satisfying the "distributive" law (13.1) where A is regarded as a subset of $\hat{A}$. Then restricting each $\omega \in \Omega_n$ to A gives the relational T-algebra A.

A morphism $f: A \to B$ of relational T-algebras is defined as a morphism

$$f: \hat{A} \to \hat{B} \quad \text{in} \quad T^{\flat}$$

satisfying the distributivity condition

$$fX = \bigcup fx, \qquad x \in X. \tag{13.2}$$

A function $\hat{A} \to \hat{B}$ satisfying (13.2) is the same thing as a relation from A into B; it is described by the subset R of $A \times B$ defined as follows:

$$R = \{(a, b) \mid b \in fa\}.$$

The relational T-algebras form a category denoted by $T^{\natural}$. The category of T-algebras is a subcategory of $T^{\natural}$. Further, the passage from A to $\hat{A}$ yields a functor $\Lambda : T^{\natural} \to T^{\flat}$.

An important fact to note is that the initial algebra A_0 for the category $T^{\flat}$ remains an initial algebra also within the larger category $T^{\natural}$. Indeed, if $A \in T^{\natural}$, then $\hat{A} \in T^{\flat}$ and we have a unique $\zeta_{\hat{A}} : A_0 \to \hat{A}$. This defines $\zeta_A : A_0 \to A$ in $T^{\natural}$, which is unique since $\zeta_{\hat{A}}$ is.

14. RELATIONAL AUTOMATA

We define a relational automaton $\mathbf{A} = (A, t)$ exactly as above, except that $A \in T^{\natural}$. The behavior is defined as

$$\mathfrak{B}\mathbf{A} = \zeta_A^{-1} t = \{x \mid x \in A_0, \zeta_A x \cap t \neq \emptyset\}.$$

It is now clear that if we define the automaton

$$\mathbf{A} = (\hat{A}, t'), \qquad t' = \{X \mid X \subset A, X \cap t \neq \emptyset\},$$

then $\mathfrak{B}\mathbf{A} = \mathfrak{B}\mathbf{A}$. We thus have the generalization of the known fact that nondeterministic automata recognize the same sets as deterministic automata.

15. POLYNOMIALS

Let T be a free theory. A *polynomial*

$$P: [n] \to [p]$$

is an n-tuple $P = (P_1, \cdots, P_n)$ where $P_1, \cdots, P_n$ are finite subsets of

$T(1, [p])$. The elements of P_i are called the *constituents* of P, and if all these constituents have degree 1, then we say that P has degree 1.

Let A be a relational T-algebra, and let $X = (X_1, \cdots, X_p)$ be a p-vector of subsets of A. We define

$$XP_i = \bigcup_{\phi \in P_i} (X_1, \cdots, X_p)\phi$$

$$XP = (XP_1, \cdots, XP_n).$$

Thus, XP is an n-vector of subsets of X. Therefore, P defines a function

$$P_A : \hat{A}^p \to \hat{A}^n.$$

In the p-fold product $\hat{A}^p$ of $\hat{A}$ we define inclusion and union coordinate by coordinate. We then have the following important property of P_A:

(15.1) If in $\hat{A}^p$ we have

$$X^0 \subset X^1 \subset \cdots \subset X^k \subset \cdots,$$

then

$$(\bigcup_k X^k)P_A = \bigcup_k (X^k P_A).$$

For the proof it suffices to consider the case $n = 1$ and $P = P_1 = \phi: I \to [p]$ is a monomial (i.e., P has a single constituent). In this case the desired relation is proved in a straightforward manner by induction on the degree of ϕ.

Property (15.1) implies that P_A is monotone; i.e., that $XP_A \subset YP_A$ whenever $X \subset Y$ in $\hat{A}^p$.

We now consider a polynomial

$$P: [n] \to [n].$$

Then the transformation

$$P_A : \hat{A}^n \to \hat{A}^n$$

may be iterated, yielding

$$P_A{}^k: \hat{A}^n \to \hat{A}^n,$$

for which (15.1) also holds. In particular, if $\emptyset \in \hat{A}^n$ is the n-tuple $(\emptyset, \cdots, \emptyset)$, we have

$$\emptyset \subset \emptyset P_A \subset \emptyset P_A{}^2 \subset \cdots \subset \emptyset P_A{}^k \subset \cdots$$