An introduction to turbulence

Turbulence, magnetic fields, and self-organization in laboratory and astrophysical plasmas

Les Houches, France
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Turbulence in lab & natural plasmas

Fusion plasmas

Space plasmas

Astrophysical plasmas

Basic plasma science

What are the fundamental principles of plasma turbulence?
Central role of plasma turbulence

Particle acceleration & propagation

Cross-field transport

Magnetic reconnection

Dynamo action

What is the role of plasma turbulence in these processes?
This lecture is meant to be an introduction; no prior knowledge of turbulence is required.

You are invited to ask questions at any time.

Lectures may explain, motivate, inspire etc., but they cannot replace individual study...
Recommended reading

Also:

P. A. Davidson
Turbulence
Some turbulence basics
Turbulence is ubiquitous...

Active fluids (dense bacterial suspensions)
According to a famous statement by Richard Feynman…

…as well as an important unsolved physics problem

A challenging topic for both basic and applied research

“Millennium Issue”
(December 1999)

TURBULENCE:
A challenging topic for both basic and applied research
WHAT IS TURBULENCE THEN?

Turbulence…

• is an intrinsically nonlinear phenomenon
• occurs only in open systems
• involves many degrees of freedom
• is highly irregular in space and time
• often leads to a statistically quasi-stationary state far from thermodynamic equilibrium

Note that these are also the features of LIFE!

Figure 4. The Energy Cascade Picture of Turbulence

This figure represents a one-dimensional simplification of the cascade process with $\beta$ representing the scale factor (usually taken to be 1/2 because of the quadratic nonlinearity in the Navier-Stokes equation). The eddies are purposely shown to be "space filling" in a lateral sense as they decrease in size. (This figure was modified with permission from Uriel Frisch. 1995. Turbulence: The Legacy of A.N. Kolmogorov. Cambridge, UK: Cambridge University Press.)

Figure 5. Time Series of Velocities in a Turbulent Boundary Layer

This time series of velocities for an atmospheric turbulent boundary layer with Reynolds number $Re \approx 2 \times 10^7$ was measured at a single location with a hot-wire anemometer. The velocity fluctuations are apparently random. (This figure is courtesy of Susan Kurien of Los Alamos, who has used data recorded in 1997 by Brindesh Dhruva of Yale University.)
WHAT IS THE CHALLENGE?

Theories that don’t apply directly:

- nonlinear dynamics
- equilibrium statistical mechanics
- nonequilibrium statistical mechanics near equilibrium

Co-existence of randomness and coherence

Two dangers constantly threaten the world: order and disorder. Paul Valéry
Supercomputers can help to unravel the “mysteries” of turbulence in the spirit of John von Neumann (1949):

„There might be some hope to 'break the deadlock' by extensive, but well-planned, computational efforts. There are strong indications that one could name certain strategic points in this complex, where relevant information must be obtained by direct calculations. This should, in the end, make an attack with analytical methods possible.“
Milestones in turbulence research I

Osborne Reynolds (1842-1912)
1883  Transition from laminar to turbulent flows (Reynolds number)
1895  Reynolds decomposition into mean and fluctuating flows

Ludwig Prandtl (1875-1953)
1904  Recognition of the importance of boundary layers
1925  Mixing length model for turbulent transport

Lewis Fry Richardson (1881-1953)
1922  Book „Weather Prediction by Numerical Process“
      Notion of turbulent eddies & (local, direct) energy cascade

Geoffrey Ingram Taylor (1886-1975)
1935  Series of papers on the „Statistical Theory of Turbulence“
Werner Heisenberg (1901-1976)
1923 „Über Stabilität und Turbulenz von Flüssigkeitsströmen“
1948 Three papers on the statistical theory of turbulence

Andrey Kolmogorov (1903-1987)
1941 K41 theory: dimensional analysis, -5/3 law (energy spectrum)
1962 K62 theory: scale invariance is broken, problem of intermittency

Robert Kraichnan (1928-2008)
1957- Field-theoretic approach: Direct Interaction Approximation
1967 Inverse energy cascade in two-dimensional fluid turbulence
1973 Field-theoretic approach: Martin-Siggia-Rose formalism
Milestones in turbulence research III

Steven Orszag (1943-2011)

1966  *Eddy-Damped Quasi-Normal Markovian* approximation

1969- Towards *Direct Numerical Simulations* via spectral methods

1972  First 3D DNS on a $32^3$ grid by S. Orszag & G. Patterson

1948- *Numerical weather prediction* by J. von Neumann & J. Charney

1963  *Large-Eddy Simulation* techniques by J. Smagorinsky

1965  *Fast Fourier Transform* algorithm by J. Cooley & J. Tukey

1977  Cray-1 at the *National Center for Atmospheric Research*

2002  Largest 3D DNS to date on a $4096^3$ grid by Y. Kaneda et al.
Turbulence: The bigger picture

Some grand challenges
• Design airplanes, ships, cars etc.
• Predict weather & climate
• Unravel role of turbulence in space & astrophysics
• Predict performance of fusion devices like ITER

