If M is a free monoid with base $\mu_1, \cdots, \mu_k$, then $\tilde{M}$ is the free theory $S_0[\mu_1, \cdots, \mu_k, \tau]$.

10. OPERATIONS ON THEORIES

An important operation on theories is the construction of the *free product* (also called *direct sum* or *coproduct*) $T = T' \oplus T''$ of two theories T' and T''. This theory is completely determined by the requirement that its algebras are to be sets A equipped with a T'-algebra structure and T''-algebra structure simultaneously, without any further conditions. The existence of such a theory can be established by choosing presentations of T' and T''. In particular, if $T' = S_0[\Omega']$ and $T'' = S_0[\Omega'']$, then $T = S_0[\Omega]$ where Ω is the disjoint union $\Omega' \cup \Omega''$. The free product $T \oplus S_0[\Omega]$ is denoted by $T[\Omega]$; this is the theory obtained from T by adjoining "freely" the operations Ω.

Let T be a theory and A^0 a T-algebra. One can construct a new theory $T[A^0]$ whose algebras will be pairs (A, f) where A is a T-algebra and $f: A^0 \to A$ is a morphism of T-algebras. A morphism $g: (A, f) \to (A', f')$ will be a morphism $g: A \to A'$ of T-algebras satisfying $gf = f'$. The theory $T[A^0]$ may be constructed by first adjoining freely all the elements of A^0 to T as operations $I \to \emptyset$ and then dividing by a suitable congruence. The theory $T[A^0]$ has the property that the pair $(A^0, 1_{A^0})$ becomes the initial algebra in category $T[A^0]^\flat$. If $A^0 = A_X$ is a free algebra on a base X, then $T[A^0]$ is nothing else than the free extension $T[\Omega]$ with $\Omega_0 = X$, $\Omega_i = \emptyset$ for $i > 0$.

11. RECOGNIZABLE SETS

Let A be a T-algebra and Q a congruence in A. We say that Q is *finite* if A has a finite number of equivalence classes mod Q, or equivalently if A/Q is a finite T-algebra. A subset X of A is said to be *closed* for Q if X is the union of congruence classes mod Q, or equivalently if $a \sim b$ and $a \in X$ imply $b \in X$. A subset X of A which is closed relative to some finite congruence Q is called *recognizable*. The class of recognizable subsets of A is closed with respect to finite Boolean operations.

If M is a monoid, then M may be viewed as a T-algebra for a variety of theories T. If we take for T the "monoid theory", then $T^\flat$ is the category of monoids and $M \in T^\flat$. A congruence Q in A is then an equivalence relation for which $m_1 \sim m_2$ implies $km_1l \sim km_2l$ for all $k, l \in M$. The same notion of congruence in M is obtained if we view M as the initial algebra for the extended theory $T[M]$.

On the other hand, M may also be viewed as the initial algebra for the theory $\tilde{M} = S_0[M, \tau]$ described in section 9. A congruence in M is then an equivalence relation for which $m_1 \sim m_2$ implies $m_1l \sim m_2l$ for all $l \in M$.

It is a known fact that both types of congruences in M lead to the same class of recognizable sets.

12. AUTOMATA

Let T be a theory. A *T-automaton* is a pair $\mathbf{A} = (A, t)$ where A is a *finite* T-algebra and t is a subset of A. The T-automata are converted into a category $T_a^\flat$ by defining a morphism $f: \mathbf{A} \to \mathbf{B}$ where $\mathbf{B} = (B, s)$ as a morphism $f: \mathbf{A} \to \mathbf{B}$ of T-algebras such that $f^{-1}s = t$.

The *behavior* $\mathcal{B}\mathbf{A}$ is defined as a subset of the initial T-algebra A_0 as follows: Let $\zeta_A : A_0 \to A$ be the unique T-morphism. Then $\mathcal{B}\mathbf{A} = \zeta_A^{-1}t$.

The morphism ζ_A defines a congruence Q in A_0 by defining $a_1 \sim a_2$ whenever $\zeta_A a_1 = \zeta_A a_2$. This congruence is finite since A is finite. Further, $\mathcal{B}\mathbf{A}$ is closed for Q.

$\mathcal{B}\mathbf{B}$ is a recognizable subset of A_0. Conversely, let X be any recognizable subset of A_0. Let then Q be a finite congruence in A_0 for which X is closed. Then $A = A_0/Q$ is a finite T-algebra and setting $t = X/Q$ we obtain a T-automaton $\mathbf{A} = (A, t)$. Further, ζ_A is the natural factorization mapping $A_0 \to A_0/Q = A$. Thus, $X = \zeta_A^{-1}t = \mathcal{B}\mathbf{A}$. This shows that the class of all the behaviors of T-automata coincides with the class of recognizable subsets of A_0.

13. RELATIONAL ALGEBRAS

In order to generalize the notion of a nondeterministic automaton, we restrict ourselves to the case that the theory T is free: $T = S_0[\Omega]$. In view of (7.7), a T-algebra A is then described by functions $(x_1, \cdots, x_n)\omega \in A$ for $x_1, \cdots, x_n \in A$ and $\omega \in \Omega_n$.

We define a *relational* T-algebra A to consist of a set A together with functions which to $x_1, \cdots, x_n \in A$ and $\omega \in \Omega_n$ assign a *subset* $(x_1, \cdots, x_n)\omega$ of A. If $X_1, \cdots, X_n$ are subsets of A, then we set

$$(X_1, \cdots, X_n)\omega = \bigcup (x_1, \cdots, x_n)\omega, \tag{13.1}$$

the union extended over all n-tuples $(x_1, \cdots, x_n)$ in A such that $x_i \in X_i$, $i = 1, \cdots, n$. In this way, the set $\hat{A}$ of all the subsets of A becomes a T-algebra.

Conversely, assume that on the set $\hat{A}$ we have a T-algebra structure