New Developments around the Synchrosqueezing Transform

S. Meignen, LJK Laboratory,
joint work with R. Behera (Post-doc), LJK, T. Oberlin, IRIT, Toulouse,
and Duong-Hung Pham (Doc), LJK.

ANR meeting

19th of November 2015

Plan

1. Theoretical analysis of second-order synchrosqueezing transform
2. Improving the synchrosqueezing transform using demodulation

Plan

Synchrosqueezing transform in the STFT framework

In the following, V_f^g denotes the short time Fourier transform (STFT) of a signal f and is defined by:

$$V_f^g(\eta, t) = \int_{\mathbb{R}} f(\tau) g(\tau - t) e^{-2i\pi\eta(\tau - t)} d\tau$$

- Computation of an instantaneous frequency (IF) estimate at time t and frequency η through:

$$\hat{\omega}_f(\eta, t) = \frac{1}{2\pi} \mathcal{I} \left(\frac{\partial_t V_f^g(\eta, t)}{V_f^g(\eta, t)} \right)$$

- The synchrosqueezing transform is defined as:

$$T_f(\omega, t) = \frac{1}{g(0)} \int_{\mathbb{R}} V_f^g(\eta, t) \delta(\omega - \hat{\omega}_f(\eta, t)) d\eta$$

Mode reconstruction for slightly modulated modes

- Let us consider a multicomponent signal defined as

$$f(t) = \sum_{k=1}^{K} f_k(t)$$

with $f_k(t) = A_k(t)e^{2i\pi\phi_k(t)}$, $A_k(t) > 0$, $\phi_k'(t) > 0$ and $\phi_{k+1}'(t) > \phi_k'(t)$.

- Define the set $\mathcal{B}_{\Delta,\epsilon}$ of multicomponent signals with modulation ϵ and separation Δ as:

$$A_k \in C^1(\mathbb{R}) \bigcap L^\infty(\mathbb{R}), \phi_k \in C^2(\mathbb{R}),$$

$$\sup_{t\in\mathbb{R}} \phi_k'(t) < \infty,\ \phi_k'(t) > 0, A_k(t) > 0,\ \forall t$$

$$|A_k'(t)| \leq \epsilon, |\phi_k''(t)| \leq \epsilon,\ \forall t \in \mathbb{R}$$

- The $f_k's$ are separated with resolution Δ:

$$\forall t,\ \phi_{k+1}'(t) - \phi_k'(t) \geq 2\Delta$$

Approximation theorem

Theorem

Consider $f \in \mathcal{B}_{\Delta,\epsilon}$ *and put* $\tilde{\epsilon} = \epsilon^{1/3}$. *Let* $g \in \mathcal{S}(\mathbb{R})$ *with* $supp(\hat{g}) \subset [-\Delta, \Delta]$. *Then, if* ϵ *is small enough, the following holds:*

(a) $|V_f^g(\eta, t)| > \tilde{\epsilon}$ *only when there exists* $k \in \{1, ..., K\}$ *s. t.* $(\eta, t) \in Z_k := \{(\eta, t), \ s.t. \ |\eta - \phi_k'(t)| < \Delta\}$.

(b) *For all* $k \in \{1, ..., K\}$ *and all* $(\eta, t) \in Z_k$ *such that* $|V_f^g(\eta, t)| > \tilde{\epsilon}$, *we have*

$$|\widehat{\omega}_f(\eta, t) - \phi_k'(t)| \leq \tilde{\epsilon}.$$

(c) *For all* $k \in \{1, ..., K\}$, *there exists* C *s.t. for all* $t \in \mathbb{R}$,

$$\left| \lim_{\delta \to 0} \left(\int_{|\omega - \phi_k'(t)| < \tilde{\epsilon}} T_f^{\delta, \tilde{\epsilon}}(\omega, t) dw \right) - f_k(t) \right| \leq C\tilde{\epsilon}.$$

Extension of the theorem

Definition

A window $g_\sigma(t) = \sigma^{-1} g(\frac{t}{\sigma})$ is said to be with quadratic decay if $\hat{g}(\eta)$ behaves like $\frac{1}{\gamma\eta^2}$ for any $|\eta| \geq \Delta$.

- If one uses such a window the above theorem is still valid provided $\sigma\tilde{\epsilon} \geq \frac{1}{\sqrt{|\gamma|\Delta}}$.
- We are going to use non compactly supported window in the Fourier domain to extend the above theorem to multicomponent signals containing modes with strong frequency modulation.

New IF estimate

- Let us put $\tilde{\omega}_f(\eta,t) = \frac{\partial_t V_f^g(\eta,t)}{2i\pi V_f^g(\eta,t)}$ and introduce: $\tilde{t}_f(\eta,t) = t - \frac{\partial_\eta V_f^g(\eta,t)}{2i\pi V_f^g(\eta,t)}$.
- Define $\widehat{\omega}_f(\eta,t) = \mathcal{R}(\tilde{\omega}_f(\eta,t))$, and $\widehat{t}_f(\eta,t) = \mathcal{R}(\tilde{t}_f(\eta,t))$.
- Estimate of frequency modulation introduced in (Oberlin et *al.*, 2015).

Definition

Let $f \in L^2(\mathbb{R})$ and consider when $V_f^g(\eta,t) \neq 0$ and $\frac{\partial_t(\partial_\eta V_f^g(\eta,t)}{V_f^g(\eta,t))} \neq 2i\pi$ the quantity

$$\tilde{q}_f(\eta,t) \quad = \quad \frac{\partial_t \tilde{\omega}_f(\eta,t)}{\partial_t \tilde{t}_f(\eta,t)}.$$

An estimate of the frequency modulation is then defined by

$$\hat{q}_f(\eta,t) = \mathcal{R}\left(\tilde{q}_f(\eta,t)\right).$$

- IF estimator finally defined by: $\phi'(t) = \hat{\omega}_f(\eta,t) + \hat{q}_f(\eta,t)(t - \hat{t}_f(\eta,t))$

Here we propose a slightly different IF estimate which allows for mathematical study.

Definition

Let $f \in L^2(\mathbb{R})$. Define the second order IF complex estimate of f as:

$$\tilde{\omega}_f^{(2)}(\eta,t) = \begin{cases} \tilde{\omega}_f(\eta,t) + \tilde{q}_f(\eta,t)(t - \tilde{t}_f(\eta,t)) & \text{if } \partial_t \tilde{t}_f(\eta,t) \neq 0 \\ \tilde{\omega}_f(\eta,t) & \text{otherwise,} \end{cases}$$

and then its real part:

$$\hat{\omega}_f^{(2)}(\eta,t) = \begin{cases} \mathcal{R}\left(\tilde{\omega}_f(\eta,t) + \tilde{q}_f(\eta,t)(t - \tilde{t}_f(\eta,t))\right) & \text{if } \partial_t \tilde{t}_f(\eta,t) \neq 0 \\ \hat{\omega}_f(\eta,t) & \text{otherwise.} \end{cases}$$

Definition of the new Fourier-based SST

We first define a new type of mode before introducing the new synchrosqueezing transform:

Definition

The set $\mathcal{B}^{(2)}_{\Delta,\epsilon}$ of multicomponent signals with second order modulation ϵ and separation Δ corresponds to multicomponent signals made of modes satisfying:

(a) function f_k s. t. A_k and ϕ_k satisfy the following conditions:

$$A_k(t) \in L^\infty(\mathbb{R}) \cap C^2(\mathbb{R}),\ \phi_k(t) \in C^3(\mathbb{R}),$$
$$\phi_k^{'}(t),\ \phi_k^{''}(t),\ \phi_k^{'''}(t) \in L^\infty(\mathbb{R}),$$
$$A_k(t) > 0,\ \inf_{t\in\mathbb{R}} \phi_k^{'}(t) > 0,\ \sup_{t\in\mathbb{R}} \phi_k^{'}(t) < \infty,$$
$$|A_k^{'}(t)| \leq \epsilon,\ |A_k^{''}(t)| \leq \epsilon,\ \text{and } |\phi_k^{'''}(t)| \leq \epsilon,$$

(b) the functions f_ks satisfy the following separation condition

$$\phi_{k+1}^{'}(t) - \phi_k^{'}(t) > 2\Delta \quad \forall t \in \mathbb{R} \quad , \quad \forall k \in \{1, ..., K-1\}.$$

Definition

Let h be a positive L^1 normalized window belonging to $C_c^\infty(\mathbb{R})$, and consider $\gamma, \delta > 0$, the second order Fourier-based SST of f with threshold γ and accuracy δ is defined by:

$$\widetilde{T}_f^{\delta,\gamma}(\omega, t) = \frac{1}{g(0)} \int_{|V_f^{g_\sigma}(\eta,t)|>\gamma} V_f^{g_\sigma}(\eta, t) \frac{1}{\delta} h\left(\frac{\omega - \hat{\omega}_f^{(2)}(\eta, t)}{\delta}\right) d\eta.$$

Theorem

Consider $f \in \mathcal{B}^{(2)}_{\Delta,\epsilon}$, $\tilde{\epsilon} = \epsilon^{1/6}$.

g *a window satisfying for all* t *and* $k = 1, \cdots, K$, $\mathcal{F}(g(\tau)e^{i\pi\sigma^2\phi_k''(t)\tau^2})(\eta)$ *behaves like* $\frac{1}{\gamma\eta^2}$ *when* $|\eta| \geq \Delta$.

Then, provided ϵ *is sufficiently small and that* $\frac{1}{\sqrt{|\gamma|\Delta}} \leq \sigma\tilde{\epsilon} \leq C$ *for some constant* C*, the following hold:*

(a) $|V_f^{g_\sigma}(\eta, t)| \geq \widetilde{\epsilon}$ *only when there exists* $k \in \{1, ..., K\}$ *such that* $(\eta, t) \in Z_k$.

(b) *For all* $k \in \{1, ..., K\}$ *and all* $(\eta, t) \in Z_k$ *such that* $|V_f^{g_\sigma}(\eta, t)| > \tilde{\epsilon}$, *we have*

$$|\widehat{\omega}_f^{(2)}(\eta, t) - \phi_k'(t)| \leq \tilde{\epsilon}.$$

(c) *Moreover, for each* $k \in \{1, ..., K\}$, *there exists a constant* C *such that*

$$\left| \left(\lim_{\delta \to 0} \frac{1}{2\pi g(0)} \int_{\mathcal{M}_{k,\widetilde{\epsilon}}} \widetilde{T}_f^{\delta,\widetilde{\epsilon}}(\omega, t) d\omega \right) - A_k(t) e^{i\phi_k(t)} \right| \leq C\widetilde{\epsilon},$$

where $\mathcal{M}_{k,\widetilde{\epsilon}} := \{\omega : |\omega - \phi_k'(t)| < \widetilde{\epsilon}\}$.

Plan

1. Theoretical analysis of second-order synchrosqueezing transform

2. Improving the synchrosqueezing transform using demodulation

- The results of the synchrosqueezing transform, highly sensitive to the choice of the window's length.
- We are interested in limiting the influence of the window on the reconstruction of the modes by means of demodulation.

Sensitivity to σ of the synchrosqueezing transform

- IF estimate $\hat{\omega}_f(\eta,t)$ is tied to two inequalities, (the window is a L^1 normalized Gaussian window):

$$\begin{aligned} |V_f^{g_\sigma}(\eta,t) - \sum_{l=1}^{K} f_l(t)\widehat{g_\sigma}(\eta-\phi_l'(t))| &\leq \epsilon\Gamma_{1,\sigma}(t) \\ |\partial_t V_f^{g_\sigma}(\eta,t) - 2i\pi\sum_{l=1}^{K} f_l(t)\phi_l'(t)\widehat{g_\sigma}(\eta-\phi_l'(t))| &\leq \epsilon(\Gamma_{2,\sigma}(t)+2\pi|\eta|\Gamma_{1,\sigma}(t)), \end{aligned}$$

  with $\Gamma_{1,\sigma}(t) = K\sigma I_1 + \pi\sigma^2 I_2 \sum_{l=1}^{K} A_l(t)$ and $\Gamma_{2,\sigma}(t) = KI_1' + \pi\sigma I_2' \sum_{l=1}^{K} A_l(t)$. From this we may say, that better approximation is obatined using a small σ.

- However, in such a case the separation between the different modes is made harder since $\hat{g}(\sigma\Delta)$ has to be negligible, which puts a great constraint on σ (it has to be large).

Algorithm for Demodulation

- To compute an estimate of the ridge $(t, \phi_k'(t))$ knowing the number K of mode, we compute the local minimum of:

$$E_f((\varphi_k)_{k=1,\cdots,K}) = \sum_{k=1}^{K} -\int_{\mathbb{R}} |T_f(t,\varphi_k(t))|^2 dt + \int_{\mathbb{R}} (\lambda \varphi_k'(t)^2 + \beta \varphi_k''(t)^2) dt,$$

  where T_f is the synchrosqueezing transform.
- We obtain a set of curves $(t, \varphi_k(t))_{k=1,\cdots,K}$ which approximates the different ridges. However, due to discretization effects these estimates cannot be directly used for demodulation.
  If the signal f is defined on $[0,1]$, $\varphi_k(t)$ will be an integer for all t.
- We use EMD to smooth the obtained ridges and thus non integer value estimates.
- We use these smoothed ridges to demodulate, we compute:

$$f_{D,k}(t) \quad = \quad f(t) e^{-i2\pi(\int_0^t \tilde{\varphi}_k(t)dx - \varphi_0(t))}$$

  where $\tilde{\varphi}_k(t) = \varphi_k(t) - h(t)$, where $h(t)$ is the first mode obtained with EMD.

Algorithm for Demodulation:continued

- Apply synchrosqueezing transform to $f_{D,k}$ and extract the kth mode
- Reconstruct the kth mode by multiplying by $e^{-i2\pi(\int_0^t \tilde{\varphi}_k(t)dx - \varphi_0(t))}$

Illustration of the demodulation process:

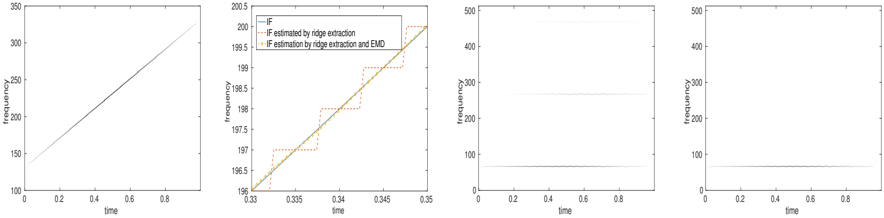
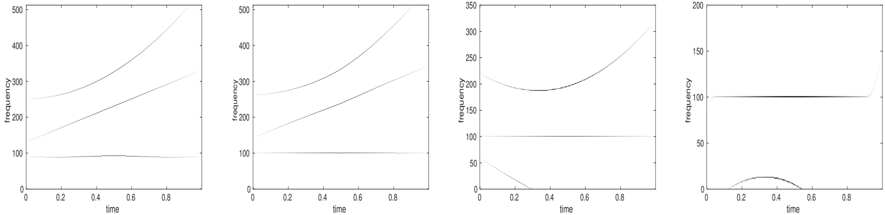


Illustration of the improved reconstruction process

Effect of the demodulation process:



Alleviating the dependence on σ: