and we define

$$\bar{P}_A = \bigcup_k \emptyset P_A^{\,k}.$$

The following fact should be regarded as well known (as well as easily provable):

(15.2) $\bar{P}_A$ is the least solution of the equation

$$XP_A = X,$$

for $X \in A^n$, as well as the least solution of the inequality

$$XP_A \subset X.$$

In the special case where $A = A_0$ is the initial algebra in $T^\natural$, we shall write $\bar{P}$ instead of $\bar{P}_{A_0}$. This special case is all-important because of

$$\text{If } \zeta_A : A_0 \to A, \qquad \text{then } \bar{P}_A = \zeta_A \bar{P}. \tag{15.3}$$

Here, ζ_A denotes the mapping $\zeta_A : \hat{A}_0^{\,n} \to \hat{A}^n$ defined by the mapping $\zeta_A : \hat{A}_0 \to \hat{A}$ given by the relation $\zeta_A : A_0 \to A$. The fact stated in (15.3) follows readily from the commutative diagram

$$\begin{array}{ccc} \hat{A}_0^{\,n} & \xrightarrow{P_{A_0}} & \hat{A}_0^{\,n} \\ {\scriptstyle \zeta_A}\big\downarrow & & \big\downarrow{\scriptstyle \zeta_A} \\ \hat{A}^n & \xrightarrow[P_A]{} & \hat{A}^n \end{array}$$

and the facts that

$$\zeta_A \emptyset = \emptyset, \qquad \zeta_A \bigcup_k X^k = \bigcup_k \zeta_A X^k.$$

16. ALGEBRAIC SETS

A subset X of the initial algebra A_0 for a free theory $T = S_0[\Omega]$ is called *algebraic* if there exists an integer n and a polynomial $P : [n] \to [n]$ such that $X = \bar{P}_1$; i.e., X is the first coordinate of the least solution of the equation $YP_{A_0} = Y$ for $Y \in \hat{A}_0^{\,n}$.

The following properties of algebraic sets should be regarded as known:

(16.1) Each element x of A_0 is an algebraic set.

(16.2) The empty set is algebraic.

(16.3) If X_1 and X_2 are algebraic sets, then so is $X_1 \cup X_2$.

(16.4) If $\phi : I \to [p]$ in T and $X_1, \cdots, X_p$ are algebraic sets, then so is $(X_1, \cdots, X_p)\phi$.

We are now in a position to state the main results of this paper.

THEOREM 1. *For each algebraic set X in the initial algebra A_0 over a free theory $T = S_0[\Omega]$, there exists an integer $n > 0$ and a polynomial $P : [n] \to [n]$ of degree 1 such that $X = \bar{P}_1$.*

The proof will be given in section 17.

Now assume that the theory $T = S_0[\Omega]$ is free on a finite base; i.e., that each of the sets Ω_k is finite and that $\Omega_k = \emptyset$ for all but a finite number of integers $k \geqq 0$.

Let A be a relational T-algebra with $[n]$ as underlying set. We associate with A a polynomial,

$$A^P : [n] \to [n]$$

of degree 1 as follows: A morphism $\phi : I \to [n]$ of degree 1 is a composition

$$I \xrightarrow{\omega} [p] \xrightarrow{x} [n],$$

where $\omega \in \Omega_p$ and $x = (x1, \cdots, xp)$, a p-tuple of elements in $[n]$; i.e., in A. We define $x\omega \in P_i$ if and only if $i \in (x1, \cdots, xp)\omega$ according to the relational T-algebra structure A. This clearly gives a bijection between the relational T-algebra structures A on $[n]$ and polynomials $P : [n] \to [n]$ of degree 1.

THEOREM 2. *If the relational T-algebra A on $[n]$ and the polynomial $P : [n] \to [n]$ of degree 1 are related as above, then*

$$\bar{P}_i = \zeta_A^{-1} i.$$

We recall here that $\zeta_A : A_0 \to A = [n]$ is a relation so that

$$\zeta_A^{-1} i = \{y \mid y \in A_0, i \in \zeta_A y\}.$$

The proof will be given in section 18.

From the two theorems asserted above, we can now prove:

THEOREM 3. *If $T = S_0[\Omega]$ is a free theory on a finite base, then in the initial T-algebra A_0 the recognizable sets and the algebraic sets coincide.*

Proof. Let $X \subset A_0$ be recognizable. Then $X = \mathfrak{B}\mathbf{A}$ where $\mathbf{A} = (A, t)$ is an automaton. Since the T-algebra A is finite, we may, without loss, assume that the underlying set of A is $[n]$ for some $n > 0$. Let $P : [n] \to [n]$ be the associated polynomial of degree 1. Then by Theorem 2

$$X = \mathfrak{B}\mathbf{A} = \zeta_A^{-1} t = \bigcup_{i \in t} \zeta_A^{-1} i = \bigcup_{i \in t} \bar{P}_i.$$

Since each $\bar{P}_i$ is algebraic, it follows from (16.3) that X is algebraic.