Some open problems beyond Homogeneous Isotropic Turbulence
• Effects of inhomogeneity, anisotropy, compressibility
  (role of walls, drive, stratification, rotation etc.)
• From fluid to magneto-/multi-fluid to kinetic turbulence

Conceptual approach
• Ab initio simulations of complicated problems are not feasible
• Seek physics understanding to construct reliable minimal models
Scientific Overview

Turbulence is perhaps the primary paradigm of complex nonlinear multi-scale dynamics. It is ubiquitous in fluid flows and plays a major role in problems ranging from the determination of drag coefficients and heat and mass transfer rates in engineering applications, to important dynamical processes in environmental science, ocean and atmosphere dynamics, geophysics, and astrophysics. Understanding turbulent mixing and transport of heat, mass, and momentum remains an important open challenge for 21st century physics and mathematics.

This IPAM program is centered on fundamental issues in mathematical fluid dynamics, scientific computation, and applications including rigorous and reliable mathematical estimates of physically important quantities for solutions of the partial differential equations that are believed, in many situations, to accurately model the essential physical phenomena.

Workshop Schedule

• Mathematics of Turbulence Tutorials: September 9 - 12, 2014.
• Workshop I: Mathematical Analysis of Turbulence. September 29 - October 3, 2014.
• Workshop II: Turbulent Transport and Mixing. October 13 - 17, 2014.
• Culminating Workshop at Lake Arrowhead Conference Center, December 7 – 12, 2014.

Participation

This program will bring together physicists, engineers, analysts, and applied mathematicians to share problems, insights, results and solutions. Enhancing communications across these traditional disciplinary boundaries is a central goal of the program.

Full and partial support for long-term participants is available. We are especially interested in applicants who intend to participate in the entire program (September 8 – December 12, 2014), but will consider applications for shorter periods. Funding is available for participants at all academic levels, though recent PhDs, graduate students, and researchers in the early stages of their careers are especially encouraged to apply. Encouraging the careers of women and minority mathematicians and scientists is an important component of IPAM's mission and we welcome their applications. More information and an application is available online.
On the physics of 3D fluid turbulence
Early observations

Leonardo da Vinci (ca. 1500)
The Navier-Stokes equations (1822)

The NSE for incompressible fluids:

\[ \partial_t v + v \cdot \nabla v = -\nabla p + \nu \nabla^2 v, \]
\[ \nabla \cdot v = 0. \]

Expressed in terms of vorticity \( \omega = \nabla \wedge v \):

\[ \partial_t \omega = \nabla \wedge (v \wedge \omega) + \nu \nabla^2 \omega \]

\[ \nabla v^2 = 2v \cdot \nabla v + 2v \wedge (\nabla \wedge v). \]
\[ \mathbf{v} = V \tilde{\mathbf{v}}, \quad \mathbf{x} = L \tilde{\mathbf{x}}, \quad t = (L/V) \tilde{t}, \quad (p/\rho) = V^2 \tilde{p} \]

\[ \partial_t \tilde{\mathbf{v}} + (\tilde{\mathbf{v}} \cdot \nabla) \tilde{\mathbf{v}} = -\nabla \tilde{p} + \frac{1}{Re} \Delta \tilde{\mathbf{v}} \quad \text{Similarity principle} \]

\[ \nabla \cdot \tilde{\mathbf{v}} = 0 \]

\[ Re = \frac{V L}{\nu} \]

Reynolds number
Conservation laws

Spatial average in a 3D periodic box:
\[ \langle f \rangle \equiv \frac{1}{L^3} \int_{B_L} f(r) dr \]

Vorticity: \[ \omega = \nabla \times \mathbf{v} \]

**Kinetic energy**

\[ E \equiv \left\langle \frac{1}{2} |\mathbf{v}|^2 \right\rangle, \quad \Omega \equiv \left\langle \frac{1}{2} |\omega|^2 \right\rangle, \]
\[ H \equiv \left\langle \frac{1}{2} \mathbf{v} \cdot \omega \right\rangle, \quad H_{\omega} \equiv \left\langle \frac{1}{2} \omega \cdot \nabla \times \omega \right\rangle \]

**Enstrophy**

**Ideal invariants**

**Helicity**

**Vortical helicity**

\[ \frac{d}{dt} E = -2v \Omega, \quad \frac{d}{dt} H = -2v H_{\omega} \]
Dissipative anomaly

Energy dissipation rate:

\[ \varepsilon \equiv -\frac{dE}{dt} \]

\[ \lim_{\nu \to 0} \nu \int_V |\omega|^2 \, d\mathbf{x} \neq 0 \]

In turbulent flows, the energy dissipation rate \( \varepsilon \) has a finite limit as the viscosity \( \nu \) tends to zero!

In this sense, the Euler and Navier-Stokes equations are fundamentally different!
Energy budget scale-by-scale

Low-/High-pass filter

\[ f(r) = \sum_k \hat{f}_k e^{ik \cdot r} \]

\[ f^\leq_K(r) = \sum_{k \leq K} \hat{f}_k e^{ik \cdot r} \]

\[ f^\geq_K(r) = \sum_{k > K} \hat{f}_k e^{ik \cdot r} \]

\[ f(r) = f^\leq_K(r) + f^\geq_K(r) \]
Energy budget scale-by-scale (cont’d)

Cumulative kinetic energy between wavenumbers 0 and K:

\[ \varepsilon_K \equiv \frac{1}{2} \langle |v_K^<|^2 \rangle = \frac{1}{2} \sum_{k \leq K} |\hat{v}_k|^2 \]

Scale-by-scale energy budget equation:

\[ \hat{c}_t \varepsilon_K + \Pi_K = -2\nu \Omega_K + \mathcal{F}_K \]

Energy flux through wavenumber K:

\[ \Pi_K \equiv \langle v_K^< \cdot (v_K^< \cdot \nabla v_K^>) \rangle + \langle v_K^< \cdot (v_K^> \cdot \nabla v_K^>) \rangle \]

Cumulative enstrophy:

\[ \Omega_K \equiv \frac{1}{2} \langle |\omega_K^<|^2 \rangle = \frac{1}{2} \sum_{k \leq K} k^2 |\hat{v}_k|^2 \]

Cumulative energy injection:

\[ \mathcal{F}_K \equiv \langle f_K^< \cdot v_K^< \rangle = \sum_{k \leq K} \hat{f}_k \cdot \hat{v}_{-k} \]
Spectral energy transfer

**Energy spectrum:**

\[ E(k) = \frac{\partial}{\partial k} \left( \frac{1}{2} |\mathbf{v}_k|^2 \right) \]

Scale-by-scale energy transfer equation:

\[ \partial_t E(k) = T(k) + F(k) - 2\nu k^2 E(k) \]

Net energy transfer spectrum:

\[ T(k) \equiv -\frac{\partial}{\partial k} \Pi_k \]

Energy injection spectrum:

\[ F(k) = \frac{\partial}{\partial k} \left( \mathbf{f}_k \cdot \mathbf{v}_k \right) \]
Spectral energy transfer (cont’d)

For homogeneous turbulence:

\[
\frac{\partial}{\partial t} E(k, t) = P_k(k, t) - \frac{\partial}{\partial k} T_k(k, t) - 2\nu k^2 E(k, t)
\]

Production  Spectral transfer  Dissipation
The Richardson cascade (real space)

„Big whorls have little whorls, little whorls have smaller whorls that feed on their velocity, and so on to viscosity“
The Richardson cascade (Fourier space)

Turbulence as a local cascade in wave number space...

Much turbulence research addresses the cascade problem. (Important note: In this context, think of an Autobahn, not of a waterfall…)
Kolmogorov’s theory from 1941

K41 is based merely on intuition and dimensional analysis – it is *not* derived rigorously from the Navier-Stokes equation

Key assumptions:

- Scale invariance – like, e.g., in critical phenomena
- Central quantity: energy flux $\varepsilon$

$$E = \frac{1}{2V} \int v^2 \, d^3x = \int_0^\infty E(k) \, dk$$

$E(k) = C \varepsilon^{2/3} k^{-5/3}$

This is the most famous turbulence result: the “-5/3” law.

However, K41 is fundamentally wrong: scale invariance is broken (anomaly)!
Intermittency & structure functions

\[ S_p(\ell) = \langle (\delta v_\parallel(\ell))^p \rangle \propto \ell^{\zeta_p} \]
Intermittency & non-Gaussian pdf's

Renner et al. 2001
financial data
(exchange rates)
Direct numerical simulations

Wilczek et al. 2008

Structure formation and broken scale invariance
Key open issues: Inertial range

- Is the inertial range physics universal (for $\text{Re} \to \infty$)?
- If so, can one derive a rigorous IR theory from the NSE?
- How should one, in general, handle the interplay between randomness and coherence? Key issue: Intermittency!

Example: Trapping of tracers in vortex filaments

Note:

The observed deviations from self-similarity can be reproduced qualitatively by relatively simple vortex models.

Wilczek, Jenko, and Friedrichs 2008
Key open issues: Drive range

- Often, one is interested mainly in the *large* scales. Here, one encounters an interesting interplay between linear (drive) and nonlinear (damping) physics. – *Is it possible to remove the small scales?*

- Candidates: LES, dynamical systems approach etc.
Turbulence: Further reading
Some literature on 3D turbulence

Hydrodynamical turbulence:


MHD turbulence: