, 1)\pi,$$

where ϵ_1 is the composition

$$I \xrightarrow{\epsilon} \emptyset \xrightarrow{\sigma} I,$$

σ being the unique morphism.

There is one more theory that is of vital interest in the theory of automata. Let M be a fixed monoid. We shall construct a theory whose algebras will be sets on which M operates on the right. To this end we consider generators $m: I \to I$ in a 1–1 correspondence with the elements of M. In the free algebra A_1 we then must equate the pairs

$$(1m)n, 1(mn) \qquad m, n \in M$$
$$1, 1\epsilon$$

where $\epsilon \in M$ is the unit element of M. Usually we adjoin to this theory an additional generator $\tau: I \to \emptyset$ without any axioms. There results a theory $\tilde{M} = S_0[M, \tau]$. An $\tilde{M}$-algebra is a set A with a right action $am \in A$ for $a \in A$, $m \in M$ satisfying

$$(am)n = a(mn), \qquad a\epsilon = a$$

and with a selected element $\tau_A \in A$. The initial algebra for this theory is the monoid M itself acting on itself by multiplication and with