Conversely, let X be an algebraic set in A_0. Then by Theorem 1 we have $X = \bar{P}_1$ for some polynomial $P\colon [n] \to [n]$ of degree 1. Let A be the relational T-algebra structure associated with P and let $\mathbf{A} = (A, t)$ be the relational automaton with $t = \{1\}$. Then by Theorem 2

$$\mathcal{B}\mathbf{A} = \zeta_A^{-1} 1 = \bar{P}_1 = X$$

so that X is recognizable.

17. PROOF OF THEOREM 1

We shall establish two auxiliary propositions.

PROPOSITION 1. *Given a polynomial* $P\colon [n] \to [n]$, *there exists a polynomial* $Q\colon [n] \to [n]$ *such that*

(i) *The constituents of* Q *are precisely the constituents of* P *of degree* >0.

(ii) $\bar{Q} = \bar{P}$.

PROPOSITION 2. *Given a polynomial* $P\colon [n] \to [n]$, *there exists a polynomial* $Q\colon [m] \to [m]$, $n \leqq m$ *such that*

(i) *All constituents of* Q *have degree* $\leqq 1$.

(ii) Q_i *has the same constituents of degree* 0 *as* P_i, $i = 1, \cdots, n$.

(iii) Q_i, $n < i < m$ *has no constituents of degree* 0.

(iv) $\bar{Q}_i = \bar{P}_i$ *for* $i = 1, \cdots, n$.

It is now clear how Theorem 1 follows from these two propositions. Given an algebraic set X in A_0, choose a polynomial $P\colon [n] \to [n]$ such that $X = \bar{P}_1$. Then apply both propositions consecutively and in either order. There results a polynomial $Q\colon [m] \to [m]$, $n \leqq m$ of degree 1 such that $\bar{Q}_1 = X$.

In the proofs that follow, it will be convenient to use the symbol $+$ for $\cup$ and Σ for $\bigcup$.

Proof of Proposition 1. We represent the polynomial P in the form $P = R + M$ where $R\colon [n] \to [n]$ consists of all the constituents of P of degree >0, while M consists of the constituents of P of degree 0. The morphisms $j\colon 1 \to [n]$ of degree 0 are $j = 1, \cdots, n$. Thus M may be represented as the $n \times n$ matrix $\{M_{ij}\}$ whose coordinates M_{ij} are 1 or 0 depending on whether $j\colon I \to [n]$ is a constituent of M_i (i.e., a constituent of P_i). We now form the matrix

$$N = E + M + M^2 + \cdots + M^k + \cdots$$

as follows: We regard $\mathcal{B} = \{0, 1\}$ as a semi-ring with the operation table

$$0 + 0 = 0, \qquad 0 + 1 = 1 + 0 = 1 + 1 = 1$$

$$11 = 1, \qquad 01 = 10 = 00 = 0$$

and regard M as a matrix with coefficients in $\mathcal{B}$. Matrices are multiplied and added in the usual fashion. E denotes the matrix with 1 on the diagonal and zero everywhere else. The sequence of matrices

$$M^{(k)} = E + M + M^2 + \cdots + M^k$$

$k = 0, 1, \cdots$ is ascending and, therefore, for some k we have $M^{(k)} = M^{(l)}$ for all $l \geqq k$. The matrix N is then defined as $M^{(k)}$, for k sufficiently large. We now define the polynomial $Q\colon [n] \to [n]$ as $Q = RN$; i.e.,

$$Q_i = \sum_k R_k N_{ki}.$$

Since $N_{ii} = 1$ we have $R_i \subset Q_i$. Condition (i) of Proposition 1 is then clearly satisfied and we now prove that $\bar{Q} = \bar{P}$.

Assume that $X = (X_1, \cdots, X_n)$ is a vector of subsets of A_0 such that $XP \subset X$. Then since $P = R + M$, we must have $XR \subset X$ and $XM \subset X$. Therefore, $XM^k \subset X$ for all k and thus $XN \subset X$. It follows that

$$XQ = (XR)N \subset XN \subset X.$$

This proves that $\bar{Q} \subset \bar{P}$. To prove the converse, assume that $XQ = X$. Since $R \subset Q$, we have $XR \subset X$. Since $NM \subset N$, we have

$$QM = RNM \subset RM = Q.$$

Therefore,

$$XP = XR + XM = XR + XQM \subset X + QX = X.$$

This shows that $\bar{P} \subset \bar{Q}$. Thus $\bar{P} = \bar{Q}$.

Proof of Proposition 2. Assume that $k > 1$ is the highest degree of the constituents of P and let $\phi\colon I \to [n]$ be a constituent of P of degree k. We may write

$$P = R + \phi M,$$

where R is a polynomial not containing ϕ as a constituent while $M = (M_1, \cdots, M_n)$ is a vector with components 0, 1 defined by $M_i = 1$ or $M_i = 0$ depending on whether or not ϕ is a constituent of P_i. The morphism ϕ has a factorization

$$I \xrightarrow{\omega} [p] \xrightarrow{\psi} [n]$$