Plasma waves and instabilities: from drift waves to kinetic MHD modes

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• Introduction to waves and instabilities – reactive and kinetic instabilities

• MHD instabilities – kinetic MHD

• Drift waves

• Non linear saturation processes (sketchy)
Plasma waves: basics

- Small sinusoidal perturbation of the electromagnetic field

\[ \phi(\mathbf{x}, t) = \phi_{k \omega} e^{i(k \cdot \mathbf{x} - \omega t)} + c.c. \]

- Linear response of current of charge and current densities

- Self-consistent problem

Plasma response: kinetic or fluid equations

\[ \text{Maxwell equations} \quad \mathbf{E}, \mathbf{B} \]

\[ \text{charge and current densities} \quad \rho, \mathbf{j} \]
Leads to a dispersion relation

• Maxwell + plasma response leads to a dispersion relation

\[ \epsilon(k, \omega) = \epsilon_r(k, \omega) + i\epsilon_i(k, \omega) = 0 \]

• Solution usually complex

\[ \omega(k) = \omega_r(k) + i\gamma(k) \]

• Real solution \( \omega_r(k) \to \text{wave} \)

• Complex solution \( \gamma(k) > 0 \to \text{instability} \)
A bit more on the dispersion relation

• Convenient to calculate the Lagrangian of the electromagnetic field

\[ L_{k\omega} = \varepsilon_0 E_{k\omega} \cdot E_{k\omega}^* - \frac{1}{\mu_0} B_{k\omega} \cdot B_{k\omega}^* + J_{k\omega} \cdot A_{k\omega}^* - \rho_{k\omega} \phi_{k\omega}^* \]

- Current density
- Charge density
- Electromagnetic field
- Wave-particle interaction

• Electric field

\[ E_{k\omega} = i\omega A_{k\omega} - ik \phi_{k\omega} \]

• Magnetic field

\[ B_{k\omega} = ik \times A_{k\omega} \]
A bit more on the dispersion relation

• Maxwell equations

\[ \frac{dL}{d\phi^*_{k\omega}} = 0 \quad \frac{dL}{dA^*_{k\omega}} = 0 \]

• Small perturbations: \( \rho_{k\omega} \) and \( J_{k\omega} \) are linear functions of \( \phi_{k\omega} \) and \( A_{k\omega} \).

• Dispersion relation

\[ L(k, \omega) = \epsilon(k, \omega) \epsilon_0 |E_{k\omega}|^2 \]
Energetics of an instability

- Energy exchanged between e.m. field and particles (real $\omega$)

$$P(k, \omega) = 2\omega \text{Im}(L_{k\omega}) = J_{k\omega} \cdot E_{k\omega}^* + J_{k\omega}^* \cdot E_{k\omega}$$

- Two types of instabilities

$P(k, \omega) = 0 \quad \rightarrow \quad \text{"reactive"}$

$P(k, \omega) \neq 0 \quad \rightarrow \quad \text{"kinetic"}$

- Marginal stability

$$P_k = P(k, \omega_r(k))$$

Reactive mode $R=0$
Damping $R>0$
"Kinetic" instability $R<0$
Kinetic instability

• Situation close marginal stability $\gamma(k) \ll \omega_r(k)$

• Taylor development of $\varepsilon(k, \omega) = 0$
  - Lowest order $\epsilon_r(k, \omega_r) = 0 \rightarrow$ pulsation $\omega_r(k)$
  - Next order $\gamma_k = -\frac{P_k}{W_k} \rightarrow$ growth rate $\gamma(k)$

• Energy density

$$W_k = \frac{d}{d\omega} \left( \omega \varepsilon_r \right) \bigg|_{\omega = \omega_r(k)}$$

• Instability $\gamma(k) > 0$ if $P_k < 0$ : energy transferred from particles to wave
Reactive instability

- Since $P(k, \omega) = 0$, if $\omega_k$ is solution, $\omega_k^*$ is solution too.
- At threshold
  
  $$\epsilon_r = 0 \quad \text{and} \quad \frac{d\epsilon_r}{d\omega} = 0$$

  $\rightarrow$ energy $W_k = 0$

- Reactive instability sometimes called “negative energy” wave

Analogy with particle motion in a potential

- Stable
- Unstable
- Neutral

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• Vlasov equation, 1D and electrostatic

\[ \partial_t f + v \cdot \partial_x f + \frac{eE}{m} \cdot \partial_v f = 0 \]

• Response function

Unperturbed dist. function

\[ \epsilon(k, \omega) = 1 + \sum_{\text{species}} \frac{\omega_{ps}^2}{k} \int_{-\infty}^{+\infty} dv \frac{\partial_v \hat{f}_{0s}}{\omega - kv + io^+} \]

Plasma frequency

\[ \omega_{ps}^2 = \frac{n_s e_s^2}{m_s \varepsilon_0} \]
Hydrodynamic limit: Buneman instability

- Hydrodynamic limit \( \omega \gg kv \) for ions and electron beam \( f_0 = n_e \delta(v-v_0) \).

- Dispersion relation

\[
\epsilon(k, \omega) = 1 - \frac{\omega_{pi}^2}{\omega^2} - \frac{\omega_{pe}^2}{(\omega - kv_0)^2} = 0
\]

- Negative energy wave is unstable if \( kv_0 \approx \omega_{pe} \rightarrow \omega = \frac{-1 + \sqrt{3}i}{2} \left( \frac{m_e}{2m_i} \right)^{1/3} \omega_{pe} \)

Baumjohann & Treumann 2012

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• Hydrodynamic limit for all thermal species $\omega >> kv$
  + hot beam

• Dielectric

$$\epsilon(k, \omega) \simeq 1 - \frac{\omega_p^2}{\omega^2} - i\pi \frac{\partial}{\partial v} \hat{f}_0 \bigg|_{v=v_{ph}}$$

• Kinetic instability

$$\gamma(k) = -\frac{\epsilon_i}{\partial \omega \epsilon_r \big|_{\omega = \omega_p}} = \omega_p \frac{\pi}{2} \frac{\omega_{pb}^2}{k^2} \frac{\partial}{\partial v} \hat{f}_0 \bigg|_{v=v_{ph}}$$

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Single fluid ideal MHD

- MHD equations
- Lagrangian derivative
- MHD displacement

\[ \rho \frac{d}{dt} \mathbf{V} = -\nabla P + \mathbf{J} \times \mathbf{B} \]

\[ \frac{\partial}{\partial t} \mathbf{B} = \nabla \times (\mathbf{V} \times \mathbf{B}) \]

\[ \frac{d}{dt} P + \Gamma P \nabla \cdot \mathbf{V} = 0 \]

Magnetic field

Velocity

Pressure

Adiabatic index

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Alfvén waves in incompressible medium $\rho=\text{cte}$

- 3 solutions: shear Alfvén wave, fast and slow magneto-acoustic waves
- Shear Alfvén wave

$$\omega = k_\parallel V_A$$

- Alfvén velocity

$$V_A = \sqrt{\frac{B^2}{\mu_0 \rho}}$$
• In single fluid MHD, two main destabilizing terms:

  1) Pressure gradient $\rightarrow$ interchange instability

  2) Current density gradient $\rightarrow$ kink mode
MHD instabilities: interchange

Flux tube 2
\( P_2, V_2 \)

Flux tube 1
\( P_1, V_1 \)

Field line length \( dl \)

\( \oint B \times d\sigma = \text{cte} \)

Interchange

Flux tube 1
\( P'_1, V_2 \)

Flux tube 2
\( P'_2, V_1 \)

\( PV^\Gamma = \text{cte} \)

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• Exchange of two flux tubes

• Released energy

\[ \delta W = \delta P \delta V \simeq \delta P \delta \int \frac{d\ell}{B} \]

• Interchange instability - \( \nabla P \) aligned with \( \nabla B \)
Energy principle

- Force balance equation
  \[ \rho \omega^2 \mathbf{\xi}_\omega = -\mathbf{F}_\omega = \mathbf{L}_\omega \cdot \mathbf{\xi}_\omega \]
  \[ \frac{1}{2} \omega^2 \int d^3x \rho |\mathbf{\xi}_\omega|^2 = \delta W_{MHD} = \int d^3x (\mathbf{\xi}_\omega^* \cdot \mathbf{L}_\omega \cdot \mathbf{\xi}) \]

- MHD energy \( \delta W \) combines wave character and instability sources
  \[ \delta W_{MHD} = \delta W_{\text{wave}} + \delta W_{\text{interchange}} + \delta W_{\text{kink}} \]

- MHD instabilities are reactive
  \[ L_{MHD} = \omega^2 \int d^3x \rho |\mathbf{\xi}_\omega|^2 - 2\delta W_{MHD} \quad \text{real for real } \omega \]
The MHD description usually fails at low frequency

- Two reasons for breakdown of the MHD description:
  1) Landau resonances \( \omega = k \cdot v \)
  2) Effects of finite orbit width \( \delta \)
- Ideal MHD approach valid when
  \[ \omega \gg k \cdot v \text{ and } k_{\perp} \delta \ll 1 \]
- Difficulties occur at low frequencies
Low frequency limit

- Momentum equation for each species

\[ \rho d_t V = -\nabla P + n e (E + V \times B) + \ldots \]

- Strong guide field B

\[ V = V_{\parallel b} + V_E + V^* + V_{pol} + \ldots \]

EXB velocity

\[ V_E = \frac{E \times B}{B^2} \]

Parallel flow

Diamagnetic velocity

\[ V^* = \frac{B}{B^2} \times \frac{\nabla P}{n e} \]

Polarization drift

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Drift wave

- Simple slab geometry, electrostatic, fluid ions
  \[ \partial_t n_i + \nabla \cdot (n_i V_E) = 0 \]
- Fast motion along field lines → adiabatic electrons
  \[ \frac{\delta n_e}{n_{eq}} = \frac{e \delta \phi}{T_e} \]
- Electro-neutrality \((k \lambda_D << 1)\)
  \[ n_e = n_i \approx n_0 \exp(e\phi/T_e) \]

Charge -\(\delta n_e e\)

Potential \(\delta \phi\)

Field line

Drift wave

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• Density gradient in the x direction, uniform B, y,z periodic

• Phase velocity = electron diamagnetic velocity

\[ v_{ph} = \frac{\omega}{k_y} = V_{ne}^* \]

\[ V_{ne}^* = -\frac{T_0}{eB_0} \frac{\partial n_{eq}}{\partial x} e_y \]
• Start with a density $n_i = n_e$ corrugation

• Fast electron response along field lines $\rightarrow$ potential adjusts $\rightarrow$ electric field $E$

• $E \times B$ drifts shifts the perturbation along $V^*_e$
The ion inertia plays a crucial role for non-linear saturation.

- **Drawbacks of the previous model:** no instability, infinity of non-linear solutions
- **Add the polarization drift (ion inertia)**

\[ \mathbf{V} = \mathbf{V}_E + \mathbf{V}^* + \mathbf{V}_{pol} + \ldots \]

Divergence of polarization current

\[ \nabla \cdot (n_e \mathbf{V}_{pol}) \simeq - \frac{n_{eq} m_i}{B^2} d_t \nabla^2 \phi \]

Lagrangian derivative

\[ d_t = \partial_t + \mathbf{V}_E \cdot \nabla \]
A paradigm: the CHM equation

- Same assumptions + polarization drift
- Charney-Hasegawa-Mima (CHM) equation

\[ d_t \left( \phi - \rho_s^2 \nabla^2 \phi \right) + \mathbf{V}_{ne}^* \cdot \nabla \phi = 0 \]

Ion gyroradius: \[ \rho_s = \frac{\sqrt{m_s T_e}}{eB} \]

- Dispersion relation

\[ \omega = \frac{k_y V_{ne}^*}{1 + k^2 \rho_s^2} \]

\[ \mathbf{v}_E = \frac{\mathbf{B}}{B^2} \times \nabla \phi \]

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• Dominant instabilities at low frequency: drift waves mainly driven by interchange

• Kinetic type: driven via resonances by electrons or ions.

• Threshold in temperature and density gradients
• Several branches may co-exist.

• Electron branch at $k_\perp \rho_s < 1$
Example of a tokamak: electromagnetic modes

- **Ballooning mode**: shear Alfvén wave coupled to interchange instability

- **Low frequency limit**: diamagnetic drifts matter

Huysmans 2009
• Drift waves dominate at low beta

\[ \beta = \frac{2P}{B^2/2\mu_0} \]

• At high beta, kinetic ballooning modes become unstable

Pueschel 2010
Hybrid formulation for kinetic MHD modes

Two options:

1) Solve the full kinetic problem (high k)

2) Hybrid formulation (low k)

\[ \rho d_t \mathbf{V} = -\nabla P + \mathbf{J} \times \mathbf{B} - \nabla \cdot \Pi_{hot} \]

Non thermal particle stress tensor

\[ \Pi_{ij} = m \int d^3v f v_i v_j \]

→ imaginary part of \( \delta W \)

→ kinetic instabilities
Alfvén waves driven by fast particles

- Driving mechanism similar to bump on tail
- Quasi-coherent modes
- In some cases, frequency chirping

Turnbull 1993
• Variety of non linear dynamics:

1) Few modes: steady saturated state, relaxation oscillations, explosive behavior, ...

2) Many coupled modes: usually evolve towards turbulence. Turbulent state is different if waves are involved.

• Bump on tail instability is the testbed here
One mode only: particle trapping

- Particle energy in the wave frame of reference

\[ E = \frac{1}{2}mv^2 + e\phi \cos(kx) \]

- Similar to a pendulum – trapping time

\[ \tau_b^{-1} = \sqrt{\frac{ekE_{k\omega}}{m}} \]
Plateau effect

- $F(E)$ is solution of the Vlasov equation
- Flattening of the distribution in the unstable region → stabilisation Berk & Breizman 97
Analogy with vortex mixing: mixing-length estimate

Mixing of the pressure profile by vortex of size $\ell$

$\rightarrow$ “mixing length estimate” of the fluctuation level

$$\frac{\delta p}{p} \approx \frac{\ell}{L_p}$$
Plateau can generate a secondary instability

- Edges of plateau can be unstable 
- Plateau splits in holes and clumps 
- Motion of holes/clumps in phase space $\rightarrow$ chirping

Berk & Breizman 97

Lilley 15
Frequency chirping observed both in experiments and simulations

Lesur 12
• Bump on tail: variety of dynamics depending on dissipation and drive

• Limit of strong drive still to be explored Zonca 15
Stochasticity

- Multiple modes: islands localized around $v = \omega_p/k$
- Trajectory becomes stochastic $\rightarrow$ ergodization $\rightarrow$ flattening

Chirikov 59, see Lichtenberg & Liberman, 1983
Quasi-Linear Theory

- Linear solution Vlasov

\[ f_{k\omega} = -i \frac{eE_{k\omega}}{m} \frac{\partial_v f_0}{\omega - kv} \]

- Evolution equation of \( f_0 \) in velocity space

\[ \partial_t f_0 + \partial_v \Gamma = 0 \]

Average distribution function \hspace{1cm} Flux in phase space

\[ f_0(v, t) = \int \frac{dx}{2\pi} f(x, v, t) \]

\[ \Gamma = \sum_{k, \omega} \frac{eE_{k\omega}^*}{m} f_{k\omega} \]
• Linear solution of Vlasov equation → flux

\[ \Gamma = -D_{QL} \partial_v f_0 \]

• Quasi-linear diffusion coefficient

\[ D_{QL} = \sum_{k,\omega} \left| \frac{e E_{k\omega}}{m} \right|^2 \frac{\gamma(k)}{\left( \omega_r(k) - kv \right)^2 + \gamma^2(k)} \]

\[ \text{Besse 2011} \]

• Often effective beyond validity conditions cf lecture Gürcan
Mode coupling is needed to compute the turbulence intensity

- Generic form of a non-linear equation in Fourier space:

\[ \phi(x, t) = \sum_k \phi_k(t) \exp \{ i k \cdot x \} \]

\[ \partial_t \phi_k(t) = -i \omega(k) \phi_k(t) + \sum_{k'k''} \Lambda_{k'k''} \phi_{k'}(t) \phi_{k''}(t) \]

- Triad: \( k' + k'' = k \)

- Coupling leads to energy transfer between waves – if coupling is "local" \( \rightarrow \) cascade
Case of drift waves

- Coupling term for CHM equation

$$\Lambda_{k'k''} = -\frac{1}{2} \frac{\rho_s^2}{1 + k_\perp^2 \rho_s^2} \mathbf{e}_z \cdot (k' \times k'') \left( k''_\perp^2 - k'_\perp^2 \right)$$

- Conserves energy and enstrophy

$$E = \frac{1}{2} \sum_k \left( 1 + \rho_s^2 k_\perp^2 \right) |\phi_k(t)|^2$$

$$\Omega = \frac{1}{2} \sum_k \left( 1 + \rho_s^2 k_\perp^2 \right) k_\perp^2 |\phi_k(t)|^2$$
• Decay of a « pump » mode \((k_0, \omega_0)\) into two « daughter » waves \((k_1, \omega_1)\), \((k_2, \omega_2)\)

• **Constraint** \(k_0 = k_1 + k_2\); \(\omega_0 = \omega_1 + \omega_2\)

• Drift waves: decay possible if \(k_1^2 < k_0^2 < k_2^2\)
• If energy is strictly conserved: pump recovers
→ dissipative processes make it irreversible

Baumjohann & Treumann 2012
Example of a parametric decay of an acoustic wave into two drift waves

- Geodesic acoustic mode driven by energetic ions
- Parametric decay into drift (ITG) waves

Drift waves

Geodesic acoustic mode
Many unstable modes: turbulence

- If many modes are unstable: the system evolves towards a turbulent state Waltz 83, Horton 86

- Can be seen as a strange attractor in the phase space \( \{\phi_k\} \)
- Frequency spectra are broad

- Predictive model of the frequency spectrum shape still an open question
• Quasi-linear theory works well for drift waves

• A dual cascade is expected CHM

• Does not fit observation in tokamaks (see lecture Vermare) – many reasons:
  - no inertial range
  - coupling to large scale flows
  - kinetic effects
A few words on wave turbulence

- Waves change the nature of turbulence see lecture Nazarenko

- Drift or Rossby waves:
  - not isotropic
  - generation of large scale shear flows $\rightarrow$ self-regulation

Dif-Pradalier 15

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Conclusions

• Linear stability is well documented – not that simple though …

• Non linear dynamics much more complex – No simple recipe!

• For turbulent states and wave/particle interaction via Landau resonances: quasi-linear theory often works well

• Predicting a level of fluctuations is trickier

• For quasi-coherent modes: variety of non linear dynamics – parameter dependent.
Some useful textbooks

• D. B. Melrose “Instabilities in Space and Laboratory Plasmas” Cambridge UP 1986

• W. Baumjohann and R.A. Treumann “Basic Space Plasma Physics” Imperial UP 2012 vol I and II


• D. Biskamp “Nonlinear Magnetohydrodynamics », 1997 Cambridge UP

• A.J. Lichtenberg and M.A. Liberman, “Regular and stochastic motion” Springer 1983

• Y. Elskens and D. Escande “Microscopic dynamics of plasmas and chaos”, IOP 2003