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An Introduction to Deformation Quantization

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Deformation quantization (quantum mechanics in phase space) is an interesting formulation of quantum mechanics as it provides a natural way to recover classical mechanics in the limit $\hbar \to 0$, while such a limit is not rigorous in the scope of the correspondence principle. In addition, one can show that the predictions of deformation quantization and of the standard formulation of quantum mechanics are the same as we shall see in this article. Interestingly enough, this formulation also provides a very natural framework to encode the possible non-commutative behavior of space at the Planck scale where gravity and the laws of quantum mechanics are both expected to be applicable.

1. Introduction

Since the discovery of the laws of quantum mechanics, several formulations and interpretations have been put forward. The standard one, called the canonical formulation, which was developed by Heisenberg, Schrödinger and Dirac, is based on operators that act on a Hilbert space. It postulates that classical observables such as position q and momentum p can be promoted to operators ($\hat{Q}$ and $\hat{P}$) acting on a Hilbert space while making a correspondence between the classical Poisson bracket $\{.,.\}$ and the commutator of operators $[.,.]$ in such a way that the Poisson bracket can be restored in the classical limit $\hbar \to 0$, *i.e.*

$$\lim_{\hbar \to 0} \frac{1}{i\hbar}[\hat{A}, \hat{B}] \hat{=} \{A, B\}\,, \tag{1}$$

where $\hat{A}$ and $\hat{B}$ are quantum observables (operators) and A and B are their classical counterparts.

Another formulation is the path-integral approach, which was motivated by Dirac and elaborated by Feynman. This approach assumes that the probability amplitude $\langle f|i\rangle$ for a particle to make a transition from some initial state $|i\rangle$ to some final state $|f\rangle$ can be obtained by summing over all possible classical configurations that interpolate between $|i\rangle$ and $|f\rangle$. This can be, roughly speaking, expressed in the following form:

$$\langle f|i\rangle \propto \sum_{\text{all paths}} \exp\left(\frac{i}{\hbar}S\right), \tag{2}$$

where $S = \int L dt$ is the classical action associated to the Lagrangian L of the particle. The equivalence between this formulation and the canonical formulation has been intensively studied.[1]

A third formulation, the bronze medal,[2] is given by the phase space formulation or deformation quantization whose pioneers are Wigner, Weyl, Groenewold and Moyal. The main advantage of this formulation, as we shall see in the following section, is the fact that it provides a natural framework for the transition from the classical Hamiltonian formalism which relies on phase space coordinates (q, p) to quantum mechanics in phase space. We will also see how this formulation gives a natural framework for the transition from Poisson brackets to commutators (Equation 1), which is only postulated and somewhat obscure in the correspondence principle within the first formulation.

We then show how classical mechanics can be smoothly generalized to a formulation of quantum mechanics in phase space (Section 2). Some similarities with the canonical formulation of quantum mechanics are mentioned and we shall see some examples where this new formulation gives exactly the same results as the canonical formulation. Finally, we argue that space can be non-commutative at the Planck scale and we show how deformation quantization applied to space can be used to modify the Schrödinger equation in the case of a non-commutative space (Section 3).

2. Quantum mechanics as a deformation theory

The main difference between quantum mechanics and classical mechanics is the Heisenberg uncertainty relation, which is a natural consequence of the non-commutativity of position and momentum operators, *i.e.* $[\hat{X}, \hat{P}] = i\hbar\mathbb{1}$. This observation suggests that the commutative product of classical observables has to be replaced with a non-commutative product in order to account for the non-commutativity between quantum observables.

2.1. Short review of classical mechanics

It is useful given the above context to review the Hamiltonian formalism in classical mechanics, since deformation quantization is a natural generalization of classical mechanics.

In classical mechanics, functions depend on phase space coordinates which are usually denoted by $(\boldsymbol{q}, \boldsymbol{p})$ in the literature. If we take two functions f and g, they can be multiplied using the ordinary commutative product "$\cdot$", *i.e.*

$$f(\boldsymbol{q}, \boldsymbol{p}) \cdot g(\boldsymbol{q}, \boldsymbol{p}) = g(\boldsymbol{q}, \boldsymbol{p}) \cdot f(\boldsymbol{q}, \boldsymbol{p}) \ .$$

However, in the classical Hamiltonian formulation, there is another way to combine two functions using the so-called canonical Poisson bracket $\{.,.\}$ defined as follows:

$$\{f, g\} \equiv \sum_{i=1}^{N} \left(\frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} \right) . \tag{3}$$

From this definition, one deduces the canonical commutation relation $\{q_i, p_j\} = \delta_{ij}$ which is further promoted in the quantum case to $[\hat{Q}_i, \hat{P}_j] = i\hbar\delta_{ij}\mathbb{1}$ using the correspondence principle of the canonical formulation. It is also clear that the Poisson bracket (Equation 3) of two functions is non-commutative (anti-symmetric under the exchange of f and g). This property will be essential to construct another non-commutative operation between classical observables in the phase space, as we will see in the next subsection.

In order to simplify the expression of the Poisson bracket (Equation 3), we shall use this notation:

$$f\overleftarrow{\partial}_{q_i} g = \frac{\partial f}{\partial q_i} g, \quad f\overrightarrow{\partial}_{p_i} g = f\frac{\partial g}{\partial p_i} \ , \ \partial_{\boldsymbol{q}}\partial_{\boldsymbol{p}} = \sum_{i=1}^{N} \frac{\partial}{\partial q_i}\frac{\partial}{\partial p_i}. \tag{4}$$

Hence, this expression can be rewritten in a simple manner which will be useful later:

$$\{f, g\} = f(\overleftarrow{\partial}_{\boldsymbol{q}}\overrightarrow{\partial}_{\boldsymbol{p}} - \overleftarrow{\partial}_{\boldsymbol{p}}\overrightarrow{\partial}_{\boldsymbol{q}})g. \tag{5}$$

Another useful way to express the canonical Poisson bracket is by using the Poisson tensor $(\Theta^{IJ}) = \begin{pmatrix} 0 & I_N \\ -I_N & 0 \end{pmatrix}$. One can then write the Poisson bracket as follows:

$$\{f, g\}(\boldsymbol{x}) = f(\Theta^{IJ}\overleftarrow{\partial}_I\overrightarrow{\partial}_J)g \ , \tag{6}$$

where $\partial_I f \equiv \partial f/\partial x_I$ and $\boldsymbol{x} \equiv (x_1, ..., x_{2N}) \equiv (\boldsymbol{q}, \boldsymbol{p})$. Here and in the following, the Einstein summation convention over repeated indices is assumed, with the summation indices i, j lying within the range $[\![1, N]\!]$ and the indices I, J taking values within the interval $[\![1, 2N]\!]$.

If we assume that the dynamics of a classical physical system is described by a Hamiltonian function $H(\boldsymbol{q}, \boldsymbol{p})$, the equations of motion in terms of the phase space coordinates are given by the Hamilton equations:

$$\dot{q}_i = \frac{\partial H}{\partial p_i} = \{q_i, H\} \ , \quad \dot{p}_i = -\frac{\partial H}{\partial q_i} = \{p_i, H\} \, . \tag{7}$$

Using the chain rule, we can show that the time evolution equation, for any classical observable $f(\boldsymbol{x})$ on the phase space, is given by

$$\dot{f} = \{f, H\}\,. \tag{8}$$

2.2. Deformation of the usual product

In the framework of deformation quantization, there is no sudden break when the transition from classical mechanics to quantum mechanics is made. The upshot is that an observable in deformation quantization is described by the same function on phase space as its classical counterpart, *e.g.* the Hamiltonian function $H(\boldsymbol{q}, \boldsymbol{p})$, where we encode its quantum behavior using a non-commutative product which will be defined in this section.

Let us consider two smooth functions f and g on phase space which will be later identified to quantum observables. One can define a non-commutative operation "$*$" called the star product, which involves the Poisson bracket of the two functions as follows:

$$f * g = f \cdot g + \frac{i\hbar}{2}\{f, g\} + \mathcal{O}(\hbar^2)\ , \tag{9}$$

where the parameter $\hbar$ is identified with the Planck constant. The term $\mathcal{O}(\hbar^2)$ is needed to insure the associativity of this operation. From this definition, one can see that the star product is non-commutative and that the non-commutativity is a result of a smooth deformation of the usual product with a small non-commutative term at the order of $\hbar$. Hence, this definition justifies the name of "deformation" quantization. Using the previous result (Equation 9), we can define a $*$-commutator of two phase space observables f and g in the following way:

$$[f, g]_* \equiv f * g - g * f = i\hbar\{f, g\} + \mathcal{O}(\hbar^2)\ , \tag{10}$$

where the last equality follows from the definition of the star product (9).

At this point, one can clearly see the relation between the Poisson bracket (Equation 3) and the quantum commutator which is defined here as a $*$-commutator (Equation 10). Interestingly enough, the classical limit (Equation 1) can be verified by using the definition of the $*$-commutator, *i.e.*

$$\lim_{\hbar \to 0} \frac{1}{i\hbar}[f, g]_* = \{f, g\}\,. \tag{11}$$

In other words, we can naturally recover classical mechanics in the limit $\hbar \to 0$. Moreover, the quantum observables can be identified with their classical counterpart functions on phase space, the only difference here is that the usual commutative product of these functions is replaced by a non-commutative star product.

Note that within the first formulation of quantum mechanics, the classical limit has to be enforced by the correspondence principle. However, in the scope of deformation quantization, there is no such break as the transition is done in a natural way (Equation 11).

The expression of the star product to all orders of $\hbar$ is given by the Groenewold-Moyal product "$*_M$"[3–6] by exponentiating the first term of the star product (Equation 9) as follows:

$$\begin{aligned} f *_M g &= f \exp\Big(\frac{i\hbar}{2}\big(\overleftarrow{\partial_{\boldsymbol{q}}}\overrightarrow{\partial_{\boldsymbol{p}}} - \overleftarrow{\partial_{\boldsymbol{p}}}\overrightarrow{\partial_{\boldsymbol{q}}}\big)\Big)g \\ &= f \exp\Big(\frac{i\hbar}{2}\Theta^{IJ}\overleftarrow{\partial_I}\overrightarrow{\partial_J}\Big)g. \end{aligned} \tag{12}$$

An important consequence of this definition is the emergence of non-locality which is a feature of quantum mechanics. One can see that its presence is implicit since the previous equation involves not only the values of the functions, but also their derivatives to arbitrary high orders.

The careful reader may wonder whether we are allowed to use the phase space coordinates as a label for observables, since the Heisenberg uncertainty relation must be verified. In fact, this is permitted within the present formulation as the uncertainty relation is encoded in the star product.[7]

In addition, one can deduce a simple expression, called the shift formula, of the Groenewold-Moyal star product (12) which will be very useful for the resolution of the Hamiltonian eigenvalue problems:

$$(f *_M g)(\boldsymbol{x}) = f(\boldsymbol{x} - \frac{\widetilde{\boldsymbol{p}}}{2})g, \tag{13}$$

where $\widetilde{p}^I \equiv \Theta^{IJ}(-i\hbar\partial_J)$. As the Taylor expansion implies $f(\boldsymbol{x} + \boldsymbol{a}) = \exp(\boldsymbol{a}\partial_{\boldsymbol{x}})f(\boldsymbol{x})$, we can take $\boldsymbol{a} = \widetilde{\boldsymbol{p}}/2$ and use the last expression of the Taylor expansion. This allows to rewrite the shift formula from Equation 13 in a more concrete manner:

$$(f *_M g)(\boldsymbol{q}, \boldsymbol{p}) = f(\boldsymbol{q} + \frac{i\hbar}{2}\partial_{\boldsymbol{p}}, \boldsymbol{p} - \frac{i\hbar}{2}\partial_{\boldsymbol{q}})g(\boldsymbol{q}, \boldsymbol{p}). \tag{14}$$

2.3. Principles of deformation quantization

2.3.1. The phase space wave function

In the canonical formulation of quantum mechanics, a state is described by a vector $|\Psi\rangle$ in a Hilbert space. The analog of such states in the phase space formulation are functions defined on the phase space which can be associated to a quantum state $|\psi\rangle$ using the Wigner functions[8] (Equation 15).

$$\begin{aligned}\rho_\psi(\boldsymbol{q}, \boldsymbol{p}) &= \frac{1}{(2\pi\hbar)^N}\int_{\mathbb{R}^N} \mathrm{d}\boldsymbol{y}\ \overline{\psi}(\boldsymbol{q} - \frac{1}{2}\boldsymbol{y})\psi(\boldsymbol{q} + \frac{1}{2}\boldsymbol{y})e^{\frac{i}{\hbar}\boldsymbol{y}\cdot\boldsymbol{p}} \\ &= \frac{1}{(2\pi)^N}\int_{\mathbb{R}^N} \mathrm{d}\boldsymbol{y}\ \overline{\psi}(\boldsymbol{q} - \frac{\hbar}{2}\boldsymbol{y})\psi(\boldsymbol{q} + \frac{\hbar}{2}\boldsymbol{y})e^{i\boldsymbol{y}\cdot\boldsymbol{p}}.\end{aligned} \tag{15}$$

One can easily verify that the Wigner function is normalized, *i.e.* $\int_{\mathbb{R}^{2N}} \mathrm{d}\boldsymbol{q}\mathrm{d}\boldsymbol{p}\ \rho_\psi(\boldsymbol{q}, \boldsymbol{p}) = 1$, and real-valued. For the sake of clarity, we will now describe the analogies between the canonical formulation and the phase space formulation to show the important relations and quantities of the phase space formulation.

2.3.2. Analogies

In the phase space formulation, an eigenstate, denoted by $\Pi_n(\boldsymbol{q}, \boldsymbol{p})$ corresponds to an eigenstate $|\psi_n\rangle$ of the canonical formulation, *i.e.* the phase space eigenstate $\Pi_n(\boldsymbol{q}, \boldsymbol{p})$, referred to as a $*$-eigenfunction, is defined as a Wigner function associated to the eigenstate $|\psi_n\rangle$. This implies that Π_n is normalized and real-valued. Note that here, n is only a label to enumerate the $*$-eigenfunctions Π_n.

The two formulations have a lot of similarities (Table 1). In addition, one can derive an analogue for the Ehrenfest evolution (Equation 8) of classical observables in the following way:

$$\frac{\mathrm{d}\langle k\rangle}{\mathrm{d}t} = \frac{1}{i\hbar}\langle [k, H]_*\rangle\,, \tag{16}$$

where "$[.,.]_*$" is the star commutator (Equation 10). The expectation value of a classical observable $k(\boldsymbol{q}, \boldsymbol{p})$ for a system described by a phase space state $\rho(\boldsymbol{q}, \boldsymbol{p})$ is given by[9]

$$\langle k\rangle = \int_{\mathbb{R}^{2N}} \mathrm{d}\boldsymbol{q}\mathrm{d}\boldsymbol{p}\ k(\boldsymbol{q}, \boldsymbol{p})\rho(\boldsymbol{q}, \boldsymbol{p}). \tag{17}$$

Finally, it should be noted that the main difference between the two formulations is the fact that the $*$-eigenfunctions Π_n are also $*$-projectors (Table 1), which is clearly not the case in the canonical formulation.

2.4. Examples

We now have the necessary tools for tackling some concrete examples, which have been already solved in the canonical formulation. In this section, we will try to illustrate this point by solving two basic problems in quantum mechanics: a free particle and a harmonic oscillator in a one-dimensional space.[9]

2.4.1. Free particle

A free particle can be defined as a particle moving in a constant potential where the latter can be shifted to zero. In this case, the classical Hamiltonian of such a particle reduces to the kinetic term as follows:

$$H(q, p) = \frac{p^2}{2m}. \tag{18}$$

According to the canonical formulation, the energy eigenvalues of the Hamiltonian operator $\hat{H} = \hat{p}^2/2m$ are given by $E = p^2/2m$. The latter corresponds to a non-localized particle with a definite momentum p. In order to recover this result in the phase space formulation, we shall solve the $*$-eigenvalue problem $(H *_M \Pi)(q, p) = E\,\Pi(q, p)$ analogously to the eigenvalue problem in the canonical formulation.

Using the shift formula (Equation 14), the $*$-eigenvalue problem can be reduced to the following equation:

$$\frac{1}{2m}\left(p - i\frac{\hbar}{2}\partial_q\right)^2 \Pi = E\Pi, \tag{19}$$

hence:

$$\frac{1}{2m}\left(p^2 - i\hbar p\partial_q - \frac{\hbar^2}{4}\partial_q^2\right)\Pi = E\Pi. \tag{20}$$

Canonical QM		Phase space QM	
State	$\|\psi\rangle$	Wigner function	ρ_ψ
Eigenstate	$\|\psi_n\rangle$	$*$-Eigenfunction	$\Pi_n(\boldsymbol{q},\boldsymbol{p})$
Eigenvalue equation	$\hat{H}\|\psi_n\rangle = E_n\|\psi_n\rangle$	$*$-Eigenvalue equation	$(H*\Pi_n)(\boldsymbol{q},\boldsymbol{p}) = E_n\Pi_n(\boldsymbol{q},\boldsymbol{p})$
Projector	$\hat{P}_n = \|\psi_n\rangle\langle\psi_n\|$	$*$-Projector	$\Pi_n(\boldsymbol{q},\boldsymbol{p})$
Orthonormality	$\hat{P}_n\hat{P}_{n'} = \delta_{n,n'}\hat{P}_n$	$*$-Orthonormality	$(\Pi_n*\Pi_{n'})(\boldsymbol{q},\boldsymbol{p}) = \delta_{n,n'}\Pi_n(\boldsymbol{q},\boldsymbol{p})$
Completeness	$\sum_n \hat{P}_n = \mathbb{1}$	$*$-Completeness	$\sum_n \Pi_n = 1$
Spectral decomposition	$\hat{H} = \sum_n E_n\|\psi_n\rangle\langle\psi_n\|$	$*$-Spectral decomposition	$H(\boldsymbol{q},\boldsymbol{p}) = \sum_n E_n\Pi_n(\boldsymbol{q},\boldsymbol{p})$

Tab. 1 Summary of the analogies between the canonical formulation and the phase space formulation. n is regarded as a label for the eigenstates ($*$-eigenfunctions), it can be either discrete or continuous.

Since Π is a real-valued function, as suggested by the definition, the imaginary part of the reaction verifies:

$$\partial_q\Pi = 0, \tag{21}$$

and the real part verifies:

$$\left(\frac{p^2}{2m} - E\right)\Pi = 0. \tag{22}$$

Therefore, the energy eigenvalues are given by $E = p^2/2m$, which is exactly the result found in the case of the canonical formulation. In fact, the imaginary part result can already be anticipated (Equation 15) since the wave function $\psi(q)$ of a free particle is delocalised in space.

Now, in order to study a system with discrete energy eigenvalues, we shall consider a non-zero potential energy which will be the case of the next example.

2.4.2. The harmonic oscillator

The Hamiltonian of a one dimensional harmonic oscillator with a frequency ω can be written

$$H = \frac{p^2}{2m} + \frac{m\omega^2}{2}q^2. \tag{23}$$

For convenience, we shall express this Hamiltonian in terms of holomorphic variables as follows:

$$a = \sqrt{\frac{m\omega}{2\hbar}}\left(q + i\frac{p}{m\omega}\right)\ ,\quad \bar{a} = \sqrt{\frac{m\omega}{2\hbar}}\left(q - i\frac{p}{m\omega}\right), \tag{24}$$

in terms of which our Hamiltonian becomes

$$H = \hbar\omega\bar{a}a. \tag{25}$$

The choice of a and $\bar{a}$ as our phase space variables instead of q and p requires a change of variables in the Groenewold-Moyal star product (Equation 12). The latter can be done using the chain rule to go from (q,p) to $(a,\bar{a})$. By doing so, we find that the star product of functions of the phase space $(a,\bar{a})$ becomes

$$f *_M g = f\exp\left(\frac{1}{2}(\overleftarrow{\partial_a}\overrightarrow{\partial_{\bar{a}}} - \overleftarrow{\partial_{\bar{a}}}\overrightarrow{\partial_a})\right)g, \tag{26}$$

where f and g are considered as functions of the phase space $(a,\bar{a})$.

Before solving the $*$-eigenvalue problem, we would like to make an interesting remark: if we calculate the star product of a, we find that

$$\bar{a} *_M a = a\bar{a} - \frac{1}{2}, \tag{27}$$

Hence, the Hamiltonian can be expressed in the following form:

$$H = \hbar\omega\left(\bar{a} *_M a + \frac{1}{2}\right). \tag{28}$$

Interestingly enough, this expression has the same form as in the canonical formulation where the Hamiltonian operator of the harmonic oscillator is expressed as

$$\hat{H} = \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right). \tag{29}$$

This analogy underlines the fact that the non-commutativity of operators in the canonical formulation, which results in the vacuum energy

$\hbar\omega/2$, is encoded in the non-commutative star product.

Now, let us try to find the energy spectrum of the harmonic oscillator. We shall proceed similarly as we did in the case of a free particle. Let Π be a $*$-eigenfunction with an energy E, *i.e.*

$$(H *_M \Pi)(a, \overline{a}) = E\Pi(a, \overline{a}). \tag{30}$$

The crucial point that must not be forgotten here is that the shift formula in terms of p and q has to be rewritten in terms of the variables $(a, \overline{a})$ using the star product (Equation 26). This transformation can be shown to result in

$$(f *_M g)(a, \overline{a}) = f(a + \frac{1}{2}\partial_{\overline{a}}, \overline{a} - \frac{1}{2}\partial_a)g(a, \overline{a}). \tag{31}$$

As a consequence, the energy $*$-eigenvalue problem can be expressed, using the last shift formula, as follows:

$$\hbar\omega(a\overline{a} - \frac{1}{4}\partial_a\partial_{\overline{a}})\Pi + \hbar\omega(\overline{a}\partial_{\overline{a}} - a\partial_a)\Pi = E\Pi. \tag{32}$$

It is clear that the second term on the left hand side is purely imaginary since Π is real-valued. Hence, it should vanish. From this constraint, it follows that $\overline{a}\partial_{\overline{a}}\Pi = a\partial_a\Pi$. This means that Π is only a function of the product $a\overline{a}$, *i.e.* $\Pi(a, \overline{a}) = \rho(a\overline{a})$. This can be shown by Taylor expanding $\Pi(a, \overline{a})$ in a and $\overline{a}$ and using the constraint $\overline{a}\partial_{\overline{a}}\Pi = a\partial_a\Pi$. For convenience, we define $u \equiv a\overline{a}$. Then, using the chain rule, the real part solution becomes

$$\left(u - \frac{1}{4}\partial_u - \frac{1}{4}u\partial_u^2 - \frac{E}{\hbar\omega}\right)\rho(u) = 0. \tag{33}$$

In the limit of large u, we expect that ρ decreases exponentially with some finite rate λ. This is suggested by the definition of the Wigner function since the states $\psi(\mathrm{q})$ in the position space representation are expected to decrease exponentially. Hence, $\partial_u\rho/(u\partial_u^2)\rho \sim \lambda/u \ll 1$. If we substitute $\rho \sim \exp(-u/\lambda)$ in the limit of large u, we find that $\lambda = 1/2$. Now setting $\rho(u) = \exp(-2u)P(u)$ gives

$$\left[\frac{1}{4}u\partial_u^2 + (\frac{1}{4} - u)\partial_u + \left(\frac{E}{\hbar\omega} - \frac{1}{2}\right)\right]P(u) = 0. \tag{34}$$

This differential equation can be solved by using a power series expansion $P(u) = \sum_{j=0}^{\infty} p_j u^j$ which leads to the following recursion relation:

$$\frac{1}{4}(j+1)^2 p_{j+1} = \left(j - \frac{E}{\hbar\omega} + \frac{1}{2}\right)p_j. \tag{35}$$

In order to ensure that $\rho(u)$ decreases exponentially at infinity, $P(u)$ must be a polynomial. This means that there exists n such that $p_{n+1} = 0$. This leads to the energy spectrum:

$$E = \hbar\omega\left(n + \frac{1}{2}\right). \tag{36}$$

The resulting polynomials $P(u)$ are Hermite polynomials. Thus, as expected, we have recovered the well-known result of the canonical formulation. If we denote the $*$-eigenfunction corresponding to this energy by Π_n, one can verify that the ground state Π_0 satisfies the property

$$a *_M \Pi_0 = \Pi_0 *_M \overline{a} = 0. \tag{37}$$

We can also check that a and $\overline{a}$ satisfy the properties

$$[a, \overline{a}]_* = 1 \ , \ [H, \overline{a}]_* = +\hbar\omega\overline{a} \ , \ [H, a]_* = -\hbar\omega a \ . \tag{38}$$

Hence, a and $\overline{a}$ can be called ladder functions as they play the same role as the ladder operators $\hat{a}$ and $\hat{a}^\dagger$ in the canonical formulation.

2.5. Further applications

Phase space quantization has several interesting applications in physics. One example is quantum optics where the Wigner function is essential to characterize interference phenomena, as famously done in the work of Nobel Laureat Serge Haroche.[10] It is also a natural language for quantum decoherence and quantum chaos.[11]

3. Non-commutative Space in Quantum Mechanics

3.1. Motivation

The idea of non-commutativity of space coordinates was first proposed by Heisenberg[7] and motivated by quantum field theory, where this idea can serve as a tool for removing the short-distance divergences.[12,13] The non-commutative behavior of the coordinates can be described by

$$[\hat{X}^i, \hat{X}^j] = \theta^{ij} \ , \tag{39}$$

where (θ^{ij}) is an antisymmetric tensor, whose components are of the order of the Planck scale squared.[13]

Analogously to the Heisenberg uncertainty relation, which is due to the non-commutativity of position and momentum, we can similarly assume that there is an uncertainty relation for the spacial coordinates. This means that space is only defined by cells of the order of the Planck scale:

$$\lambda_p = \sqrt{G\hbar/c^3} \approx 1.6 \times 10^{-33} \text{ cm}.$$

The idea behind the existence of a minimal scale (Planck scale) can be put forward as follows: if we consider the Compton wavelength $\lambda_c = \hbar/Mc$ representing the minimum size of the region in which a mass M can be localized, then the Planck mass M_p is defined such that λ_c is at the order of the corresponding Schwarzschild radius $r_s = 2\mathrm{GM}/\mathrm{c}^2$, *i.e.*

$$\lambda_c \sim r_s,$$

which implies that $M_p \sim \sqrt{\hbar c/G}$. The Planck scale is then defined as

$$\lambda_p = \frac{\hbar}{M_p c} \sim \sqrt{\frac{G\hbar}{c^3}} \sim r_s = \frac{2GM_p}{c^2}. \tag{40}$$

As a result, if we attempt to probe a distance at the order of the Planck scale, we would need to provide an energy equal to the Planck mass inside this scale, which would trigger the formation of a micro black hole since the Planck scale corresponds to the Schwarzschild radius of the Planck mass M_p (Equation 40). In addition, no further localization would be possible under the Planck scale as the Heisenberg uncertainty principle imposes a minimal delocalization length, *i.e.* the Compton wavelength of the Planck mass M_p. Thus, these two effects (both classical and quantum) would prevent us from probing smaller distances with respect to the Planck scale.[14] This interesting observation puts forward the idea that space has a minimal scale, which suggests it can be non-commutative.

3.2. Illustration of a non-commutative space

A very nice way to motivate space non-commutativity in quantum mechanics is by considering strong constant magnetic field described by a vector potential $\boldsymbol{A} = (A_x, A_y, A_z)$.[15] The latter field is applied to a charged particle confined to the xy plane. The Lagrangian of such particle reads

$$L = \frac{1}{2m}(\dot{x}^2 + \dot{y}^2) + \frac{e}{c}(\dot{x}A_\mathrm{x} + \dot{y}A_\mathrm{y}) - V(x,y) \ . \tag{41}$$

Here, we choose to work in the Landau gauge, *i.e.* $\boldsymbol{A} = (0, Bx, 0)$, which implies that $\boldsymbol{B} = (0,0,B)$. If we work in the limit of a strong magnetic field, then the kinetic term of the Lagrangian becomes negligible. The Lagrangian reduces to

$$L_\infty = \frac{e}{c}Bx\dot{y} - V(x,y) \ . \tag{42}$$

One can see that eBx/c can be identified with the conjugate momentum of y. Thus, upon quantization, the canonical Hilbert space operators $\hat{x}$ and $\hat{y}$ obey the commutation relation

$$[\hat{y}, \frac{eB}{c}\hat{x}] = i\hbar\mathbb{1}, \tag{43}$$

$$[\hat{x}, \hat{y}] = -i\frac{\hbar c}{eB}\mathbb{1} \ . \tag{44}$$

Hence, the two dimensional space is quantized.[15] This implies that it is defined by elementary space cells whose surfaces are at the order of $\hbar c/(eB)$. The latter can be interpreted as a quantization of the magnetic flux within every single elementary cell, *i.e.*

$$\Phi = B \, \|[\hat{x}, \hat{y}]\| \sim \frac{\hbar c}{e} \sim \Phi_0, \tag{45}$$

where $\|[\hat{x}, \hat{y}]\|$ denotes the typical surface of an elementary cell and $\Phi_0 = hc/2e$ is the quantum magnetic flux.

In the next example, we will see how we can implement the non-commutativity of space in the Schrödinger equation and how non-commutativity of space can suggest the presence of a strong magnetic field.

3.3. Modification of the Schrödinger equation

In the approach of deformation quantization, the quantum states are replaced by phase space functions while substituting the usual product with a non-commutative star product.

In the present instance, we consider quantum states as functions in the position representation of the canonical approach, and we implement space non-commutativity, by replacing the usual product by a non-commutative star product of functions that depend on space coordinates $(x_1, ..., x_N)$. *i.e.* for two smooth functions f and g labeled by space coordinates, the star product on space is defined by

$$(f * g)(\boldsymbol{x}) = f \exp(\frac{i}{2}\theta^{ij}\overleftarrow{\partial_i}\overrightarrow{\partial_j})g(\boldsymbol{x}), \tag{46}$$

which is similar to the star product (Equation 12), the only difference being that $\boldsymbol{x}$ is a spatial coordinate and (θ^{ij}) is a constant antisymmetric $N \times N$ tensor. Thus, the only modification to the Schrödinger equation

$$i\hbar\frac{\partial\psi(\boldsymbol{x},t)}{\partial t} = \left(\frac{\hat{\boldsymbol{p}}^2}{2m} + V(\hat{\boldsymbol{x}})\right)\psi(\boldsymbol{x},t) \tag{47}$$

is to replace $V(\hat{\boldsymbol{x}})\psi(\boldsymbol{x},t)$ by

$$V(\hat{\boldsymbol{x}}) * \psi(\boldsymbol{x},t) = V(\hat{\boldsymbol{x}} - \frac{\hat{\tilde{\boldsymbol{p}}}}{2})\psi(\boldsymbol{x},t), \tag{48}$$

where $\hat{\tilde{p}}_i = \theta^{ij}\hat{p}_j$ and $\hat{p}_j = -i\hbar\partial_j$ is the momentum operator in the position representation.[16]

Now consider a one dimensional harmonic oscillator in a two dimensional non-commutative space such that $[\hat{x}, \hat{y}] = \theta\mathbb{1}$.[17] The Hamiltonian of the one dimensional harmonic oscillator $H = \hat{p}_x^2/2m + k\hat{x}^2/2$ becomes

$$\hat{H}_* = \frac{\hat{p}_x^2}{2m} + \frac{k}{2}\left(\hat{x} - \theta\frac{\hat{p}_y}{2}\right)^2 = \frac{\hat{p}_x^2}{2m} + \frac{1}{2m^*}\left(\hat{p}_y - \hat{x}eB^*\right)^2, \tag{49}$$

where $m^* = 4/(k\theta^2)$ and $eB^* = 2/\theta$. Remarkably, the latter Hamiltonian is similar to the Hamiltonian of a particle in two dimensions moving under the influence of a perpendicular magnetic field B^*. This means that the dynamics of this harmonic oscillator in a non-commutative two dimensional space is analogous to the motion of a charged particle under the influence of a very strong magnetic field.[17] We have $eB^* = 2/\theta$ and the value of θ is expected to very small; typically at the order of the Planck scale, this suggests that B^* is a very strong magnetic field. It turns out that this interesting idea has a deep connection with non-commutativity in string theory and the effect of strong magnetic fields.[18,19]

3.4. Further applications

This exotic approach has many applications in the realm of condensed matter physics as it successfully explained the fractional quantum Hall effect.[20] In addition, this idea is a subject of intense research in quantum gravity and also in string theory where, roughly speaking, the characteristic length of the vibrating strings is of the order of the Planck scale where we expect that space is non-commutative.[13,19]

4. Conclusion

In this note, we have presented a short introduction to the phase space quantization formalism as a deformation of the classical theory and have seen how it is related to the canonical formulation of quantum mechanics.[2] We have also applied the star product deformation in order to account for space non-commutativity in the Schrödinger equation, where the effect of space non-commutativity has a close relation with the effect of a strong magnetic field.

Nowadays, non-commutative gravity is a candidate theory of quantum gravity whose goal is to unify the Einstein's field equations with the laws of quantum mechanics. This approach of quantum gravity is based on the tools of non-commutative geometry which are used to describe the Planck scale.[21]

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References

(1) Perepelitsa, D. V. Path integrals in quantum mechanics, http://web.mit.edu/dvp/www/Work/8.06/dvp-8.06-paper.pdf.

(2) Curtright, T. L.; Zachos, C. K. *Asia Pac. Phys. Newslett.* **2012**, *1* , 37–46.

(3) Groenewold, H. *Physica* **1946**, *12* (7) , 405–460.

(4) Moyal, J. E. In *Mathematical Proceedings of the Cambridge Philosophical Society*, 1949; Vol. 45, pp 99–124.

(5) Bayen, F.; Flato, M.; Fronsdal, C.; Lichnerowicz, A.; Sternheimer, D. *Ann. Phys.* **1978**, *111* (1) , 61–110.

(6) Bayen, F.; Flato, M.; Fronsdal, C.; Lichnerowicz, A.; Sternheimer, D. *Ann. Phys.* **1978**, *111* (1) , 111–151.

(7) Curtright, T.; Zachos, C. K. *Mod. Phys. Lett.* **2001**, *A16* , 2381–2385.

(8) Wigner, E. *Phys. Rev.* **1932**, *40* , 749–759.

(9) Curtright, T. L.; Fairlie, D. B.; Zachos, C. K. , *A concise treatise on quantum mechanics in phase space*; World Scientific Publishing Company: 2013.

(10) Haroche, S. *Rev. Mod. Phys.* **2013**, *85* (3) , 1083.

(11) Zurek, W. H.; Paz, J. P. *Phys. Rev. Lett.* **1994**, *72* , 2508–2511.

(12) Snyder, H. S. *Phys. Rev.* **1947**, *71* (1) , 38.

(13) Szabo, R. J. *Phys. Rept.* **2003**, *378* , 207–299.

(14) Hawking, S. *MNRAS* **1971**, *152* , 75.

(15) Delduc, F.; Duret, Q.; Gieres, F.; Lefrancois, M. *J. Phys. Conf. Ser.* **2008**, *103* , 012020.

(16) Gamboa, J.; Loewe, M.; Rojas, J. C. *Phys. Rev.* **2001**, *D64* , 067901.

(17) Mezincescu, L. **2000** .

(18) Abouelsaood, A.; Callan Jr., C. G.; Nappi, C. R.; Yost, S. A. *Nucl. Phys.* **1987**, *B280* , 599–624.

(19) Seiberg, N.; Witten, E. *J. High Energy Phys.* **1999**, *09* , 032.

(20) Dayi, O. F.; Jellal, A. *J. Math. Phys.* **2002**, *43* , [Erratum: J. Math. Phys. **2004**, 45, 827], 4592.

(21) Chamseddine, A. H.; Connes, A. *Fortsch. Phys.* **2010**, *58* , 553–600.